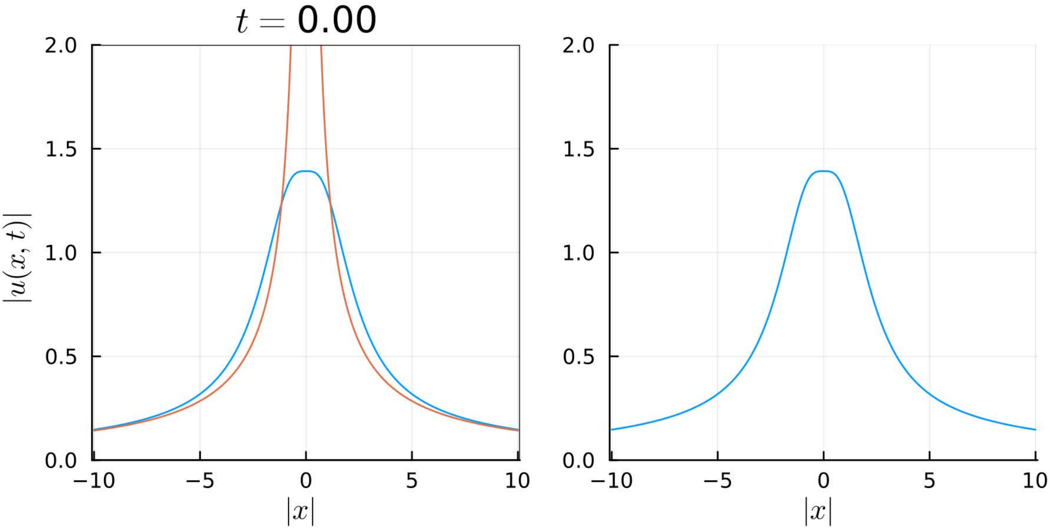
Self-similar blowup for the Nonlinear Schrödinger Equation and the Complex Ginzburg-Landau Equation

Joel Dahne

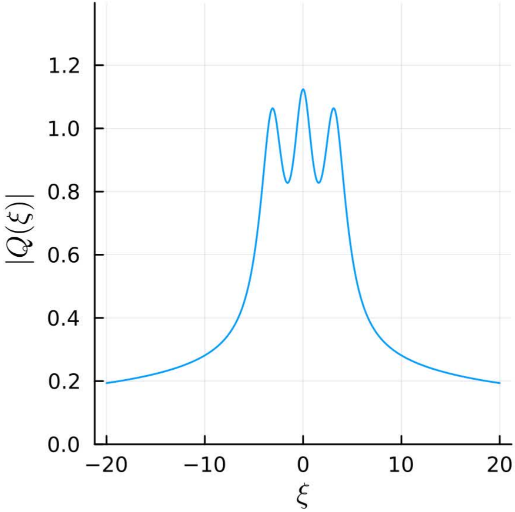
University of Minnesota

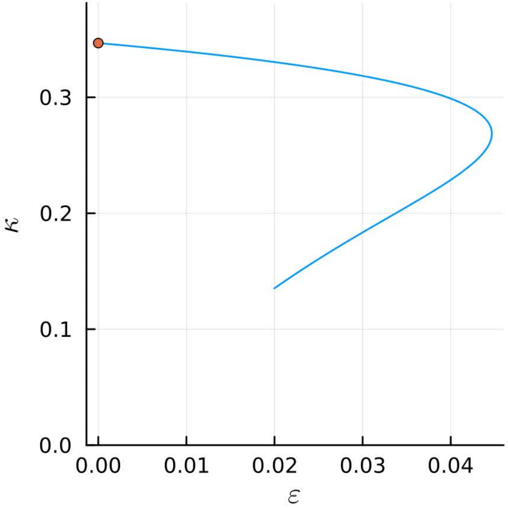
Joint with Jordi-Lluís Figueras

Self-similar blowup



Branches with self-similar blowup





The Nonlinear Schrödinger Equation (NLS)

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{2\sigma} u = 0$$

$$u(x,t) : \mathbb{R}^d \times (0, T) \to \mathbb{C}$$

$$u(x,0) = u_0(x)$$

Parameters $d \in \mathbb{Z}_{\geq 1}$, $\sigma \geq 0$

The Complex Ginzburg-Landau Equation (CGL)

$$i\frac{\partial u}{\partial t} + (1 - i\epsilon)\Delta u + (1 + i\delta)|u|^{2\sigma} u = 0$$

