THERMALIZATION UNIVERSALITY CLASSES

Sergej FLACH
Institute for Basic Science
Daejeon South Korea

- integrable limits, perturbations, network classes
- Lyapunov spectrum scaling

PRE 95 060202 (2017)
PRL 122 054102 (2019)
PRE 100 032217 (2019)
PRE 104 014218 (2021)
Chaos 32 063113 (2022)
PRL 128 134102 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)
Chaos 34 033107 (2024)
PRR 7 023149 (2025)
LTP 51 870 (2025)
PRE 112 014206 (2025)
PRE 113 L012201 (2025)
Chaos 35 113127 (2025)
arXiv:2603.26042

PCS Center for Theoretical Physics of Complex Systems

ibs Institute for Basic Science

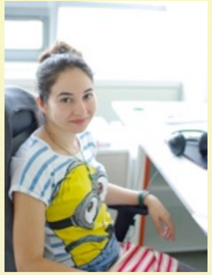
People in chronological project order

David Campbell

Carlo Danieli

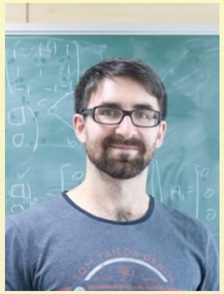
Mithun Thudiyangal

Yagmur Kati



Alexander Cherny

Thomas Engl



Mikhail Fistul

Boris Altshuler

Merab Malishava

Weihua Zhang

Barbara Dietz

Emil Yuzbashyan

Gabriel Lando

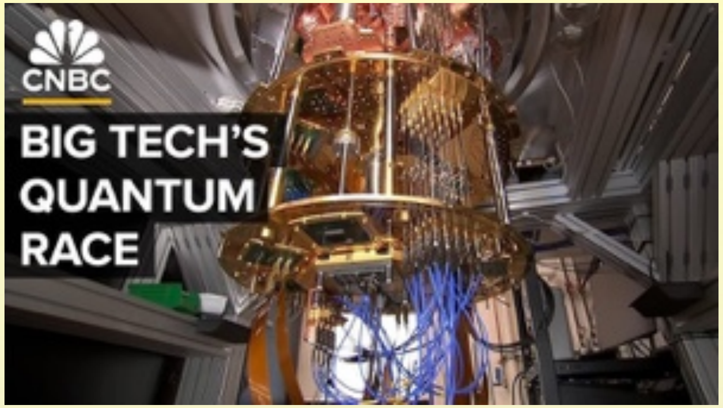
Aniket Patra

Budhaditya Bhattacharjee

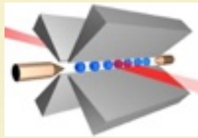
Alexei Andreanov

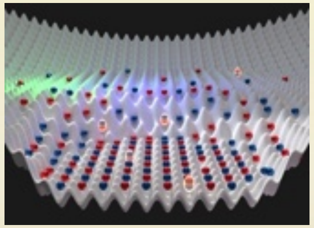
Xiaodong Zhang

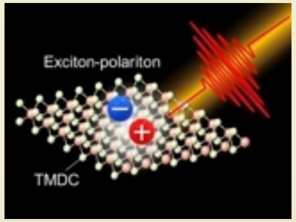
WHY THERMALIZATION DYNAMICS OF MANY BODY SYSTEMS



- quantum computer platforms
- quantum computing
- quantum simulators
- superconducting qubits
- trapped ions
- ultracold atomic gases
- Photonics
- exciton-polaritons

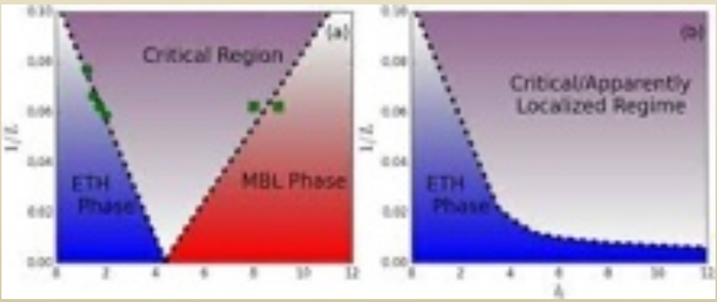


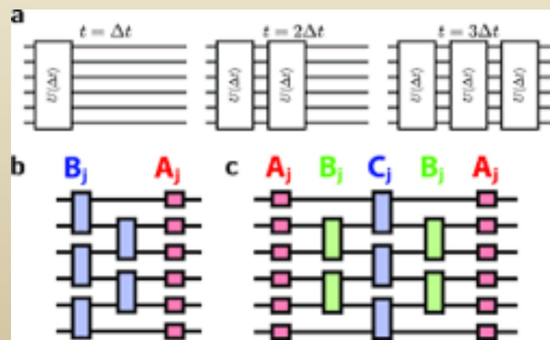


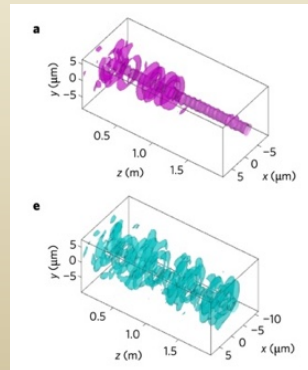


Fundamental Physics:

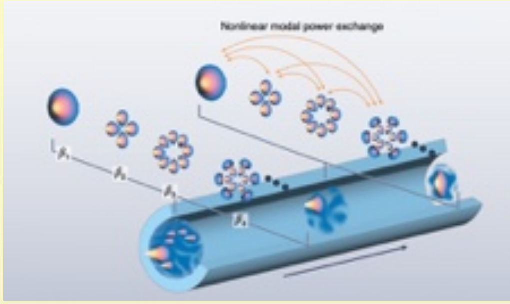
- many body localization
- ETH (eigenstate thermalization hypothesis)
- slowing down close to integrability
- Trotterization impact

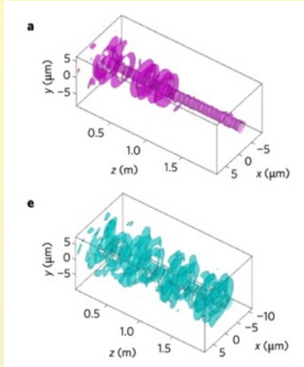


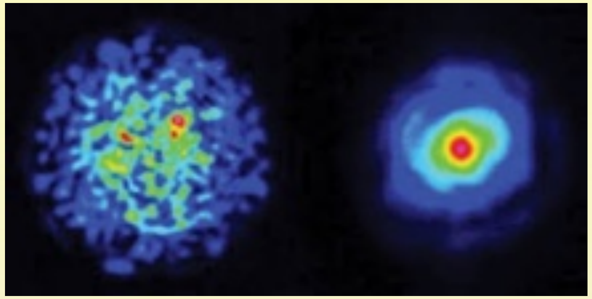


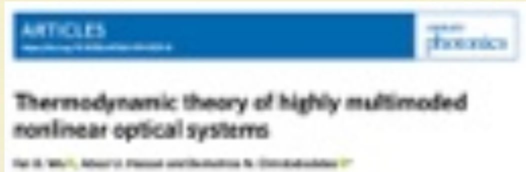


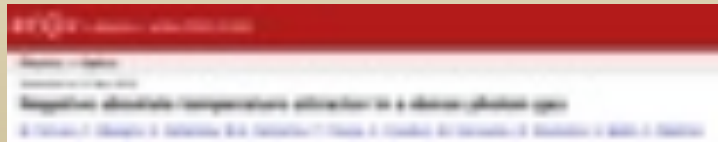
THERMALIZATION DYNAMICS WITH PHOTONICS











Goals

Thermalization dynamics slowing down of many body system in proximity to integrable limit:

- **Use unique action-angle coordinates**
- **Identify different classes of nonintegrable perturbations networks**
- **Quantify thermalization process**
- **Identify novel dynamical regimes**

Integrable System: measure Ø!

