Dynamical Reductions for Solitonic Filaments

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Road map

- Introduction
    - Motivation: BEC physical experiments

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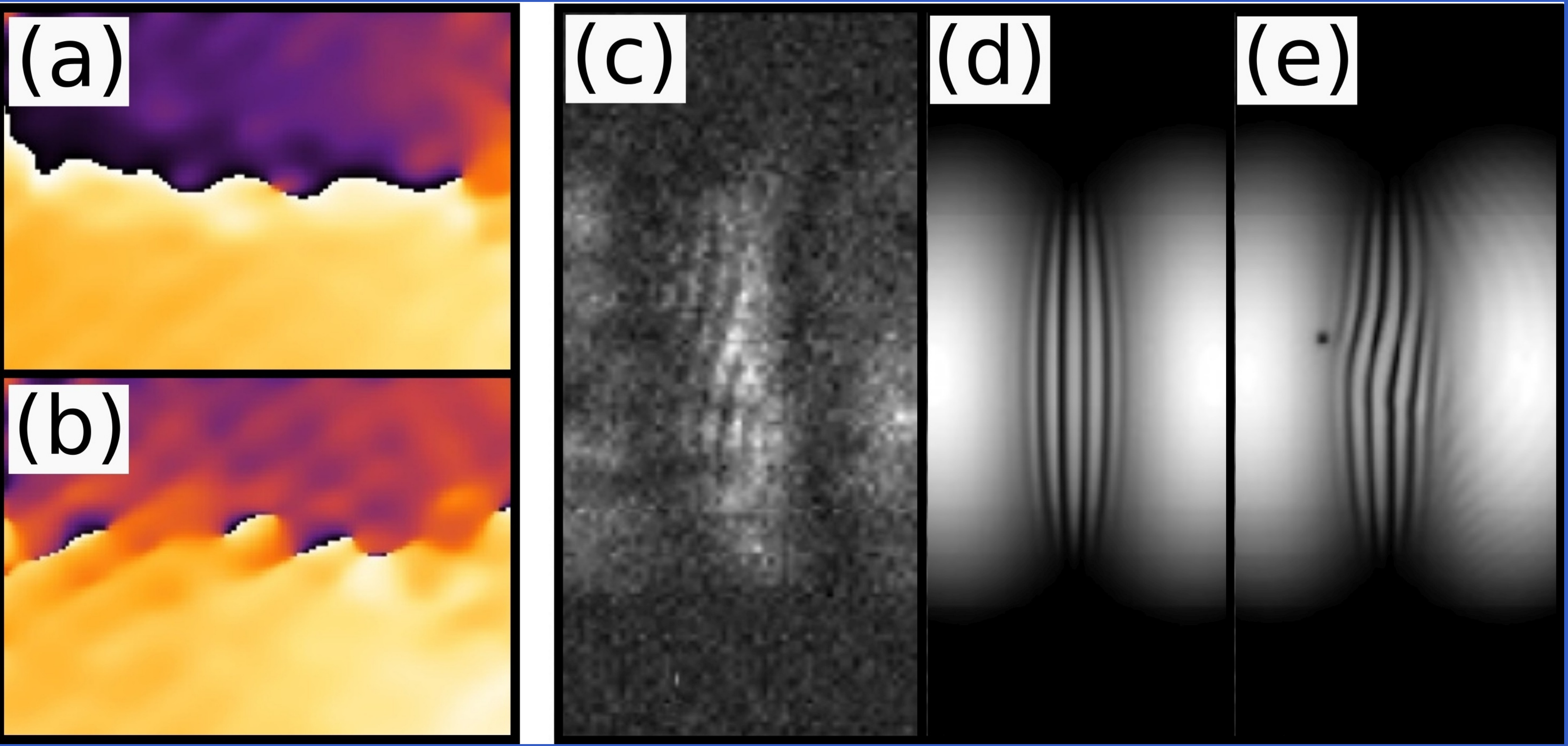
- Recap and Outlook