$$u : \mathbb{R}^d \times (0, T) \to \mathbb{C}$$

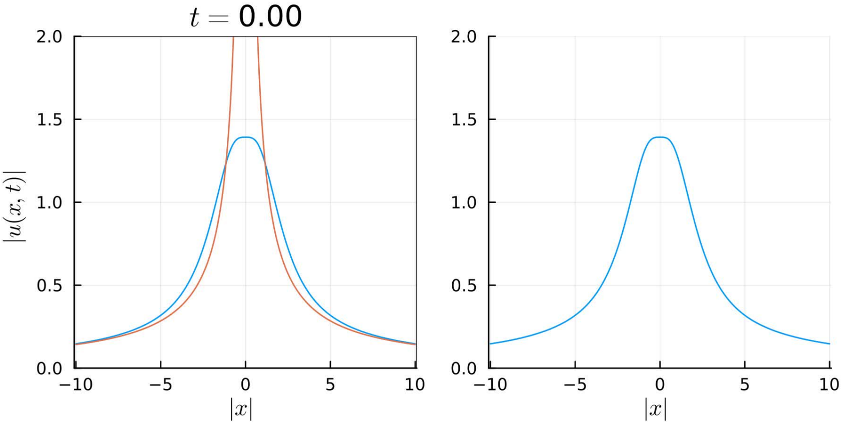
$$u(x, 0) = u_0(x)$$

Parameters $d \in \mathbb{Z}_{\geq 1},\ \sigma, \epsilon, \delta \geq 0$

NLS $\epsilon = \delta = 0$

Self-similar solutions

$$u(x,t) = \frac{1}{(2\kappa(1-t))^{\frac{1}{2}\left(\frac{1}{\sigma}+i\frac{\omega}{\kappa}\right)}} Q\left(\frac{|x|}{(2\kappa(1-t))^{\frac{1}{2}}}\right)$$



Self-similar solutions

$$u(x,t) = \frac{1}{(2\kappa(1-t))^{\frac{1}{2}\left(\frac{1}{\sigma}+i\frac{\omega}{\kappa}\right)}} Q\left(\frac{|x|}{(2\kappa(1-t))^{\frac{1}{2}}}\right)$$

$$\begin{cases} (1-i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1+i\delta)|Q|^{2\sigma}Q = 0 \\ Q'(0) = 0 \\ Q(\xi) \sim \xi^{-\frac{1}{\sigma}-i\frac{\omega}{\kappa}} \text{ as } \xi \to \infty \end{cases} .$$

Parameters $\kappa, \omega > 0$

Previous work - NLS

Book Sulem, Sulem (1999)

$$s_c = \frac{d}{2} - \frac{1}{\sigma}$$

$s_c < 0$ Global existence

$s_c = 0$
- Merle, Raphaël (CMP (2004), Inventiones (2004), Annals (2005), JAMS (2006))

$0 < s_c \ll 1$
- Merle, Raphaël, Szeftel (2010)
- Bahri, Martel, Raphaël (2021)

$s_c = \frac{1}{2}$
- Donninger, Schörkhuber (2024) (3D cubic NLS)

Sulem, Sulem (1999) - Chapter 7: Supercritical collapse

> *We are interested in complex solutions Q of (7.1.2) with a monotonically decreasing amplitude $|Q|$ and zero Hamiltonian, which provide the limiting profiles of singular solutions of the NLS equation. We call such solutions "admissible solutions".*

Conjecture (based on numerics by Budd, Chen, Russel (1999))

The supercritical NLS equation has a countable number of nontrivial radial self-similar singular solutions. In certain regimes, these solutions are characterized by the number, j, of monotone intervals of the profile $|Q|$. Except for $j = 1$, these solutions are all unstable.

Previous work - NLS

Theorem (Donninger, Schörkhuber (arXiv 2024) (CPAM 2026))

There exist a nontrivial, radial function $Q \in L^4(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ *and a* $\kappa > 0$ *such that*

$$u(x,t) = \frac{1}{(2\kappa(1-t))^{\frac{1}{2}(1+\frac{i}{\kappa})}} Q\left(\frac{x}{(2\kappa(1-t))^{\frac{1}{2}}}\right)$$

is a self-similar singular solution to the 3D cubic NLS equation

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^2 u = 0.$$

Previous work - CGL

Self-Similar blowup

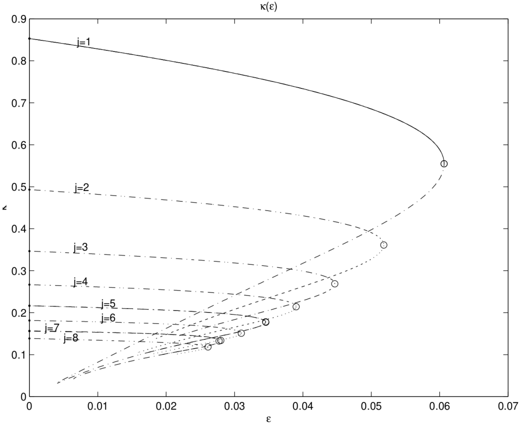
- Plecháč and Šverák (2001)

Other types of blowup

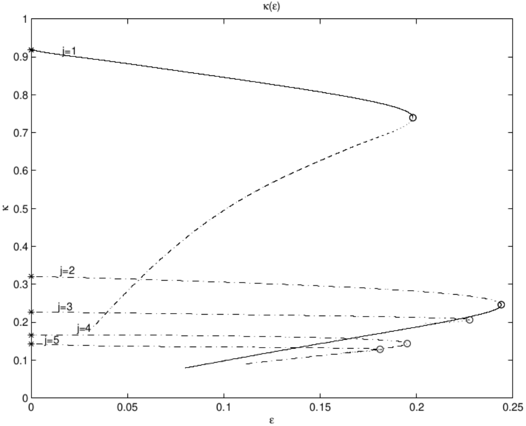
- Masmoudi, Zaag (2008)
- Cazenave, Dias, Figueira (2014)
- Duong, Nouaili, Zaag (2020, 2022, 2023)