$\nu = N$, canonical transformation: $\vec{p}, \vec{q} \Rightarrow \vec{J}, \vec{\Theta}$

$\Rightarrow H(\vec{J})$, $\dot{\vec{J}} = 0$, $\dot{\vec{\Theta}} = H_{,\vec{J}} = \vec{\omega}$, $\dot{\vec{\omega}} = 0$

$\Rightarrow$ trajectory on N-dimensional torus, embedded in 2N-dimensional phase space

$\vec{\omega}$ components incommensurate: torus densely covered (measure 1)

$\vec{\omega}$ components commensurate (measure Ø) trajectory periodic, closes after finite time

nonintegrability in proximity to integrability

Choose unique action-angle coordinates at the integrable limit

Analyze network spanned by the nonintegrable perturbation

We find two different classes of weak nonintegrability

LRN: long range network, each action couples to all others,
distance is not growing with system size

SRN: short range network, distance grows with system size

nonintegrability in proximity to integrability

Choose unique action-angle coordinates at the integrable limit

Analyze network spanned by the nonintegrable perturbation

We find two different classes of weak nonintegrability

LRN: ordered systems
weak nonlinearity (weak two-body interaction)
all to all nonintegrable interaction between actions

SRN: weak finite range lattice hopping
short range nonintegrable interaction between actions
but also
disorder (Anderson localization) and weak nonlinearity

example: Josephson junction networks in d=1,2,3

$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

Number of sites (volume) : N

Energy density: h = H/N

SRN: E_J / h → 0

LRN: h / E_J → 0

example: Long Range Network

Josephson junction network, energy density h: $h/E_J \ll 1$

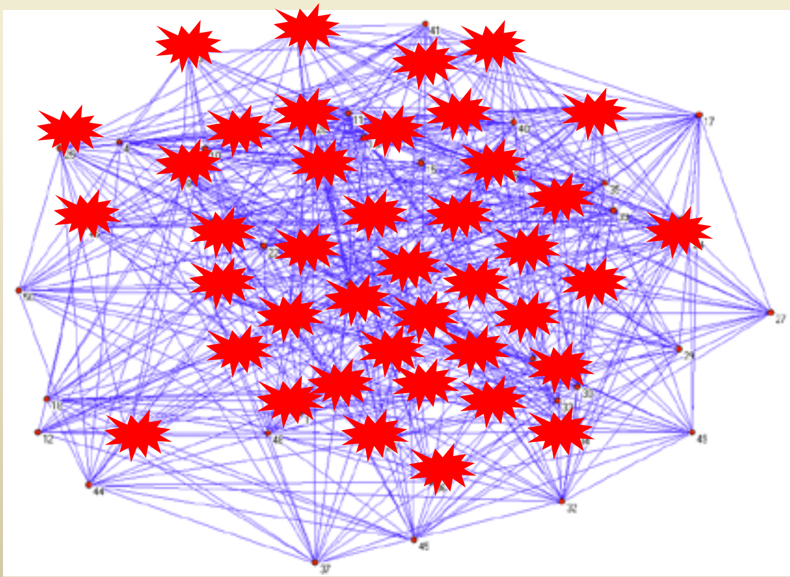
$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

$$H_0 = \sum \frac{p_n^2}{2} + \frac{E_J}{2}(q_n - q_{n-1})^2 \ : \ \text{harmonic chain}$$

$$H_1 = -\frac{E_J}{4}\sum (q_n - q_{n-1})^4 \ : \ \text{quartic anharmonicity}$$

$L = 3,\ R \sim N^2$

$\longrightarrow$ Long Range Network



long range network:

$Q_q = \sqrt{2J_q}\sin\Theta_q\ ,\ P_q = \omega_q\sqrt{2J_q}\cos\Theta_q$

$\dot{J}_q = -E_J\sum_{q_1,q_2,q_3}\omega_{q_1}\omega_{q_2}\omega_{q_3}A_{q,q_1,q_2,q_3}\sqrt{J_qJ_{q_1}J_{q_2}J_{q_3}}cos\Theta_q\sin\Theta_{q_1}\sin\Theta_{q_2}\sin\Theta_{q_3}$

example: Long Range Network

→ **exists due to weak nonlinear all-to-all interactions between normal modes**

→ **nonintegrable nonlinearity local in real space, normal modes extended**

long range network:

$$Q_q = \sqrt{2J_q} \sin\Theta_q \ , \ P_q = \omega_q \sqrt{2J_q} \cos\Theta_q$$

$$\dot{J}_q = -E_J \sum_{q_1,q_2,q_3} \omega_{q_1}\omega_{q_2}\omega_{q_3} A_{q,q_1,q_2,q_3} \sqrt{J_q J_{q_1} J_{q_2} J_{q_3}} cos\Theta_q \sin\Theta_{q_1} \sin\Theta_{q_2} \sin\Theta_{q_3}$$

example: Short Range Network

Josephson junction network, $h/E_J \gg 1$

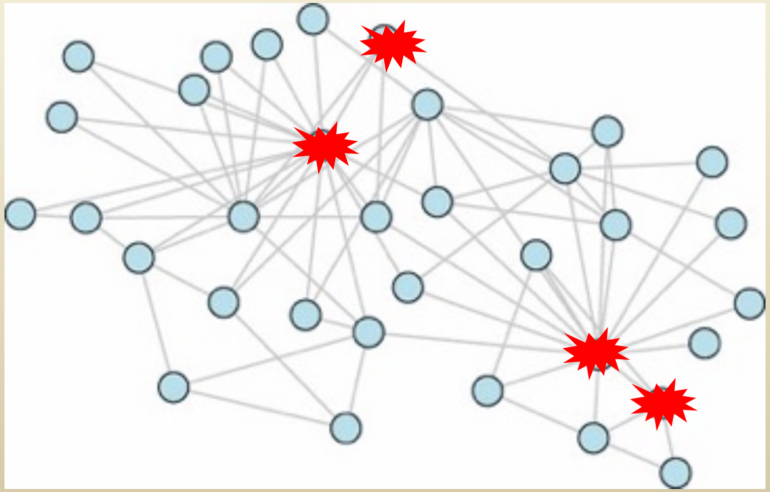
$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

$$H_0 = \sum \frac{p_n^2}{2} \; : \text{ free rotors}$$

$$H_1 = E_J \sum [1 - \cos(q_n - q_{n-1})] \; : \text{ nearest neighbour coupling}$$

$L = 1,\ R = 2$

$\longrightarrow$ Short Range Network



short range network:

$q_n = \Theta_n \;,\; p_n = J_n$

$\dot{J}_n = -E_J \left(\sin(\Theta_n - \Theta_{n-1}) + \sin(\Theta_n - \Theta_{n+1})\right)$

example: Short Range Network

→ **exists due to underlying lattice model structure**

→ **nonintegrable lattice coupling is local, rotor actions are local**

short range network:

$$q_n = \Theta_n \ , \ p_n = J_n$$

$$\dot{J}_n = -E_J \left(\sin(\Theta_n - \Theta_{n-1}) + \sin(\Theta_n - \Theta_{n+1}) \right)$$

Measuring Thermalization : measuring time scales !

Large macroscopic systems: beyond the horizons of the KAM regime

Is there structure in this chaotic world?

Are all chaotic systems thermalizing similarly?

Thermalization is not about waiting until getting to a thermal state,
but waiting until the system forgets where it came from

Measuring Thermalization : measuring time scales !

- **Measure dynamics of observables (***aka functions on phase space***)**
- **Measure and compare their time averages with ensemble averages**
- **Extract ergodization time scales T_E**
- *Pros: - can be measured in experiments and extended to quantum systems*
- *Cons: - ambiguity in observable choice*
  - basis dependent, not universal
  - lack of aesthetic satisfaction

- **Measure Lyapunov spectra**
- **Invert to obtain Lyapunov times T_L**
- *Pros: - no ambiguity in observable choice*
  - no basis choice dependence, universal
  - aesthetically satisfying
- *Cons: - not easy to measure in experiments*
  - not clear how to extend to quantum systems

Approaching integrable limits

- Time scales will diverge (length scales perhaps as well)
- How will they diverge? How many will diverge? Which ones will diverge?
- Are there different universality classes?
- Can we observe and compute critical exponents?
- Are there further universal quantities?

Nutshell summary

Integrable system H_0

- countable number of N DoF
- unique canonical choice of actions J and angles θ
- $H_0(J)$ is a function of the actions only

Nonintegrable perturbation εH_1

- $H_1(J,\theta)$
- spans a short or long range network of nonintegrable interactions among the actions J
- number of DoF N is large
- we quantify chaotic dynamics beyond KAM horizons

Nutshell summary

Lyapunov spectrum

- $\Lambda_1 > \Lambda_2 > \ldots > \Lambda_{2N}$ come in +/- pairs
- one zero pair per integral of motion
- rescaled Lyapunov spectrum

$$\bar{\Lambda}(\rho) = \Lambda_i / \Lambda_1, \;\; \rho = i/N$$

- rescaled Kolmogorov-Sinai entropy

$$\kappa = \int_0^1 \bar{\lambda}(\rho) d\rho$$

Lyapunov exponent computation

$$r(t) = \ln\left(\frac{|\mathbf{X}(t)|}{|\mathbf{X}(\mathbf{t}-\mathbf{dt})|}\right) \;, \;\; \Lambda(t) = \frac{1}{t}\int_0^t r(t')dt'$$