Solitonic filaments

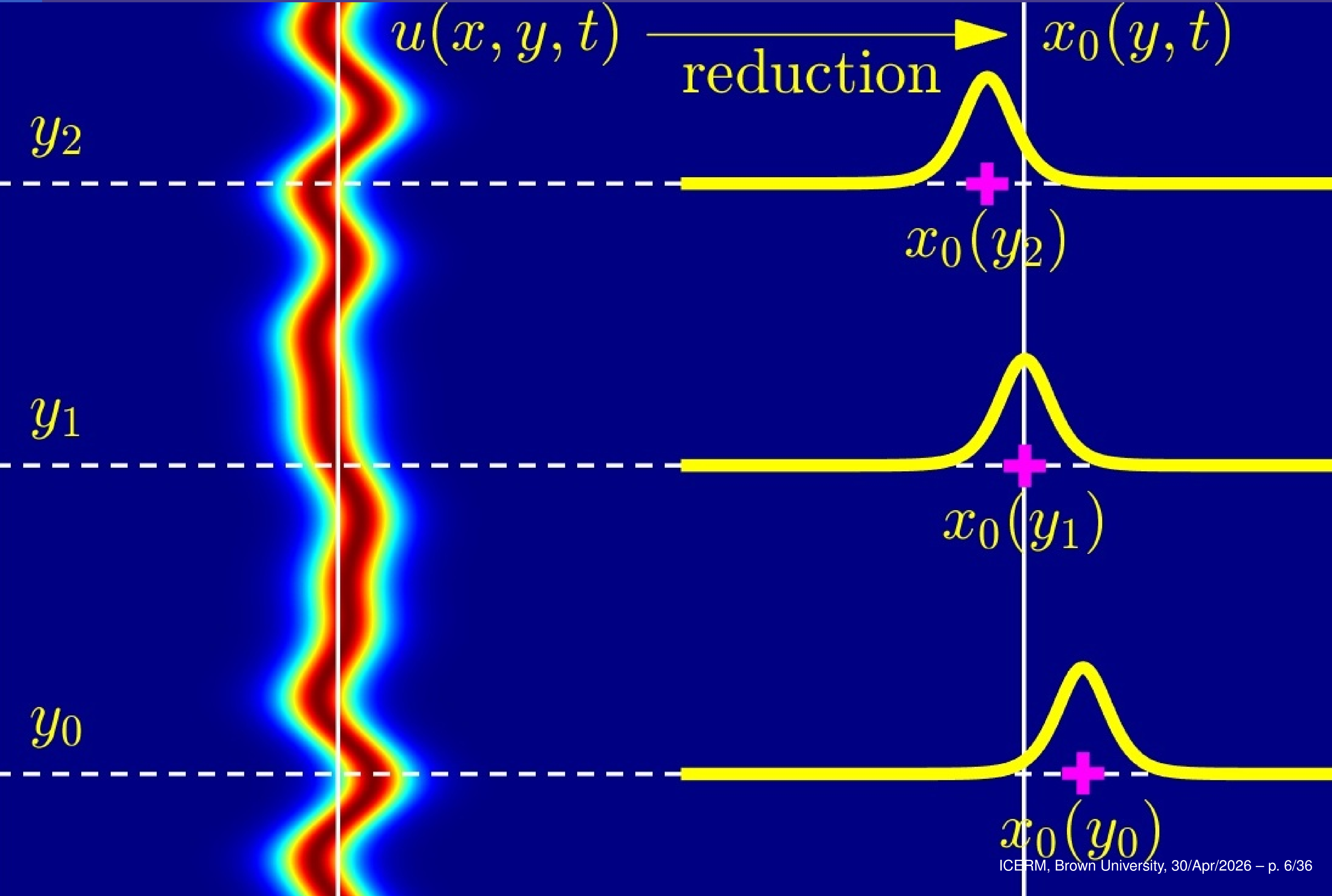
in NLS-type systems

Solitonic filaments: experimental motivation



- (a) & (b): Exciton-polariton BEC, Daniele Sanvitto, Lecce.
- (c)–(e): DSSs-vortex in atomic BEC, Brian Anderson, Arizona.

Dynamical reduction for filaments in a picture:



Reduction approach (AI+VA+EL+pert.+Galerkin+...)

- Want to solve for dynamics of a stripe in 2D: $\boxed{\mathrm{PDE}[u(x,y,t)] = 0}$

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- Use robust 1D solution ansatz u_{1D} (bright/dark soliton, kink, breather)
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- Evaluate $H_{2\mathrm{D}}$ on the stripe: $H_{2\mathrm{D}} = \iint_{-\infty}^{\infty} \mathcal{H}_{2\mathrm{D}}(u_{1\mathrm{D}}(x - x_0(y,t)))\, dx\, dy$.

