Mathlib, Analysis, and the Index Theorem

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The Mathlib Initiative

11 May 2026

Plan

I. Background and motivation

II. Proof via heat equation following [1]

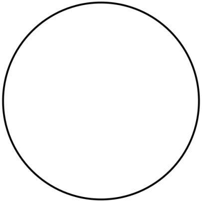
III. Analysis wish list for Mathlib

1. Roe, J. "Elliptic operators, topology and asymptotic methods".

I. Background and motivation

The (analytic) index and deformations (1/2)

Consider the eigenvalue problem for $i\partial_x$ on the circle:



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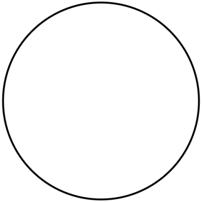

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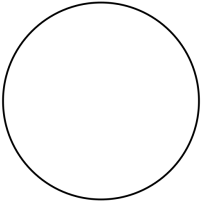

For $\lambda \in \mathbb{C}$ let D_λ, acting on functions $S^1 \to \mathbb{C}$, be:

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What is $\dim \ker D_\lambda$?

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Regard $f : S^1 \to \mathbb{C}$ as $f : \mathbb{R} \to \mathbb{C}$ with period 2π and calculate:

$$\begin{aligned} f \in \ker D_\lambda &\iff \partial_x f = -i\lambda f \\ &\iff f(x) = Ce^{-i\lambda x} \end{aligned}$$

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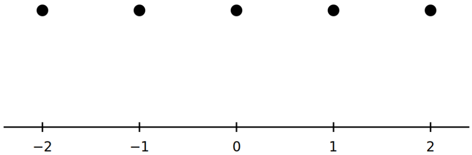

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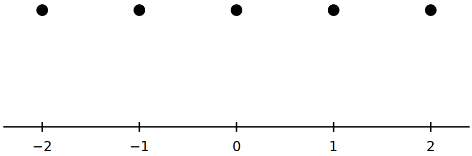

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constant.

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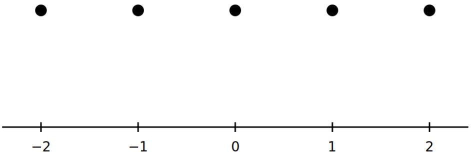

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Ellipticity means $\sigma_D(\xi)$ is an equivalence $E_x \simeq F_x$ whenever $\xi \neq 0$.

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This is the topological index. (Similar definition for higher order.)

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For an elliptic differential operator D on a smooth, closed oriented manifold, the analytic index equals the topological index:

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(Also many, many generalisations.)

Examples and applications (1/4)

Given a compact oriented smooth manifold M, choose a Riemannian metric and define the de Rham operator:

$$D = d + d^* : \Omega^{\text{even}} \to \Omega^{\text{odd}}$$

Get:

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Using Chern-Weil theory: $e(TM)[M] = \frac{1}{(2\pi)^{n/2}} \int Pf(K)$.

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E.g., for K3 we easily get $s = -16$ from just $\chi = 24$.

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Given compact complex manifold M and holomorphic vector bundle V, choose Hermitian metrics on TM and V and define Dolbeault operator:

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- integrality of $\hat{A}$ for smooth spin manifold
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- Dimension of self-dual instantons (non-linear equations!)
- ...

II. Proof via heat equation

The Dirac operator

Using data:

- Oriented even-dimensional Riemannian manifold M
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Sufficient to prove the index theorem for Dirac operators.

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Thus, using **functional calculus**, consider the operators:

$$e^{-t\Delta_\pm} : C^\infty(S_\pm) \to C^\infty(S_\pm)$$

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Choose **kernels** $k^t_\pm \in C^\infty(M \times M, \mathrm{End}(S_\pm))$ so that:

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And so obtain:

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Using the Euclidean heat kernel (Fourier, c.1822):

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but hard / impossible to calculate higher $\Theta_i^\pm$ in general.

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(on non-compact space $\mathbb{R}$).

III. Analysis wish list

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- Test-case API: prove index is locally constant
- Thanks to Bhavik Mehta we already have the Fredholm alternative (even over the p-adics)

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- What form will Meyers-Serrin theorem take? What about $p = \infty$?

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($\|\cdot\|_s$ is a L^2_s Sobolev norm).

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  $$\|v\|_{k+1} \leq C_k \left(\|Dv\|_k + \|v\|_k\right)$$

  ($\|\cdot\|_s$ is a L^2_s Sobolev norm).
- Construction of corresponding Fredholm operators on Sobolev spaces.

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- Schwartz kernel theorem + existence of Friedrich's mollifiers?

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- Prove: trace-class $\subseteq$ Hilbert-Schmidt $\subseteq$ compact $\subseteq$ bounded.

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- Test ergonomics by proving some classic expansion, e.g., for Γ function or $Ei(t) = \int_t^\infty \frac{e^{-u}}{u} du \sim e^{-t}\left(\frac{1}{t} - \frac{1}{t^2} + \frac{2!}{t^3} - \cdots\right)$.

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Useful exercise in something more concrete.