Nutshell summary

Long range network, ε ➔ 0 , Λ_1 ➔ 0

- **rescaled spectrum invariant**

$$\bar{\Lambda}(\rho)|_{\epsilon \to 0} \to F(\rho)$$

- **one time scale controls all others, κ finite**

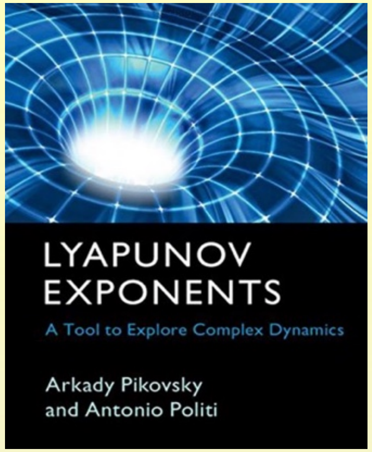
Short range network, ε ➔ 0 , Λ_1 ➔ 0

- **exponential vanishing of spectrum**

$$\bar{\Lambda}(\rho) \approx \mathrm{e}^{-\beta\rho}, \;\beta \to \infty$$

- **exponential thermalization slowing down**
  κ vanishes

The Lyapunov Spectrum



Number of Lyapunov exponents = phase space dimension

Lyapunov exponents come in $\pm\lambda$ pairs

Per each integral of motion: two zero Lyapunov exponents

TITLE	CITED BY	YEAR
Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part 1: Theory G Benettin, L Galgani, A Giorgilli, JM Strelcyn Meccanica 15 (1), 9-20	[illegible]	1980

TITLE	CITED BY	YEAR
The Lyapunov characteristic exponents and their computation C Skokos Lecture Notes in Physics 790, 63-135	[illegible]	2010

[illegible]

The Lyapunov Spectrum

PRL 128 134102 (2022)
Chaos 32 063113 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)
LTP 51 870 (2025)
PRE 112 014206 (2025)
arXiv:2506.08657

- **initial thermal state: use proper Gibbs distributions**
- **run the trajectory**
- **add small perturbation X, linearize and obtain the tangent map**
- **run 2N trajectories in the tangent map TM**
- **TM1: obtain LLE**

$$r(t) = \ln\left(\frac{|\mathbf{X}(t)|}{|\mathbf{X}(\mathbf{t}-\mathbf{dt})|}\right) \ , \ \Lambda(t) = \frac{1}{t}\int_0^t r(t')dt'$$

The Lyapunov Spectrum

PRL 128 134102 (2022)
Chaos 32 063113 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)
LTP 51 870 (2025)
PRE 112 014206 (2025)
arXiv:2506.08657

- initial thermal state: use proper Gibbs distributions
- run the trajectory
- add small perturbation X, linearize and obtain the tangent map
- run 2N trajectories in the tangent map TM
- TM1: obtain LLE
- TM2: project $\perp$ to TM1 and obtain 2^{nd} largest LE
- … and so on

N DoF (sites), $2N$ sorted LEs: $\Lambda_i > \Lambda_j$ with $\{i < j\} \in 1, ..., 2N$

total norm conserved: $\Lambda_N = \Lambda_{N+1} = 0$

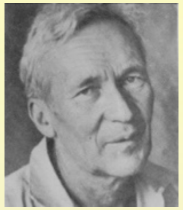
unitary evolution: $\Lambda_{i<N} = -\Lambda_{2N-i+1}$

LLE : Λ_1

Irreducible normalized LEs: $\bar{\Lambda}(\rho) = \Lambda_i / \Lambda_1$, $\rho = i/N$

The Lyapunov Spectrum: Kolmogorov, Sinai, Osedelets, Pesin

Kolmogorov-Sinai entropy: loosely speaking the entropy of return probabilities (1958-1959)




Osedelets: uniqueness of the Lyapunov spectrum (1968)

Pesin: the KS entropy is the sum over all positive Lyapunov exponents (1977)

Rescaled KS entropy

$$\kappa = \int_0^1 \bar{\lambda}(\rho) d\rho$$

The Lyapunov Spectrum: Random Matrix Theory Insights

Commun. Math. Phys. 103, 121–126 (1986)
Communications in Mathematical Physics

The Distribution of Lyapunov Exponents: Exact Results for Random Matrices

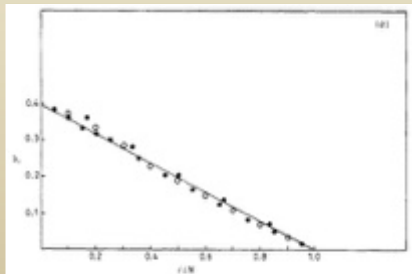
Charles M. Newman*

- **random matrix ➔ linear spectrum**
- **temporal correlations ➔ bending**

J. Phys. A: Math. Gen. 19 (1986) 1881–1888. Printed in Great Britain

Scaling law and asymptotic distribution of Lyapunov exponents in conservative dynamical systems with many degrees of freedom

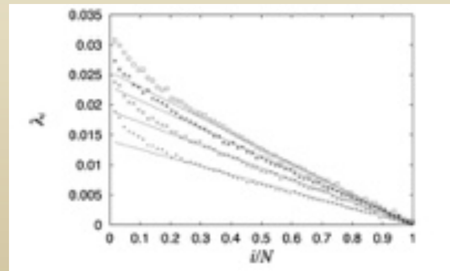
Giovanni Paladin and Angelo Vulpiani



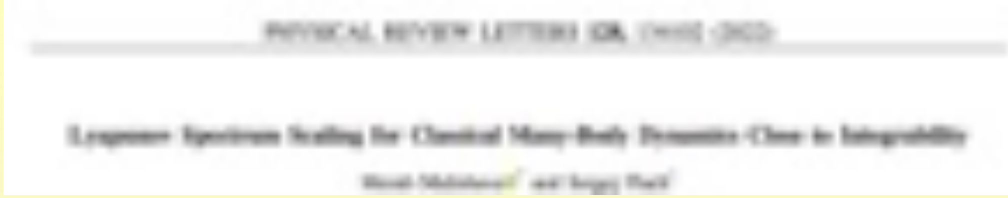
J. Phys. A: Math. Gen. 31 (1998) 195–207. Printed in the UK
PII: S0305-4470(98)84278-9

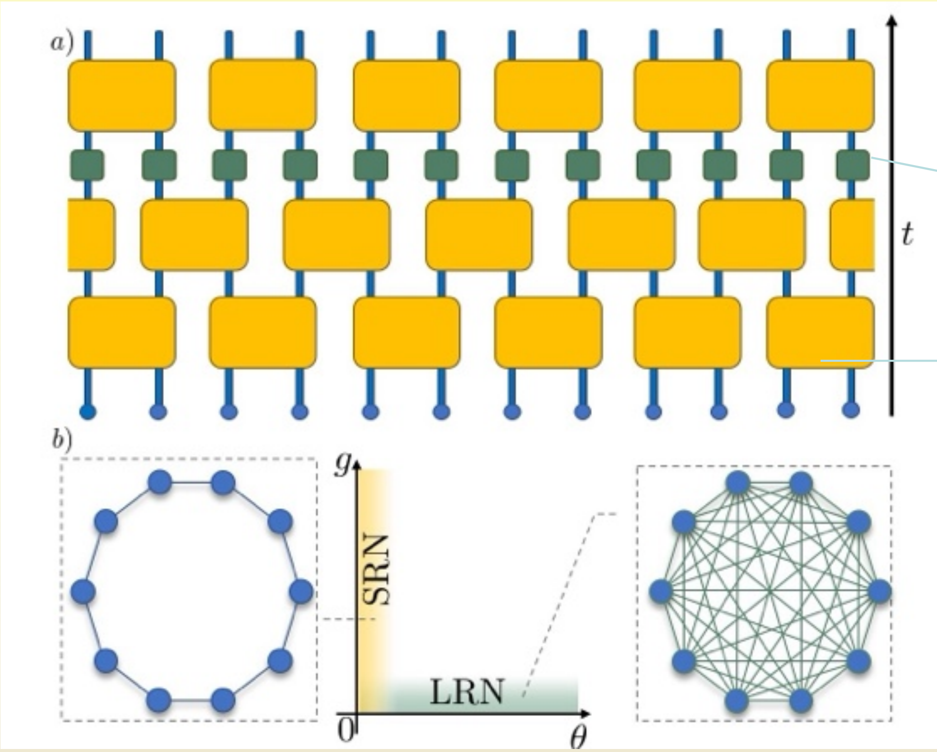
New universality of Lyapunov spectra in Hamiltonian systems

Yoshiyuki Y Yamaguchi†



Unitary Circuits for Thermalization





$$\hat{G}_n = e^{ig|\psi_n|^2}|n\rangle\langle n|$$

$$\hat{C}_{n,n+1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{\Psi}(t+1) = \widehat{U}\vec{\Psi}(t)$$

$$g \to 0 : \ \mathrm{LRN}$$

$$\theta \to 0 : \ \mathrm{SRN}$$

- **Fast numerical evolution due to parallelization**
- **No time discretization roundoff errors (except for roundoff errors)**
- **Versatile, highly efficient unitary map toolbox for long time evolution**

Unitary Circuits for Thermalization

$$\vec{\Psi}(t) = \{\psi_n^A(t), \psi_n^B(t)\}_{n=1}^{N/2}$$

$$\alpha_n^A(t) \equiv \cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t)$$
$$+ \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t)$$

$$\alpha_n^B(t) \equiv \sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t)$$
$$+ \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t) \,.$$

$$\psi_n^A(t+1) = e^{ig|\alpha_n^A|^2}\left[\cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t)\right.$$
$$\left.+ \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t)\right]$$

$$\psi_n^B(t+1) = e^{ig|\alpha_n^B|^2}\left[\sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t)\right.$$
$$\left.+ \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t)\right]$$