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- Massage/blend/simmer ($\int dx$, by parts, EL, ...) and $dH/dt=0 \Rightarrow$

$$\boxed{\mathrm{PDE}[x_0(y,t)]=0} \quad \leftarrow \text{ Lower dimensional than original !!!}$$

Adiabatic Invariant (AI) Approach : 1D : dark soliton

- 1D NLS for external potential V:

$$iu_t = -\frac{1}{2}u_{xx} + |u|^2 u + V(x)\,u.$$

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  Hamiltonian: $H_{1\text{D}} = \frac{1}{2}\int_{-\infty}^{\infty} \left[|u_x|^2 + \left(\mu - |u|^2\right)^2 \right] dx.$

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- Bring back V → AI: consider background *slowly* varying ($u_{xx} \sim 0$) → steady state: $u(x,t) = w(x)e^{-i\mu t} \Rightarrow \mu w \sim w^3 + V\, w \Rightarrow w^2 \sim \mu - V$.

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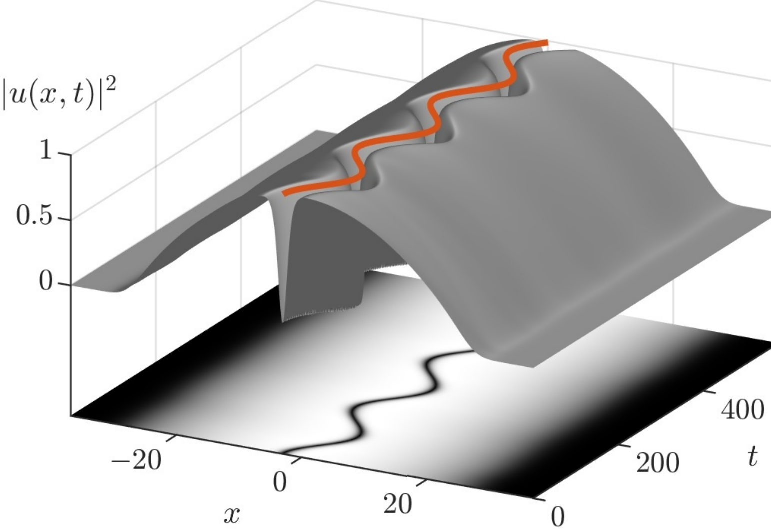
- Thus background $w^2 \sim \mu - V$ is captured with $\mu \rightarrow \mu - V(x)$:

$$H_{1\mathrm{D}} \simeq \frac{4}{3}\left(\mu - V(x_0) - \dot{x}_0^2\right)^{3/2} \;\Rightarrow\; \ddot{x}_0 \simeq -\frac{1}{2}V'(x_0),$$