Branches of self-similar singular solutions

Case I: $d = 1$, $\sigma = 2.3$, $\delta = 0$ and $\omega = 1$



Case II: $d = 3$, $\sigma = 1$, $\delta = 0$ and $\omega = 1$



Plecháč and Šverák (2001)

Equation to solve

$$\begin{cases} (1-i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1+i\delta)|Q|^{2\sigma}Q = 0 \\ Q'(0) = 0 \\ Q(\xi) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}} \text{ as } \xi \to \infty \end{cases} .$$

Fix d, σ, δ, ω

Case I $d = 1$, $\sigma = 2.3$, $\delta = 0$, $\omega = 1$

Case II $d = 3$, $\sigma = 1$, $\delta = 0$, $\omega = 1$

Vary κ, ϵ

Results - NLS

Theorem (D, Figueras, 2024)

Consider the equation

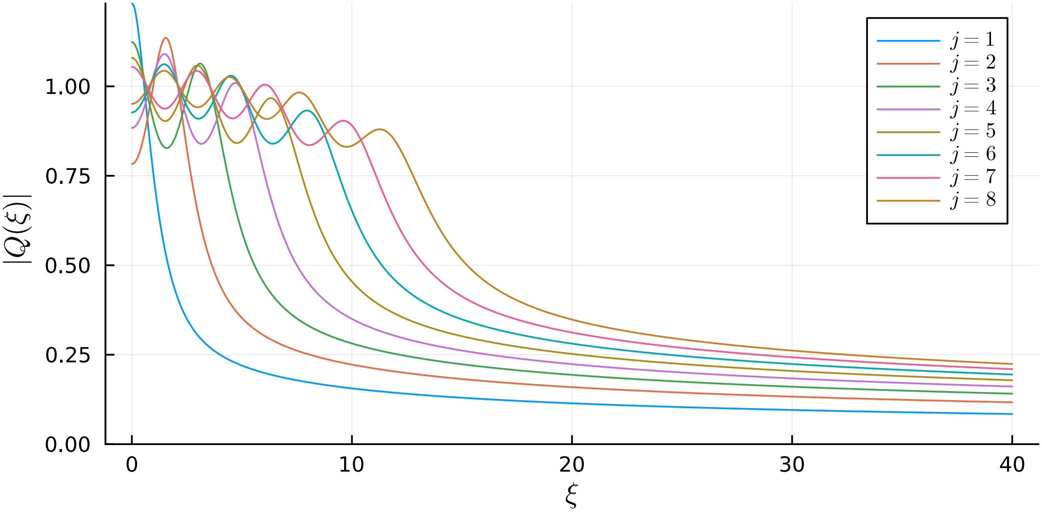
$$\begin{cases} Q'' + \frac{d-1}{\xi} Q' + i\kappa\xi Q' + i\frac{\kappa}{\sigma} Q - \omega Q + |Q|^{2\sigma} Q = 0 \\ Q'(0) = 0, \, Q(\xi) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}} \end{cases} .$$

Case I *There exist solutions for at least 8 values of* κ.

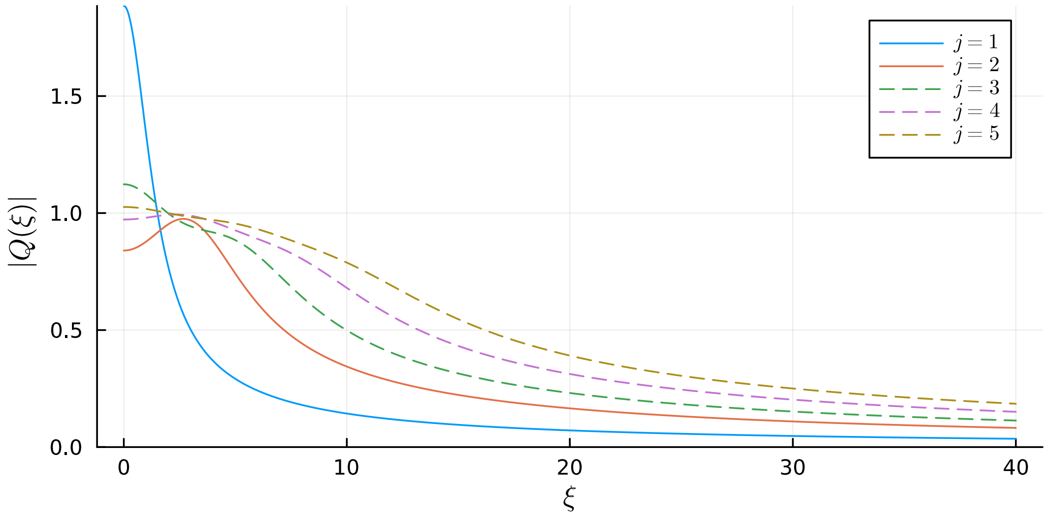
Case II *There exist solutions for at least 2 values of* κ.