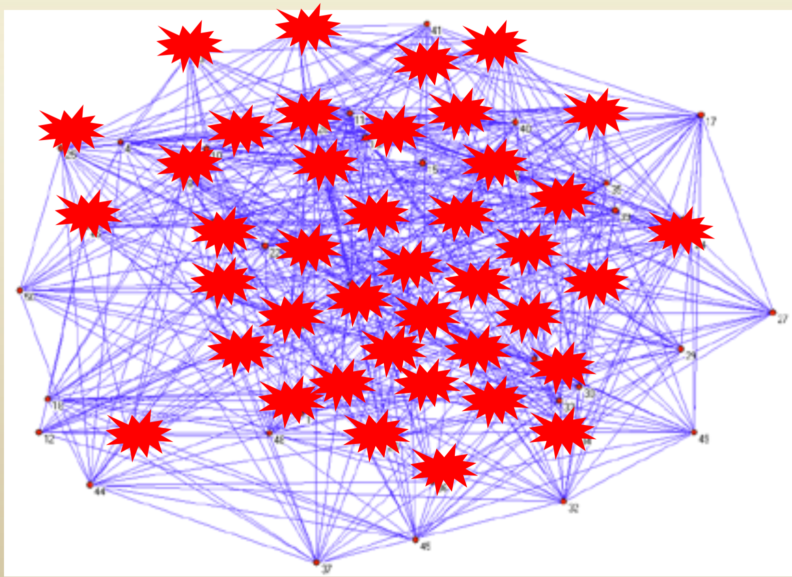
Long Range Network (small g)

$$\left(\psi_k^A(t), \psi_k^B(t)\right)^T = e^{-i(\omega_k t - kx)}\left(\psi_k^A, \psi_k^B\right)^T \qquad \vec{\Psi}(t) = \sum_k c_k^r(t)\vec{\Psi}_k^r$$

$$\omega(k) = \pm \arccos\left[\cos^2\theta + \sin^2\theta \cos k\right]$$

$$c_k^r(t+1) = e^{i\omega_k} c_k^r(t) + \frac{ig}{N} \sum_{\substack{r_1, r_2, r_3 \\ k_1, k_2, k_3}} e^{i(\omega_{k_1}^{r_1} + \omega_{k_2}^{r_2} - \omega_{k_3}^{r_3})} I_{k, k_1, k_2, k_3}^{r, r_1, r_2, r_3} c_{k_1}^{r_1}(t) c_{k_2}^{r_2}(t) \left(c_{k_3}^{r_3}(t)\right)^*$$

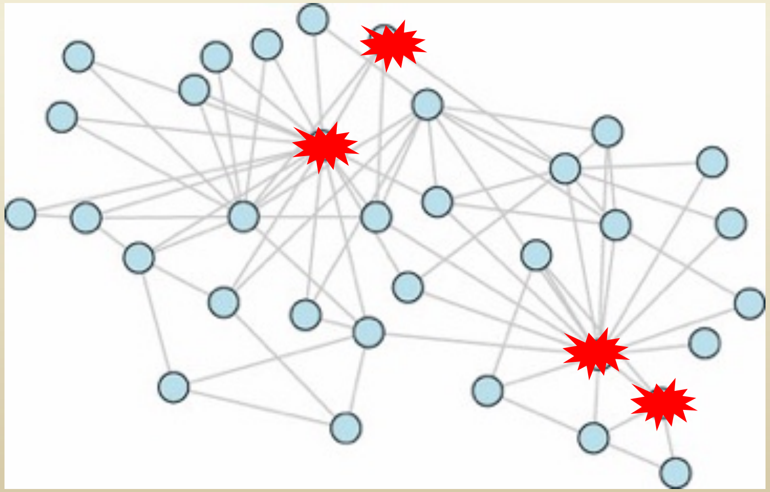
$$I_{k, k_1, k_2, k_3}^{r, r_1, r_2, r_3} = \delta_{k_1 + k_2 - k_3 - k, 0} \sum_P \psi_{k_1}^{r_1, P} \psi_{k_2}^{r_2, P} (\psi_{k_3}^{r_3, P})^* (\psi_k^{r, P})^*.$$



Short Range Network (small ϴ)

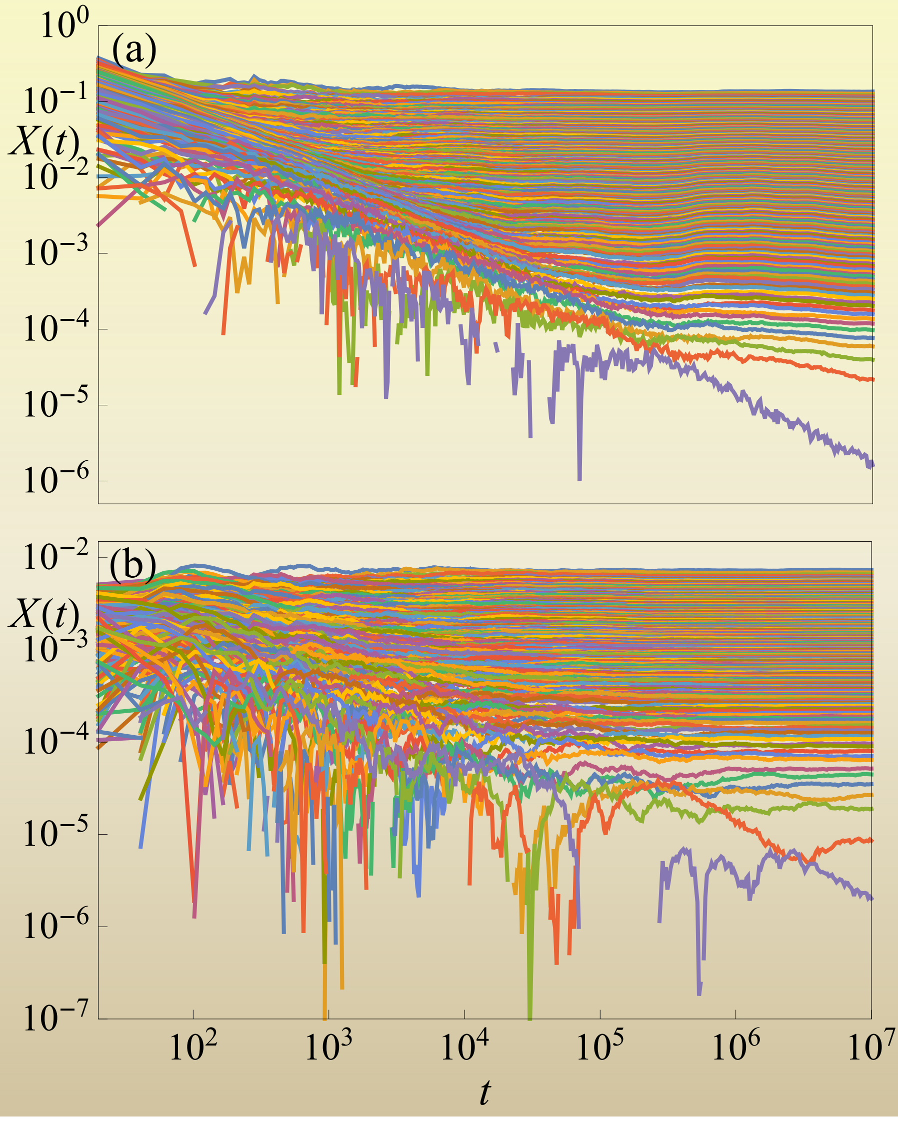
$$\psi_n^A(t+1) = e^{ig|\alpha_n^A|^2}\left[\psi_n^A(t) - \theta(\psi_{n-1}^B(t) - \psi_n^B(t))\right]$$

$$\psi_n^B(t+1) = e^{ig|\alpha_n^B|^2}\left[\psi_n^B(t) + \theta(\psi_{n+1}^A(t) - \psi_n^A(t))\right].$$



The Lyapunov Spectrum

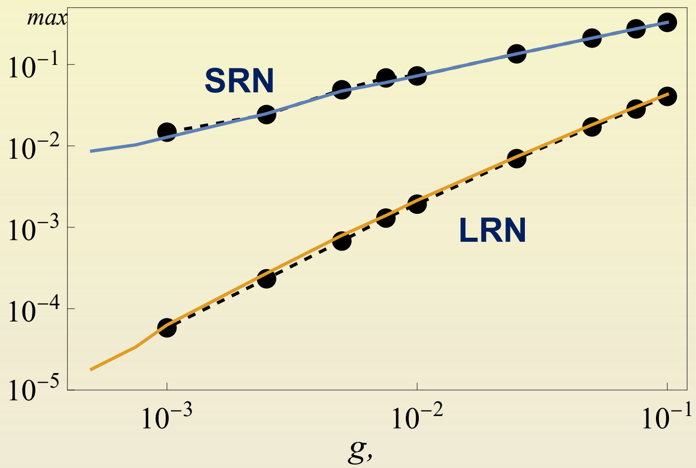
raw data:



Unitary Circuits for Thermalization

Merab Malishava, SF
PRL 128 134102 (2022)
Chaos 32 063113 (2022)

processed data: LLE

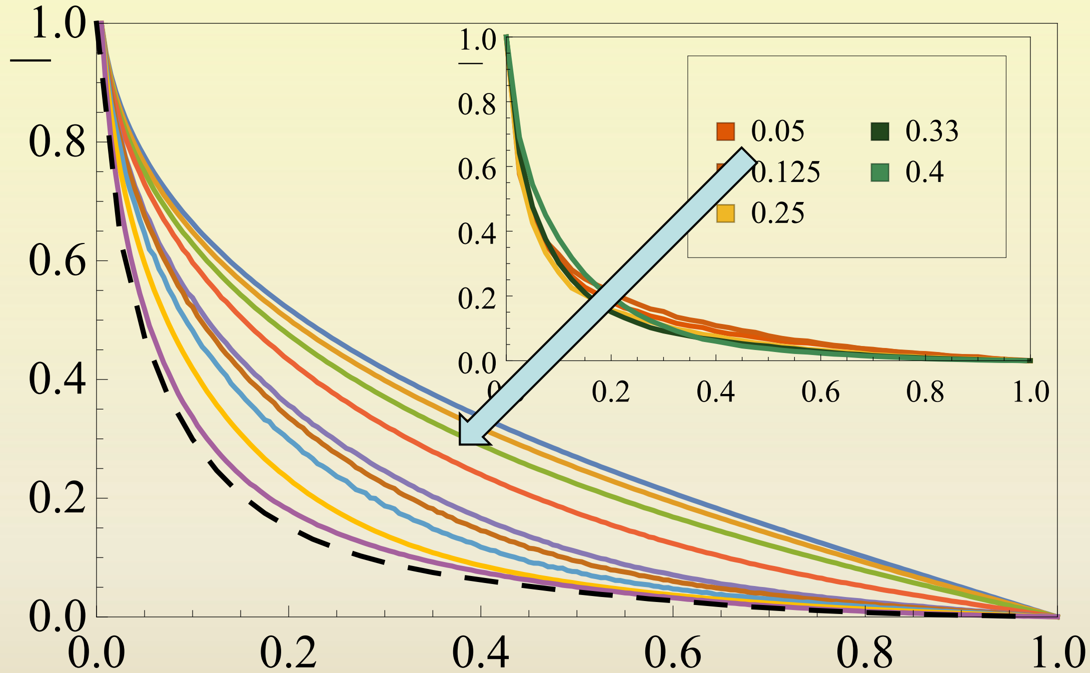


integrable limits

Unitary Circuits for Thermalization

Merab Malishava, SF
PRL 128 134102 (2022)
Chaos 32 063113 (2022)

LRN, one diverging scale



$\bar{\Lambda}(\rho)$: analytic function in the LRN limit

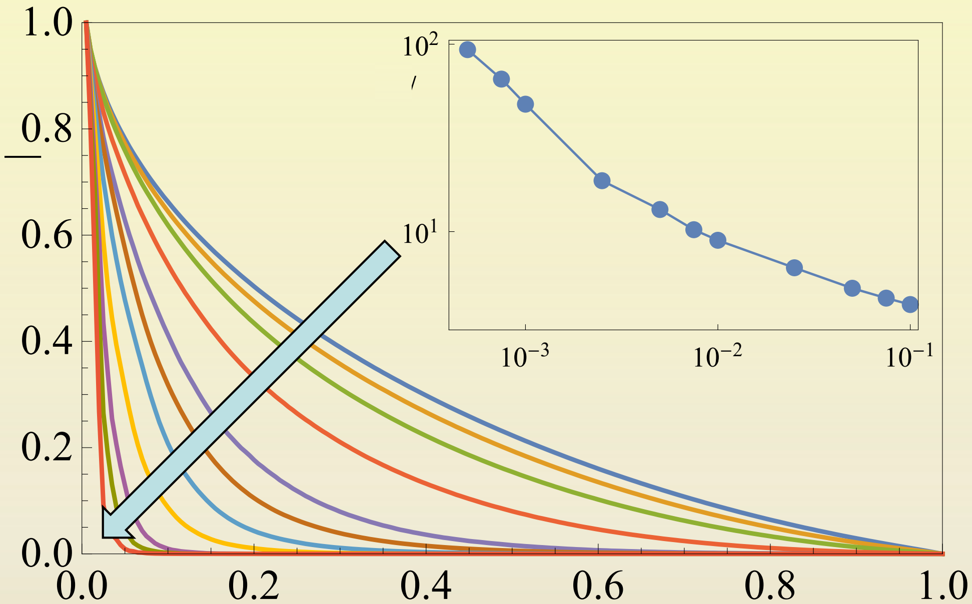
$\bar{\lambda}(\rho) = (1-\rho)e^{-\beta\rho}$

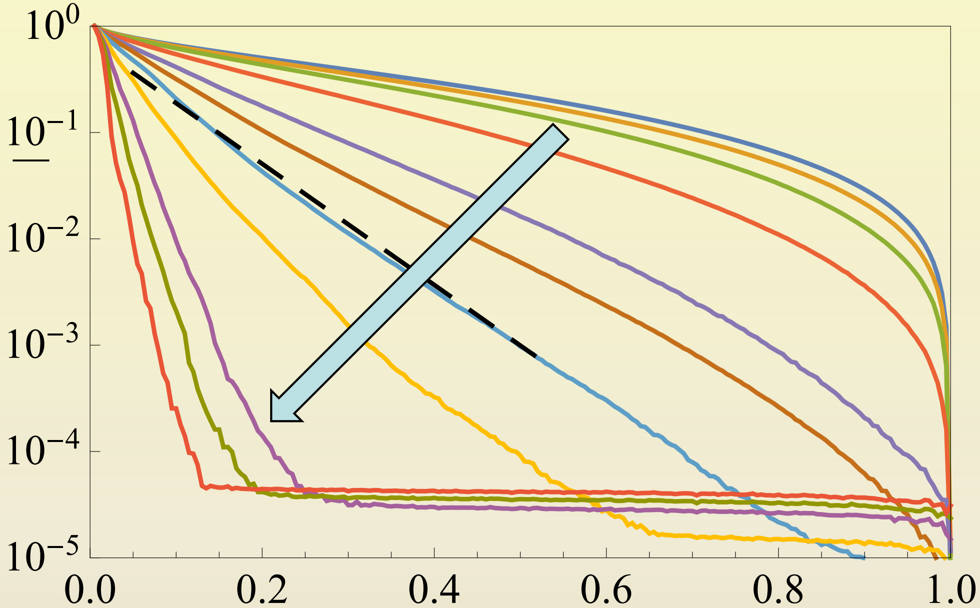
α, β and κ stay constant nonzero for $g \to 0$

Unitary Circuits for Thermalization

Merab Malishava, SF
PRL 128 134102 (2022)
Chaos 32 063113 (2022)

SRN, two diverging scales





$\bar{\Lambda}(\rho)$: non-analytic function in the SRN limit

$$\bar{\lambda}(\rho) = (1-\rho)e^{-\beta\rho}$$

β diverges and κ vanishes for $\theta \to 0$

Josephson junction networks in d=1,2,3

PHYSICAL REVIEW E 108, L062301 (2023)