Adiabatic Invariant (AI) Approa



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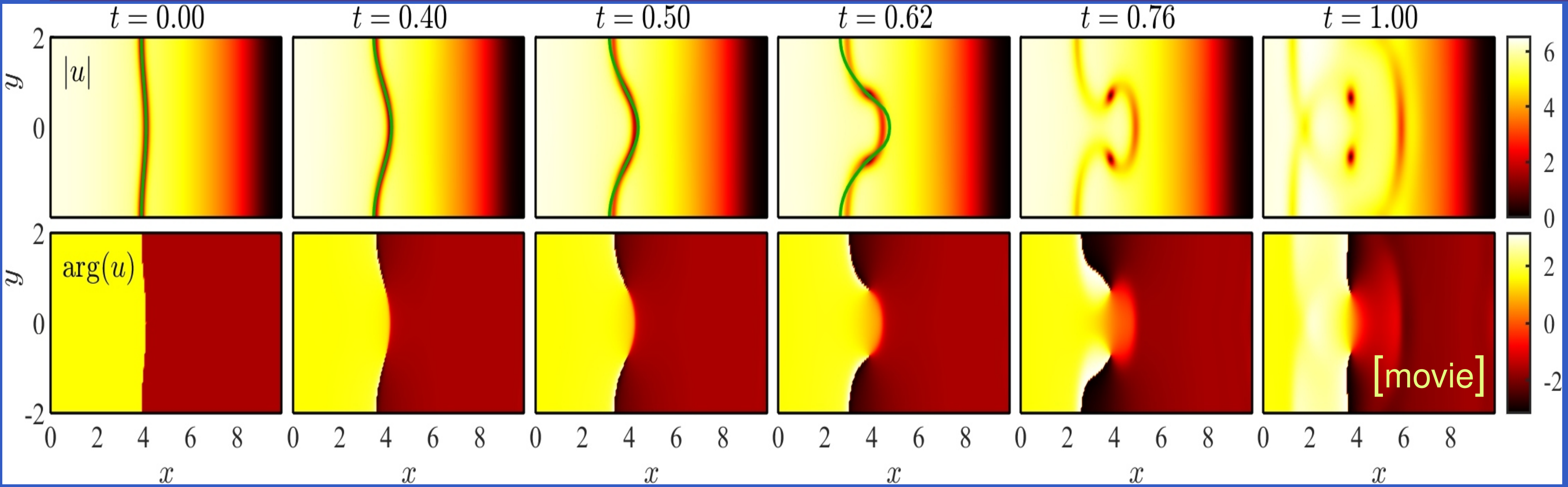
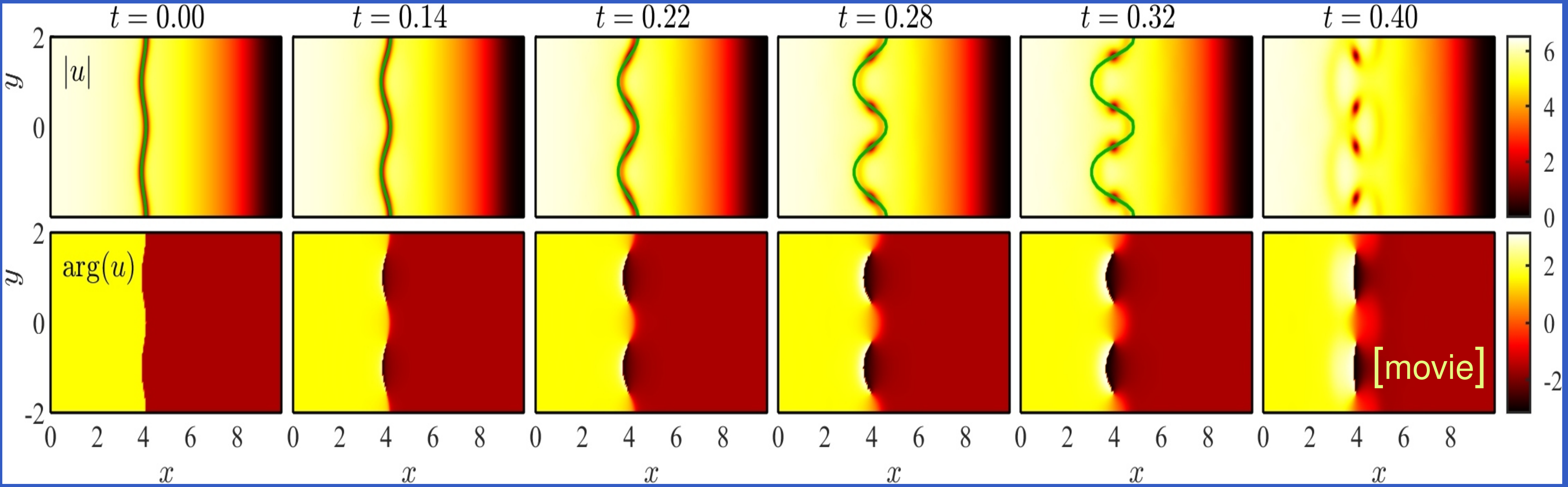
- At linear level with $V = 0$ one recovers: $x_{0tt} + \frac{\mu - v^2}{3} x_{0yy} = 0$.

DS snaking for $V \neq 0$: $V(x,y) = \frac{1}{2}\Omega^2 x^2$: PDE vs. AI

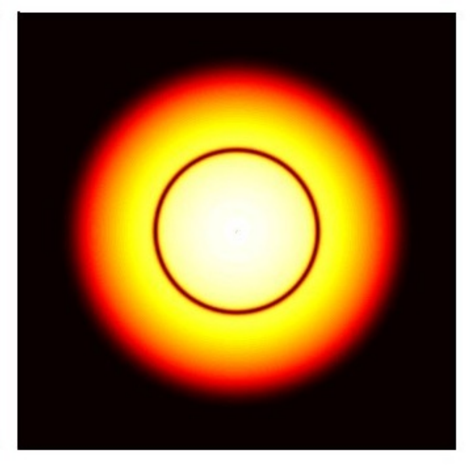


[movie]