The number of intervals of monotonicity is given by the index j *of the solution.*

Solutions for Case I



Solutions for Case II



Results - CGL

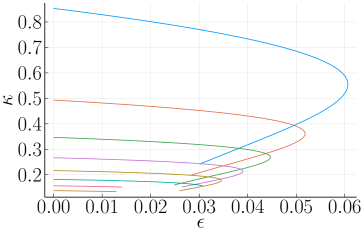
Theorem (D, Figueras, 2024)

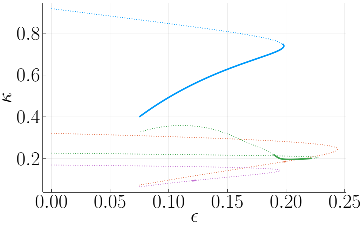
Consider the equation

$$\begin{cases} (1-i\epsilon)\left(Q''+\frac{d-1}{\xi}Q'\right)+i\kappa\xi Q'+i\frac{\kappa}{\sigma}Q-\omega Q+(1+i\delta)|Q|^{2\sigma}Q=0 \\ Q'(0)=0,\, Q(\xi)\sim \xi^{-\frac{1}{\sigma}-i\frac{\omega}{\kappa}} \end{cases}.$$

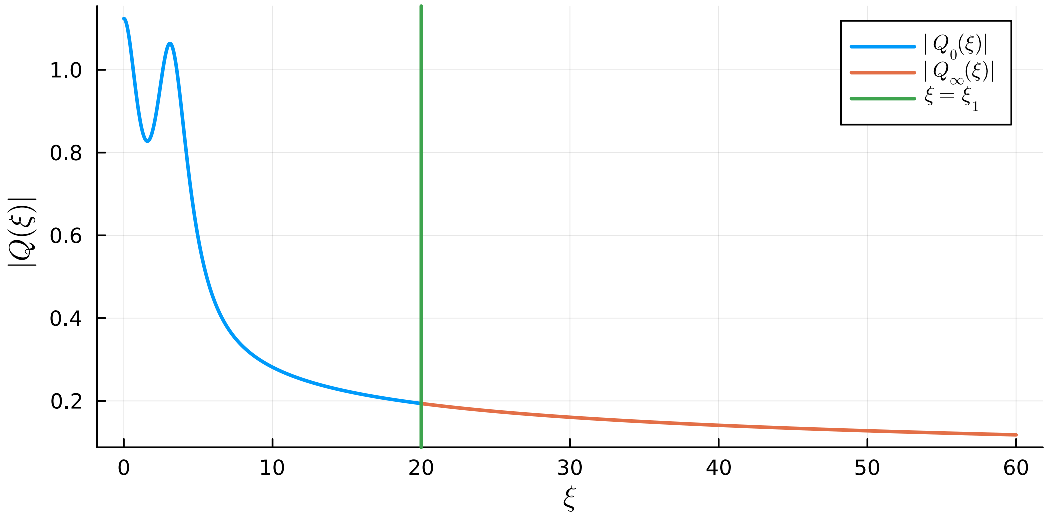
Case I *There exists at least 8 branches of solutions in ϵ. The number of intervals of monotonicity is constant along these branches.*

Case II *There exists branches of solutions in ϵ.*





Proof idea - Matching solutions



Q_0 and Q_∞

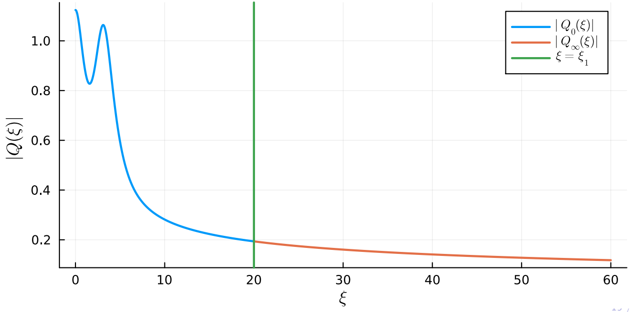
$$\begin{cases} (1-i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1+i\delta)|Q|^{2\sigma}Q = 0 \\ Q'(0) = 0,\ Q(0) = \mu > 0, \lim_{\xi\to\infty} Q(\xi) = 0 \end{cases}$$

$Q_0(\xi) = Q_0(\mu, \kappa; \xi)$ Solution with $Q_0'(0) = 0, Q_0(0) = \mu$

$Q_\infty(\xi) = Q_\infty(\gamma, \kappa; \xi)$ Solution with $Q_\infty(\xi) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}}$ as $\xi \to \infty$

Fix $\xi_1 > 0$. Want to find μ, γ, κ s.t.