Letter

Thermalization slowing down in multidimensional Josephson junction networks

Gabriel M. Lando and Sergej Flach

$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

Number of sites (volume) : N

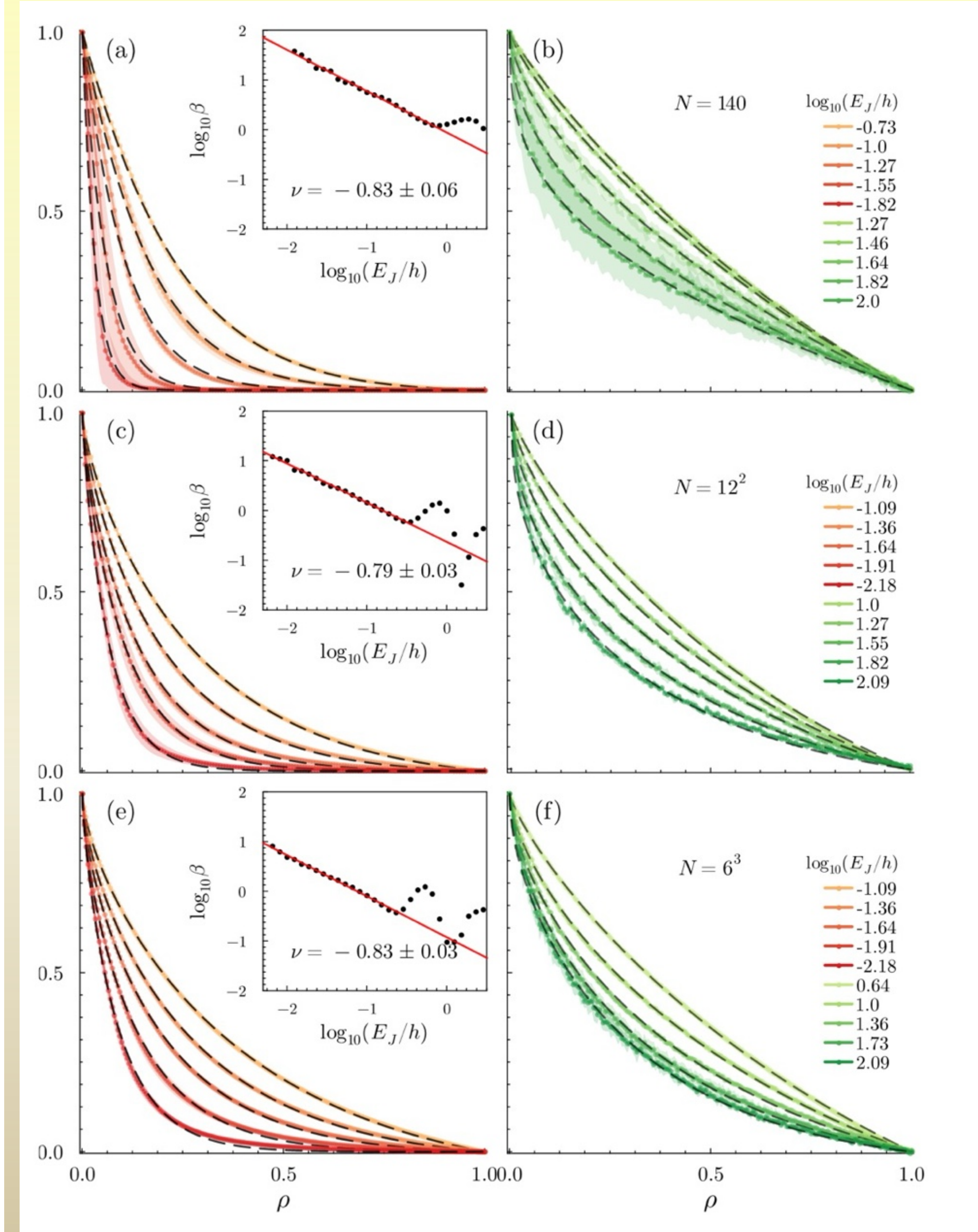
Energy density: h = H/N

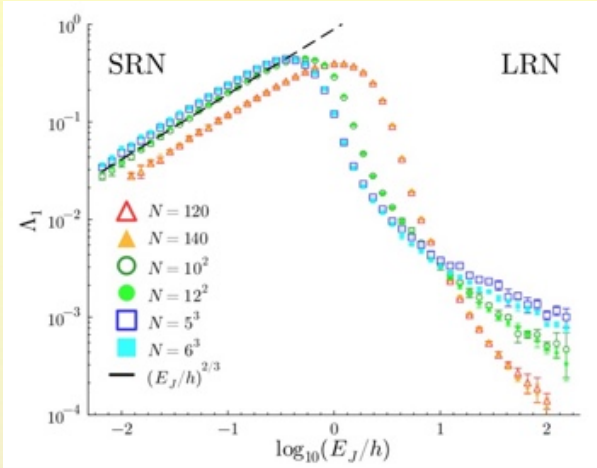
SRN: E_J / h → 0

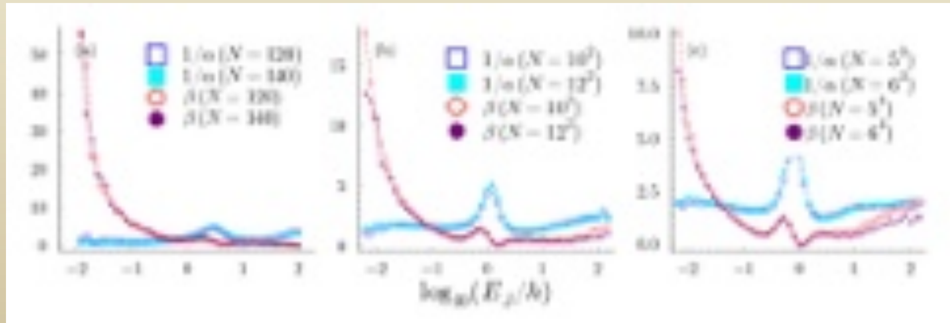
LRN: h / E_J → 0

Rescaled Lyapunov spectrum for d=1,2,3

Gabriel Lando, SF
PRE 108 L062301 (2023)








PHYSICAL REVIEW RESEARCH 6, L012064 (2024)

Letter

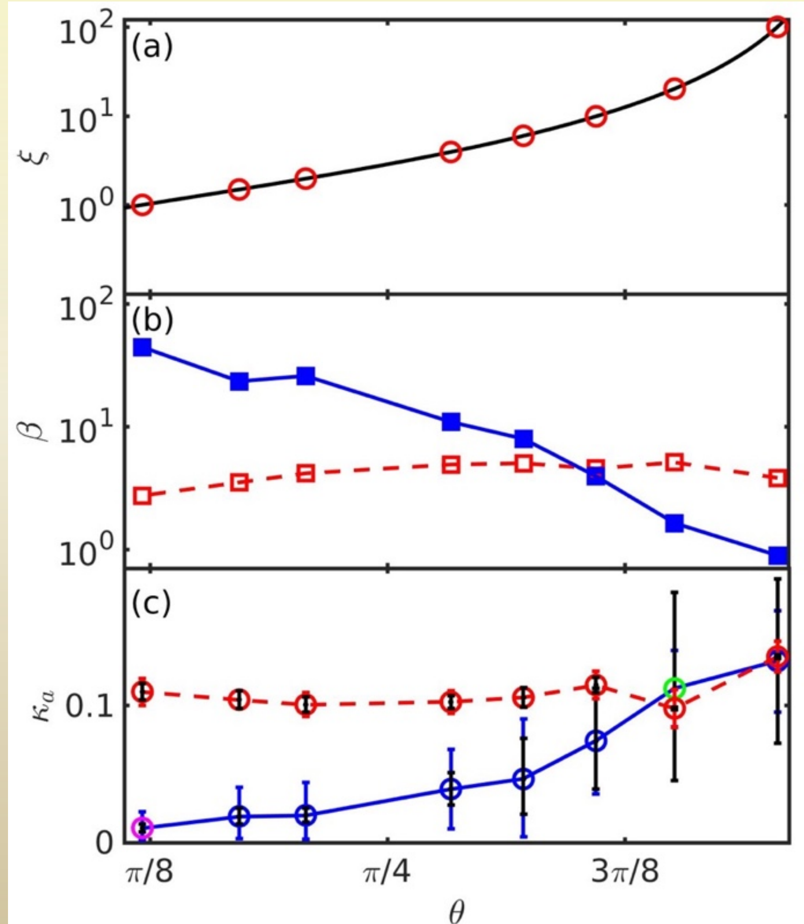
Thermalization universality-class transition induced by Anderson localization

Weihua Zhang, Gabriel M. Lando, Barbara Dietz, and Sergej Flach

Disorder and Anderson Localization:

$$\hat{G}_n = e^{i\phi_n} e^{ig|\psi_n|^2 |n\rangle\langle n|} \ , \ \phi_n \in [0, 2\pi]$$

$$g = 0 \ : \ \xi^{-1} = |\ln(|\sin\Theta|)|$$



prethermalization


Low Temperature Physics
ARTICLE
pubs.aip.org/aip/ltp

Observation of prethermalization in weakly nonintegrable unitary maps

Cite as: Fiz. Nizk. Temp. 51, 870–880 (June 2025); doi: 10.1063/10.0036801
Submitted: 22 April 2025

Xiaodong Zhang, Gabriel M. Lando, Barbara Dietz, and Sergej Flach

- **SRN and LRN regimes for unitaries**
- **compare statistics of the largest Lyapunov exponent**

$$r(t) = \ln\left(\frac{|\mathbf{X}(t)|}{|\mathbf{X}(\mathbf{t}-\mathbf{dt})|}\right)\ , \ \Lambda(t) = \frac{1}{t}\int_0^t r(t')dt'$$