[movie]

AI : Ring dark soliton (RDS) → dynamics



- Start with NO azimuthal perturbations, i.e. radius given by $R = R(t)$. [Kamchatnov+Korneev PLA'10 and Konotop+Pitaevskii PRL'04] :

$$E = 2\pi R(\mu - V(R) - \dot{R}^2)^{3/2}$$

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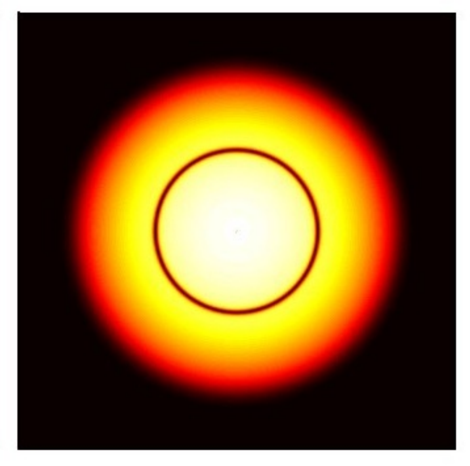

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- Generalize to RDS filament : $R(t) \rightarrow R(\theta, t)$. Energy:

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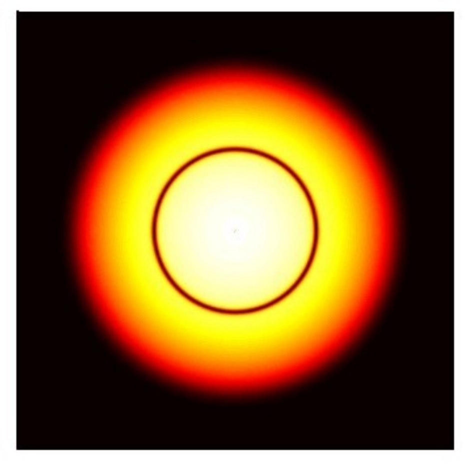

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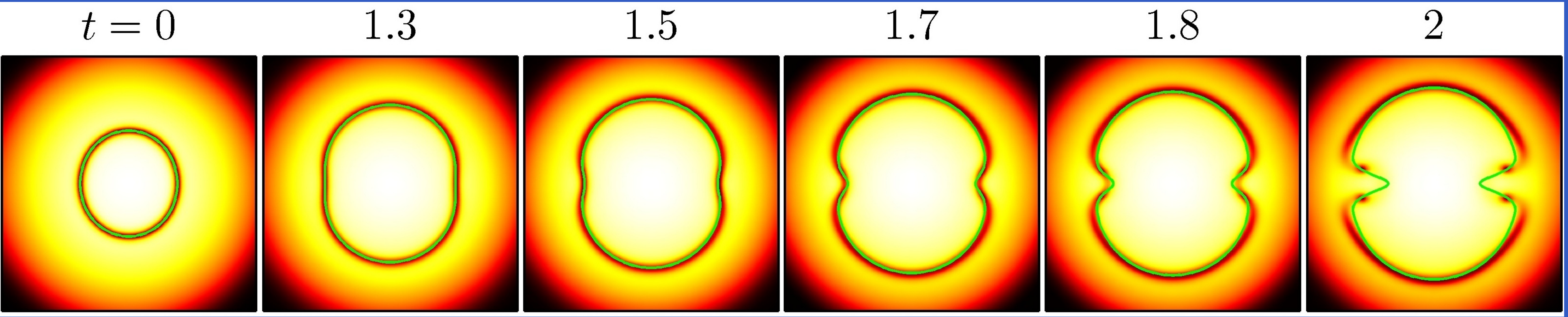
- Apply the AI methodology ($dE/dt = 0$) → PDE for $R(\theta, t)$:

$$\boxed{CD - \frac{R_{\theta\theta}}{R}C = -\frac{R_\theta}{R}\left(\frac{3}{2}V'(R)R_\theta + 3R_tR_{t\theta}\right) + RD\left(\frac{3}{2}V'(R) + 3R_{tt}\right)}$$

with $C \equiv \mu - V(R) - R_t^2$ and $D \equiv 1 + \frac{R_\theta^2}{2R^2}$.