- $Q_0(\mu, \kappa; \xi_1) = Q_\infty(\gamma, \kappa; \xi_1)$
- $Q_0'(\mu, \kappa; \xi_1) = Q_\infty'(\gamma, \kappa; \xi_1)$



Proving matching condition

$$G(\mu,\gamma,\kappa) = (Q_0(\mu,\kappa;\xi_1) - Q_\infty(\gamma,\kappa;\xi_1), Q_0'(\mu,\kappa;\xi_1) - Q_\infty'(\gamma,\kappa;\xi_1))$$

$$G(\mu, \operatorname{Re}\gamma, \operatorname{Im}\gamma, \kappa) : \mathbb{R}^4 \to \mathbb{R}^4$$

1. Find a numerical approximation
2. Rigorously verify approximation using the interval Newton method
   - Requires computing rigorous interval enclosures for
     - $Q_0(\mu,\kappa;\xi_1)$, $Q_0'(\mu,\kappa;\xi_1)$, $Q_\infty(\gamma,\kappa;\xi_1)$, $Q_\infty'(\gamma,\kappa;\xi_1)$
     - Derivatives w.r.t. μ, γ, κ

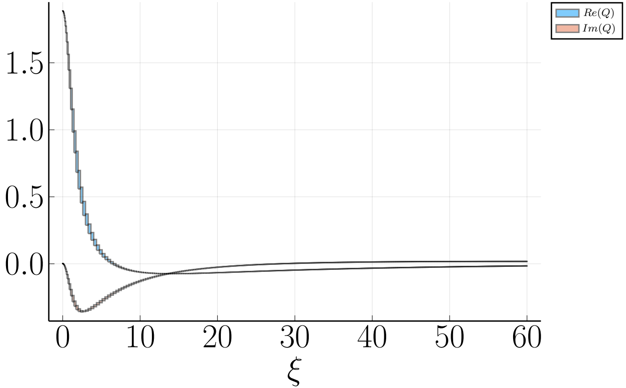
j	μ_j	γ_j	κ_j	ξ_1
1	1.88565^{73}_{67}	$1.71360^{13}_{05} - 1.49179^{42}_{35}i$	0.91735^{63}_{59}	60
2	0.8399^{62}_{57}	$13.852^{78}_{46} + 6.034^{59}_{44}i$	0.3212^{41}_{39}	140

Computing Q_0

$$\begin{cases} (1 - i\epsilon)\left(Q_0'' + \frac{d-1}{\xi} Q_0'\right) + i\kappa\xi Q_0' + i\frac{\kappa}{\sigma} Q_0 - \omega Q_0 + (1 + i\delta)|Q_0|^{2\sigma} Q_0 = 0 \\ Q_0'(0) = 0,\ Q_0(0) = \mu \end{cases}$$

Approximately Classical numerical ODE solver, e.g. `RK4`, `TSit5` or `Vern7`.

Rigorously Rigorous numerical ODE solver `CAPD`.



Computing Q_∞ - Linear equation

$$(1 - i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q = 0$$

- Confluent hypergeometric equation after change of variables.
- Two solutions given by
  - $P(\xi) = U(a, b, c\xi^2) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}}$
  - $E(\xi) = e^{c\xi^2}U(b-a, b, -c\xi^2) \sim e^{\frac{-i\kappa}{2(1-i\epsilon)}\xi^2}\xi^{\frac{1}{\sigma} - d + i\frac{\omega}{\kappa}}$

Computing Q_∞

$$(1-i\epsilon)\left(Q_\infty'' + \frac{d-1}{\xi}Q_\infty'\right) + i\kappa\xi Q_\infty' + i\frac{\kappa}{\sigma}Q_\infty - \omega Q_\infty = -(1+i\delta)|Q_\infty|^{2\sigma}Q_\infty$$

- Method of variation of parameters

$$Q_\infty(\xi) = c_1(\xi)P(\xi) + c_2(\xi)E(\xi)$$

$$\begin{aligned} Q_\infty(\xi) = \gamma P(\xi) + P(\xi)\int_{\xi_1}^{\xi} \frac{1+i\delta}{1-i\epsilon}E(\eta)W(\eta)^{-1}|Q_\infty(\eta)|^{2\sigma}Q_\infty(\eta)\ d\eta \\ + E(\xi)\int_{\xi}^{\infty} \frac{1+i\delta}{1-i\epsilon}P(\eta)W(\eta)^{-1}|Q_\infty(\eta)|^{2\sigma}Q_\infty(\eta)\ d\eta \end{aligned}$$

Computing Q_∞

$$T[Q](\xi) = \gamma P(\xi) + P(\xi)\int_{\xi_1}^{\xi} \frac{1+i\delta}{1-i\epsilon} E(\eta) W(\eta)^{-1} |Q(\eta)|^{2\sigma} Q(\eta)\; d\eta$$
$$+ E(\xi)\int_{\xi}^{\infty} \frac{1+i\delta}{1-i\epsilon} P(\eta) W(\eta)^{-1} |Q(\eta)|^{2\sigma} Q(\eta)\; d\eta$$