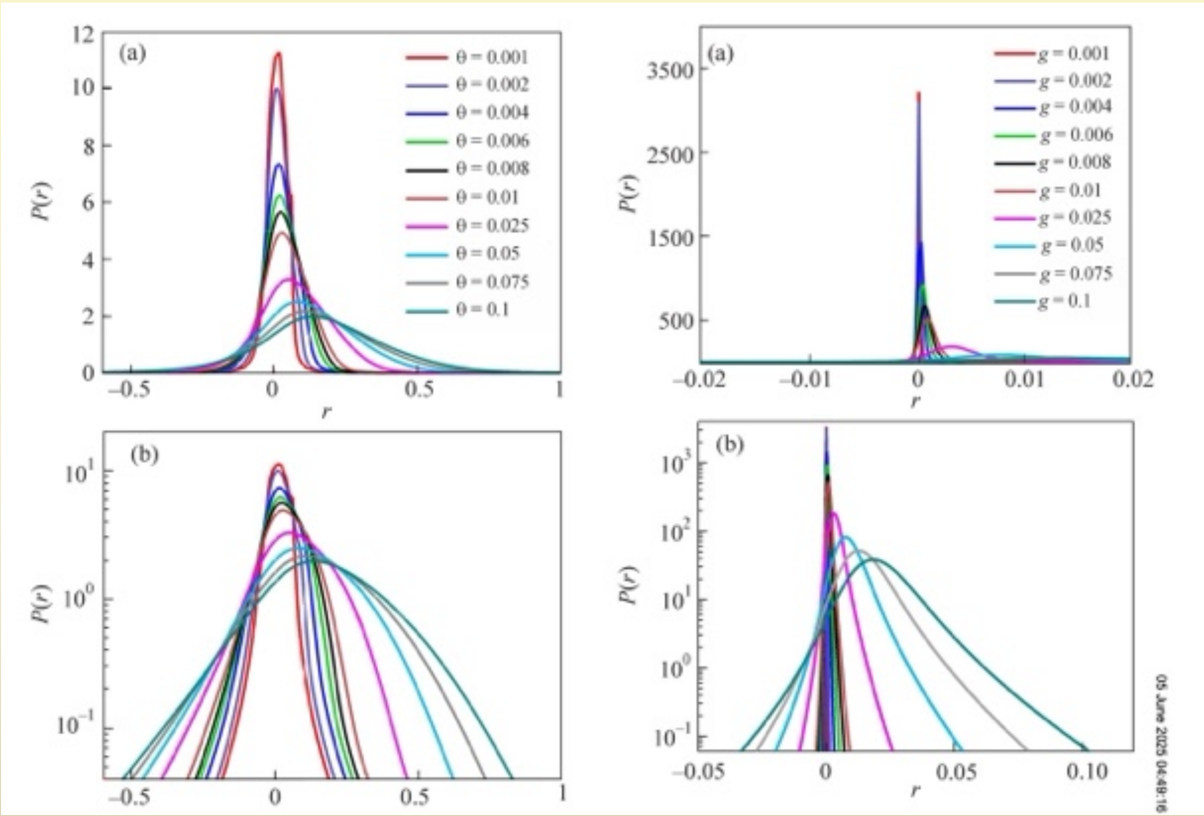
prethermalization

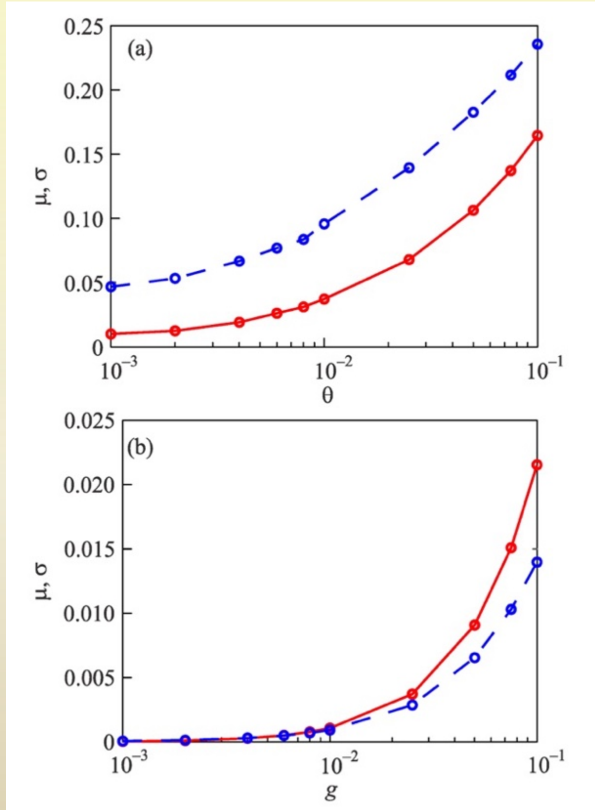

Low Temperature Physics ARTICLE pubs.aip.org/aip/ltp

Observation of prethermalization in weakly nonintegrable unitary maps

Cite as: Fiz. Nizk. Temp. 51, 870–880 (June 2025); doi: 10.1063/10.0036801
Submitted: 22 April 2025

Xiaodong Zhang, Gabriel M. Lando, Barbara Dietz, and Sergej Flach





prethermalization

Low Temperature Physics
ARTICLE
pubs.aip.org/aip/ltp

Observation of prethermalization in weakly nonintegrable unitary maps

Cite as: Fiz. Nizk. Temp. 51, 870–880 (June 2025); doi: 10.1063/10.0036801
Submitted: 22 April 2025

Xiaodong Zhang,[1,2,a)] Gabriel M. Lando,[1,b)] Barbara Dietz,[1,3,c)] and Sergej Flach[1,3,d)]

LRN

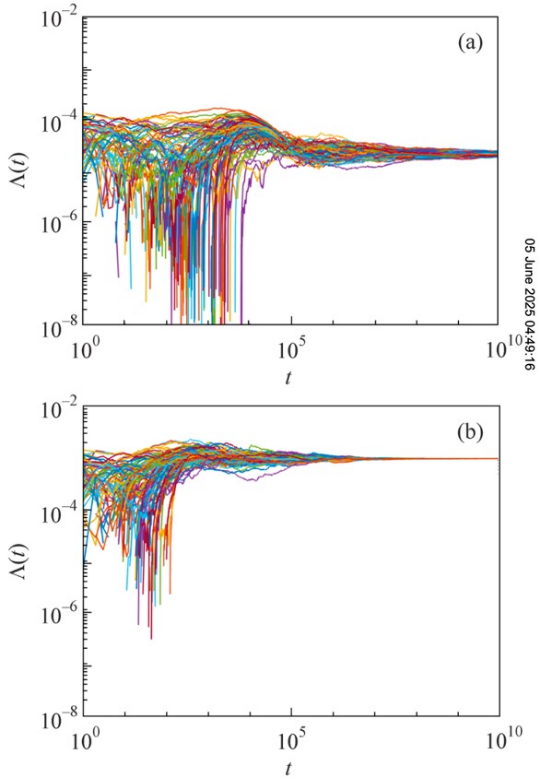


FIG. 5. Time evolution of the largest Lyapunov exponent Λ(t) on a $\log_{10}$ scale for up to 10^{10} iterations for $g = 0.001$ (a) and 0.01 (b) in the LRN case for $\theta = 0.33\pi$. Each panel shows 100 trajectories obtained from randomly chosen initial conditions. The system consists of $N = 50$ unit cells.

SRN

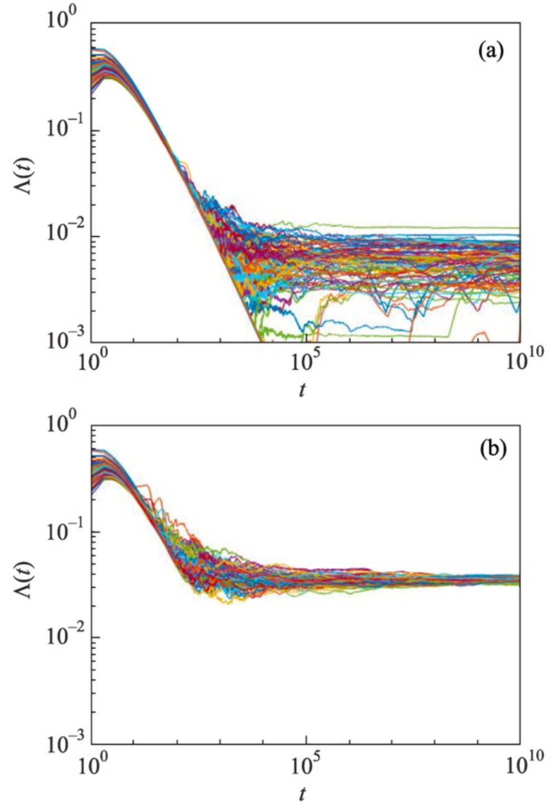


FIG. 6. Same as Fig. 5 for the SRN case with $g = 1$ and $\theta = 0.001$ (a) and 0.01 (b).