Ring dark soliton : PDE vs AI



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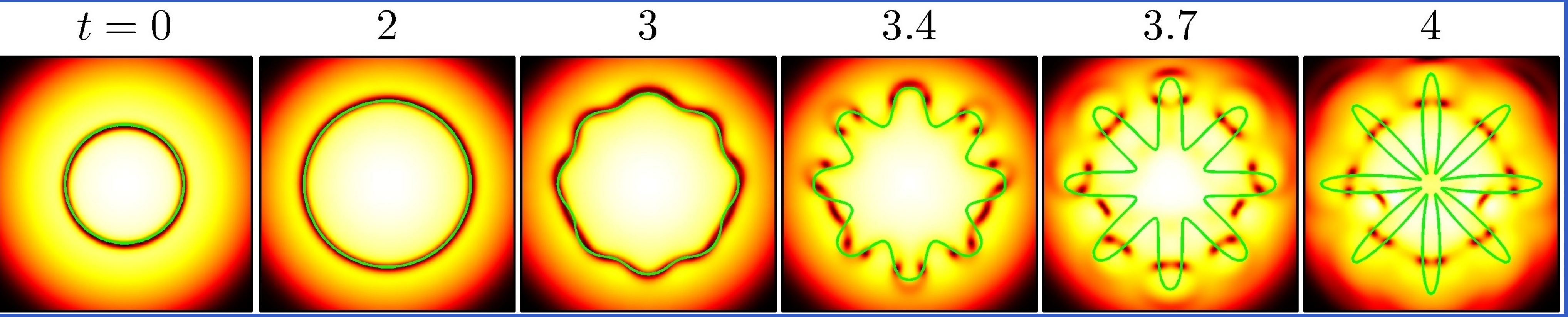

[movie]

AI : Ring dark soliton (RDS) $\rightarrow$ stability

- Steady state for generic $V(r)$: $3R_0V'(R_0) = 2(\mu - V(R_0))$
- Stability: perturb steady state : $R = R_0 + \varepsilon\, e^{i(n\theta+\omega t)}$:

$$\omega^2 \quad = \quad \frac{V'(R_0)}{2R_0}\left[\frac{5}{3} - n^2 + \frac{R_0V''(R_0)}{V'(R_0)}\right].$$

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- For parabolic trap $V(r) = \frac{1}{2}\Omega^2 r^2$ we get:

$$R_0 = \sqrt{\frac{\mu}{2\,\Omega^2}} = \frac{1}{2}R_{\rm TF} \quad \text{and} \quad \frac{\omega^2}{\Omega^2} = \frac{1}{2}\left(\frac{8}{3} - n^2\right).$$

Therefore: $n > \sqrt{8/3} = 1.633 \Rightarrow$ azimuthal INSTABILITIES.