1. Look for fixed point of T: $Q_\infty = T[Q_\infty](\xi)$
   - $\|Q\|_v = \sup_{\xi \geq \xi_1} \xi^{\frac{1}{\sigma} - v} |Q(\xi)|$
   - $\|T[Q]\|_v \leq C_P |\gamma| \xi_1^{-v} + C_{T,1} \xi_1^{-2+2\sigma v} \|Q\|_v^{2\sigma+1}$
   - $\|T[Q_1] - T[Q_2]\|_v \leq C_{T,2} \|Q_1 - Q_2\|_v (\|Q_1\|_v^{2\sigma} + \|Q_2\|_v^{2\sigma})$
2. Improve bounds using bootstrapping

Interval Newton method - One variable

Classical Newton

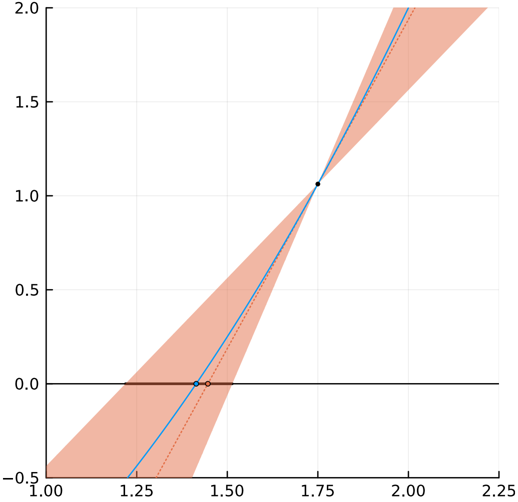
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Interval Newton

$$\boldsymbol{x_0} = [\underline{x}_0, \overline{x}_0]$$

$$\boldsymbol{x_1} = x_0 - \frac{f(x_0)}{f'(\boldsymbol{x_0})} = \left\{ x_0 - \frac{f(x_0)}{f'(x)} : x \in \boldsymbol{x_0} \right\}$$

Success if $\boldsymbol{x_1} \subseteq \boldsymbol{x_0}$



Interval Newton method - Multiple variables

$$f : \mathbb{R}^n \to \mathbb{R}^n,\ \ J_f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

$$x_0 = (x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(n)}) \in \mathbb{R}^n$$

$$\boldsymbol{x_0} = [\underline{x_0}, \overline{x_0}] = [\underline{x}_0^{(1)}, \overline{x}_0^{(1)}] \times [\underline{x}_0^{(2)}, \overline{x}_0^{(2)}] \times \cdots \times [\underline{x}_0^{(n)}, \overline{x}_0^{(n)}] \subset \mathbb{R}^n$$

$$\boldsymbol{x_1} = x_0 - J_f(\boldsymbol{x_0})^{-1} f(x_0)$$

Success if $\boldsymbol{x_1} \subseteq \boldsymbol{x_0}$

Interval Newton method - Applying it

$$G(\mu, \operatorname{Re}\gamma, \operatorname{Im}\gamma, \kappa) : \mathbb{R}^4 \to \mathbb{R}^4$$

$$x_0 = (1.88565696502, 1.7136009546, -1.4917938718, 0.91735611859) \in \mathbb{R}^4$$

$$\boldsymbol{x_0} = 1.88565_{6707}^{7268} \times 1.71360_{0550}^{1269} \times -1.49179_{3570}^{4112} \times 0.91735_{5982}^{6304} \subset \mathbb{R}^4$$

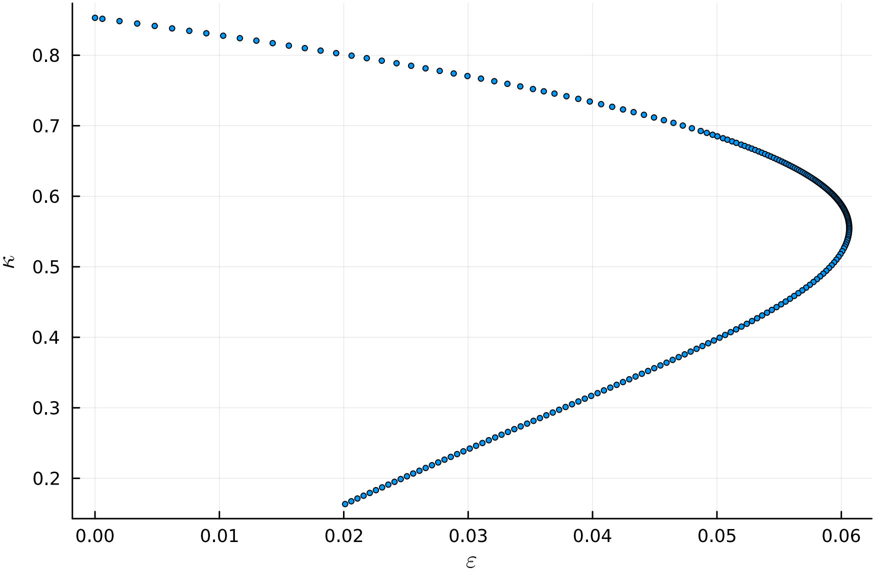
$$G(x_0) \subseteq [\pm 2.7 \cdot 10^{-11}] \times [\pm 2.6 \cdot 10^{-11}] \times [\pm 1.4 \cdot 10^{-9}] \times [\pm 1.4 \cdot 10^{-9}]$$