prethermalization


Low Temperature Physics

ARTICLE

pubs.aip.org/aip/ltp

Observation of prethermalization in weakly nonintegrable unitary maps

Cite as: Fiz. Nizk. Temp. 51, 870–880 (June 2025); doi: 10.1063/10.0036801

Submitted: 22 April 2025

Xiaodong Zhang, Gabriel M. Lando, Barbara Dietz, and Sergej Flach

LRN

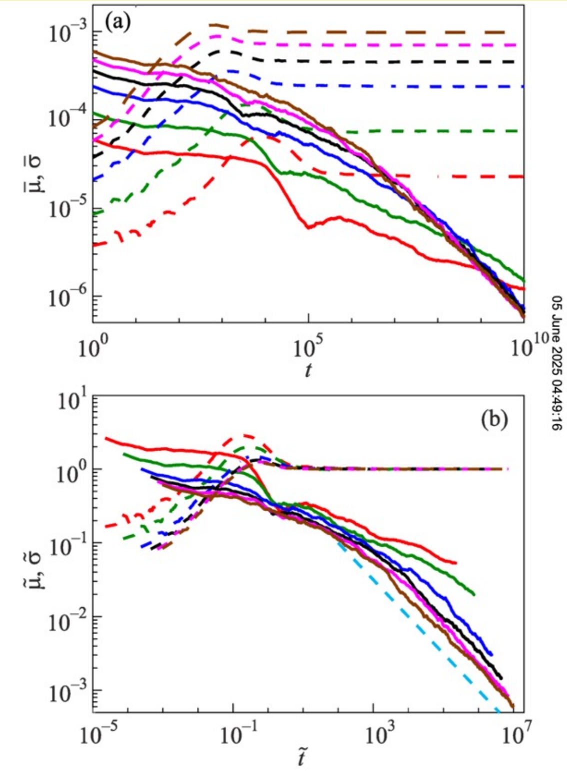


FIG. 7. The unscaled (a) and scaled (b) average values of the mean and standard deviation of $\Lambda(t)$ on a $\log_{10}$ scale for the LRN case with $\theta = 0.33\pi$ for different values of g (see main text for more information). The colors red, green, blue, black, magenta, and brown correspond to g = 0.001, 0.002, 0.004, 0.006, 0.008, respectively. Averaging is performed over 100 trajectories at each time step. In (b), the cyan dashed line indicates the temporal decay $\sigma \sim t^{-1/2}$. The time evolution is plotted on a $\log_{10}$ scale, extending up to 10^{10}. The number of unit cells is $N = 50$.

SRN

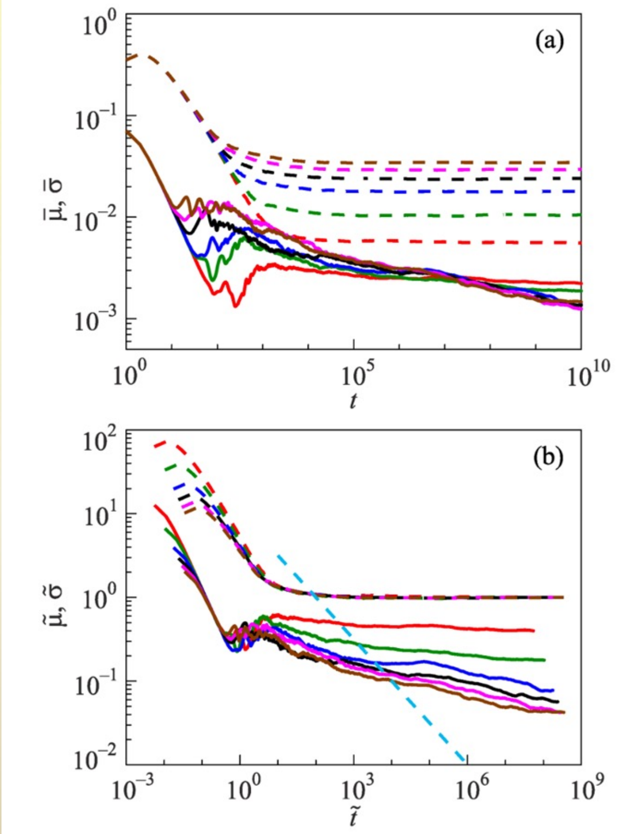


FIG. 8. Same as Fig. 7 for the SRN case with $g = 1$ for different values of θ. The colors red, green, blue, black, magenta, and brown correspond to θ = 0.001, 0.002, 0.004, 0.006, 0.008, 0.01, respectively.

$$z := \bar{\mu}(t = 10^{10})$$

$$\tilde{t} = zt,\ \tilde{\mu}(t) = \bar{\mu}(t)/z,\ \tilde{\sigma}(t) = \bar{\sigma}(t)/z$$

prethermalization

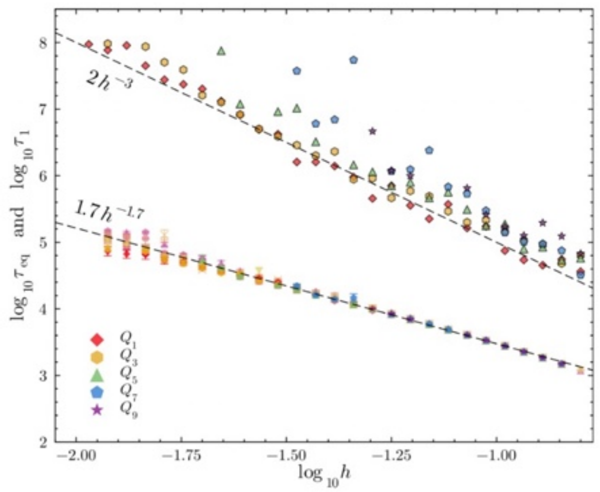


FIG. 9. Comparison of Lyapunov times in Fig. 7, where here both panels are superposed, and its corresponding linear fit. The equipartition times for root modes in Fig. 5 are also displayed, together with a linear fit for those of Q_1, showing that they are orders of magnitude longer than the Lyapunov times for all energy densities.

PHYSICAL REVIEW E **112**, 014206 (2025)

Prethermalization in Fermi-Pasta-Ulam-Tsingou chains

Gabriel M. Lando and Sergej Flach

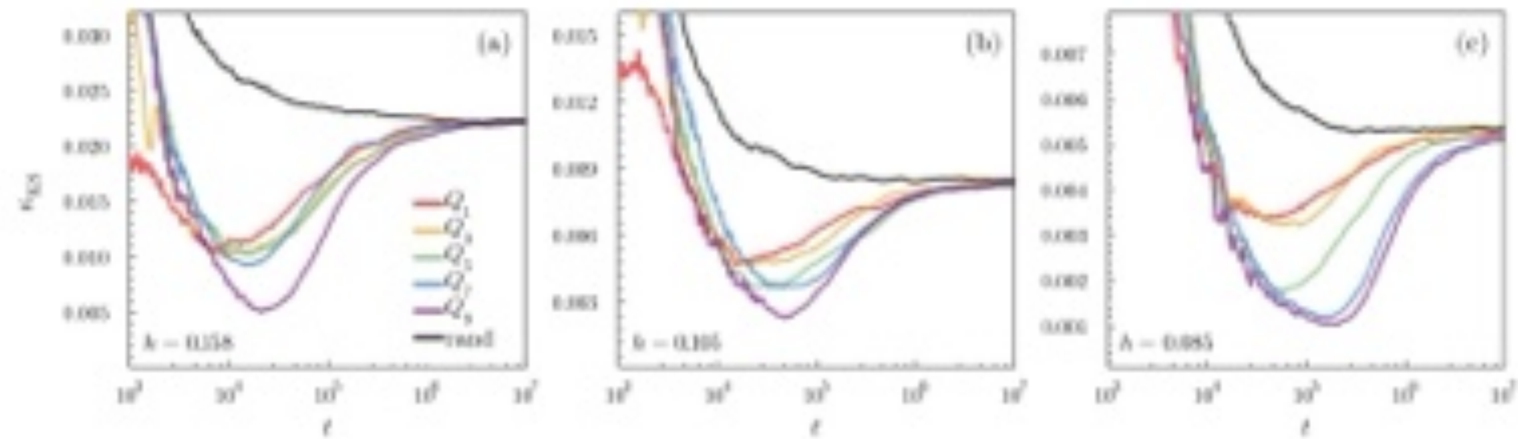


FIG. 8. Time-dependent Kolmogorov-Sinai entropy in an α-FPUT chain with $N = 63$ and $\alpha = 1/4$ computed from roots (rainbow colors) and from a random state (black). Each panel is computed for initial states with decreasing energy densities, h, shown in the left corner.

SUMMARY

- We observe two distinct universality classes for thermalization slowing down upon approaching integrability
- Classifier: nonintegrable perturbation network type
- Long range - one time scale controls all thermalization time scales
  - rescaled Lyapunov spectrum: analytic function at integrable limit
  - rescaled KS entropy finite at integrable limit
  - single parameter scaling
- Short range - exponential vanishing of rescaled Lyapunov spectrum
  - rescaled KS entropy vanishes at integrable limit
  - absence of single parameter scaling
- Disorder and Anderson localization induce transition from LRN to SRN
- Explains finite time average observations (ergodicity)
- Works in any lattice dimension
- Quantization: SRN results in MBL ???
- Impact of topology?

PRE 95 060202 (2017)
PRL 122 054102 (2019)
PRE 100 032217 (2019)
PRE 104 014218 (2021)
Chaos 32 063113 (2022)
PRL 128 134102 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)
Chaos 34 033107 (2024)
PRR 7 023149 (2025)
LTP 51 870 (2025)
PRE 112 014206 (2025)
PRE 113 L012201 (2025)
Chaos 35 113127 (2025)
arXiv:2603.26042