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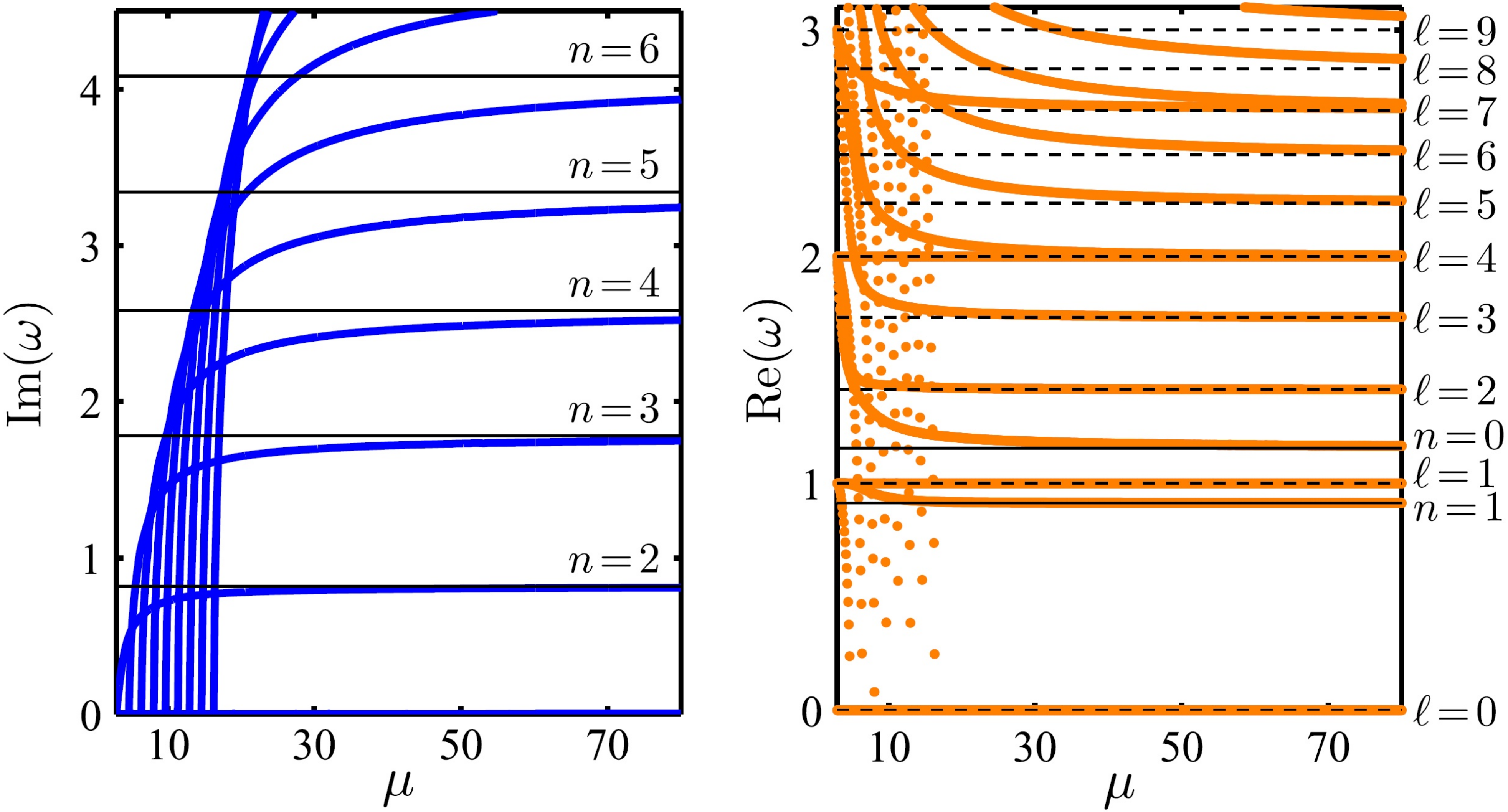
- Complete spectrum with the (stable) modes of background: [Stringari PRL'77, Kevrekidis+Pelinovsky PRA'10]

$$\omega = \pm\Omega(\ell + 2k(1+\ell) + 2k^2)^{1/2}.$$

Ring dark soliton : PDE vs AI – Stability spectrum



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- ℓ's → modes of background.