$$J_G(\boldsymbol{x_0}) \subseteq \ldots$$

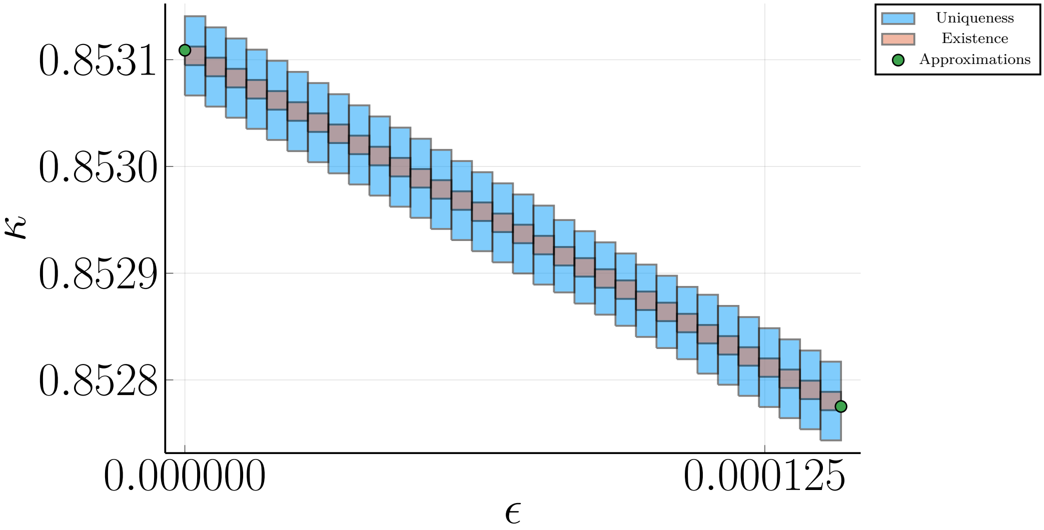
$$\begin{aligned} x_0 - J_G(\boldsymbol{x_0})^{-1} G(x_0) &\subseteq 1.88565_{6717}^{7258} \times 1.71360_{0562}^{1256} \times -1.49179_{3579}^{4103} \times 0.91735_{5988}^{6298} \\ &\subseteq 1.88565_{6707}^{7268} \times 1.71360_{0550}^{1269} \times -1.49179_{3570}^{4112} \times 0.91735_{5982}^{6304}. \end{aligned}$$

Success!

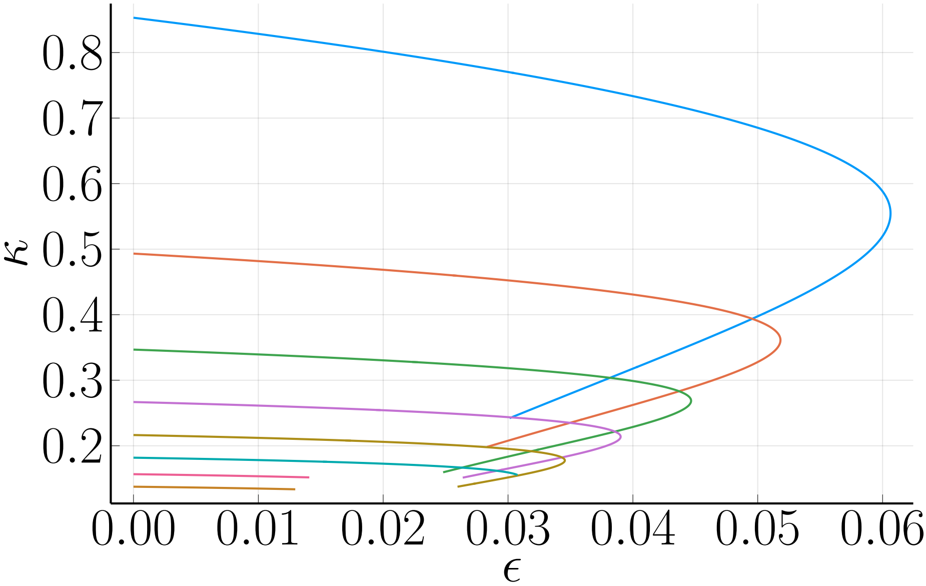
Branches - Pointwise



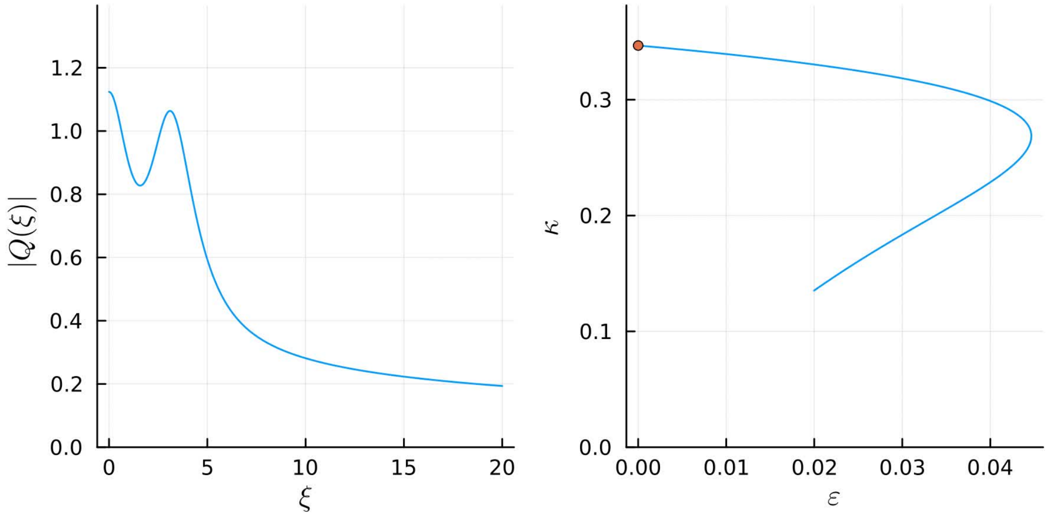
Branches - Enclosure



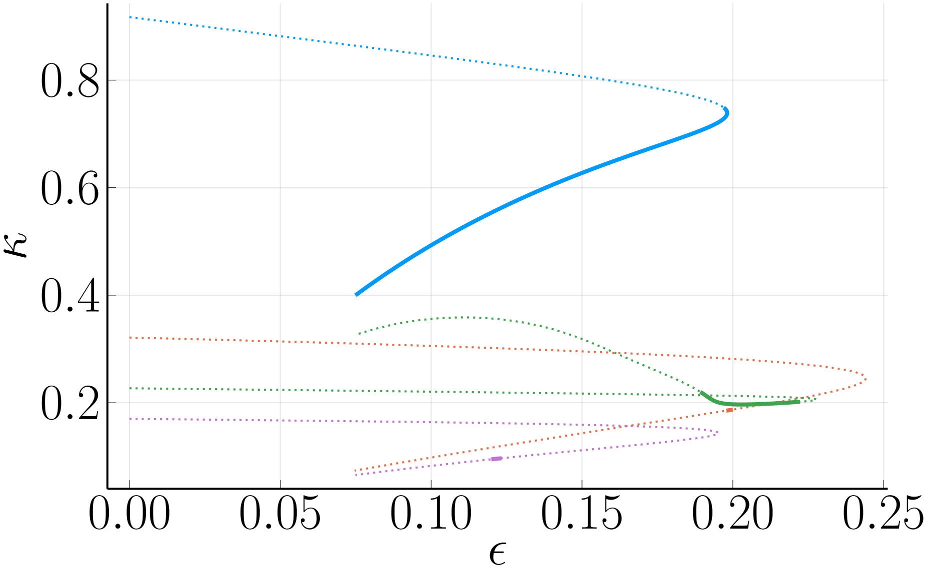
Branches Case I



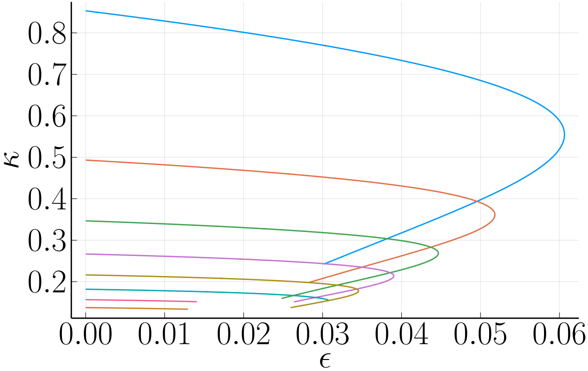
Solution for Case I along branch $j = 3$

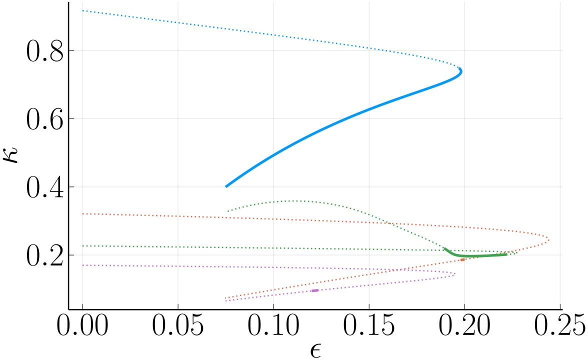


Branches Case II



Thank you!


0.8
0.7
0.6
0.5
0.4
0.3
0.2
κ
0.00
0.01
0.02
0.03
0.04
0.05
0.06
ϵ


0.8
0.6
0.4
0.2
κ
0.00
0.05
0.10
0.15
0.20
0.25
ϵ