Solitonic filaments

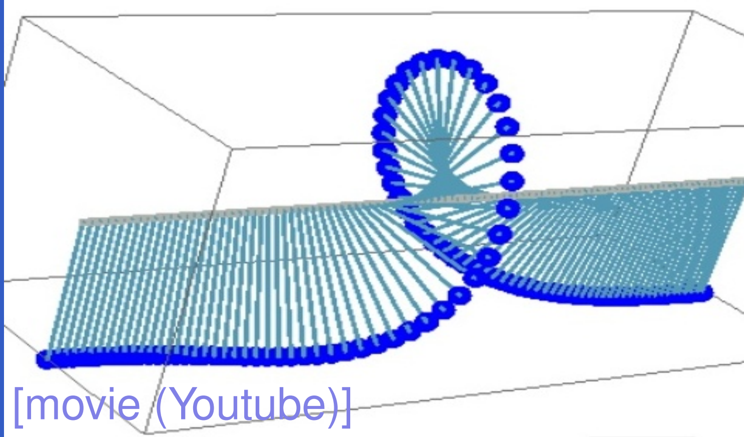
in Klein-Gordon systems

Klein-Gordon (KG) : sine-Gordon (sG) and ϕ^4

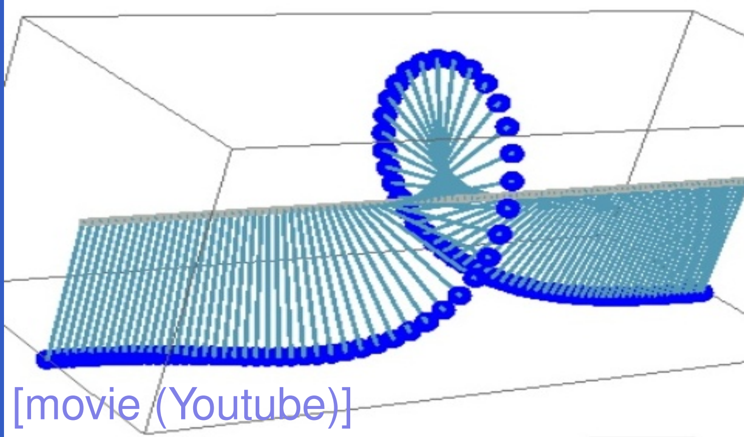
- KG in 2D with external potential $V_{\mathrm{ext}}(x,y)$:

  $$u_{tt} = \nabla^2 u - (1 + V_{\mathrm{ext}}(x,y))V'(u),$$

  where $V(u)$ is the intrinsic potential:
  sG: $V(u) = 1 - \cos(u)$
  ϕ^4: $V(u) = (u^2 - 1)^2/2$.


[movie (Youtube)]

Klein-Gordon (KG) : sine-Gordon (sG) and ϕ^4



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- KG in 2D with external potential $V_{\text{ext}}(x,y)$:

  $$u_{tt} = \nabla^2 u - (1 + V_{\text{ext}}(x,y))V'(u),$$

  where $V(u)$ is the intrinsic potential:
  sG: $V(u) = 1 - \cos(u)$
  ϕ^4: $V(u) = (u^2-1)^2/2$.

- Filament ansatz $u(x,y,t) = f(x - x_0(y,t))$ with position $x_0(y,t)$ where:
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  Namely, exact KINK solutions when $V_{\text{ext}} = 0$.

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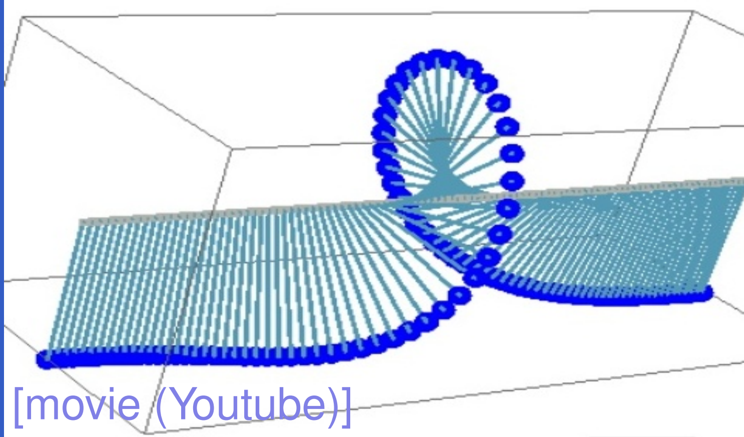

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- $dH/dt = 0 \rightarrow$ PDE for $x_0(y,t)$:

  $$\boxed{x_{0tt} - x_{0yy} = -\frac{1}{M}P'(x_0)}$$

  with $P'(x_0) = \int_{-\infty}^{\infty} V'_{\rm ext}(x)\, V(f(x-x_0,t))\, dx$
  and $M = \int_{-\infty}^{\infty} [f'(s)]^2\, ds$ is the kink's effective mass.

AI : Klein-Gordon (KG) : Localized potential

- Choosing a localized external potential: $V_{\text{ext}}(x) = A\,\text{sech}^2(x)$ yields:
  sG : $P'(x_0) = -4A\,\text{csch}^4(x_0)\,((2+C)2x_0 - 3S)$, and
  ϕ^4 : $P'(x_0) = -\frac{A}{3}\text{csch}^6(x_0)\,(T - 36x_0 - 24x_0C + 28S)$,
  where $C \equiv \cosh(2x_0)$, $S \equiv \sinh(2x_0)$, and $T \equiv \sinh(4x_0)$ for x_0 PDE:

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→ dynamics: see next...

- Stability using periodic perturb.

  with wavenumber k on $[-L_y, L_y]$:

$$\text{sG} \;:\; \omega_k = \pm\sqrt{-\frac{32A}{15M} + \left(\frac{k\pi}{L_y}\right)^2},$$

$$\phi^4 \;:\; \omega_k = \pm\sqrt{-\frac{64A}{105M} + \left(\frac{k\pi}{L_y}\right)^2}.$$

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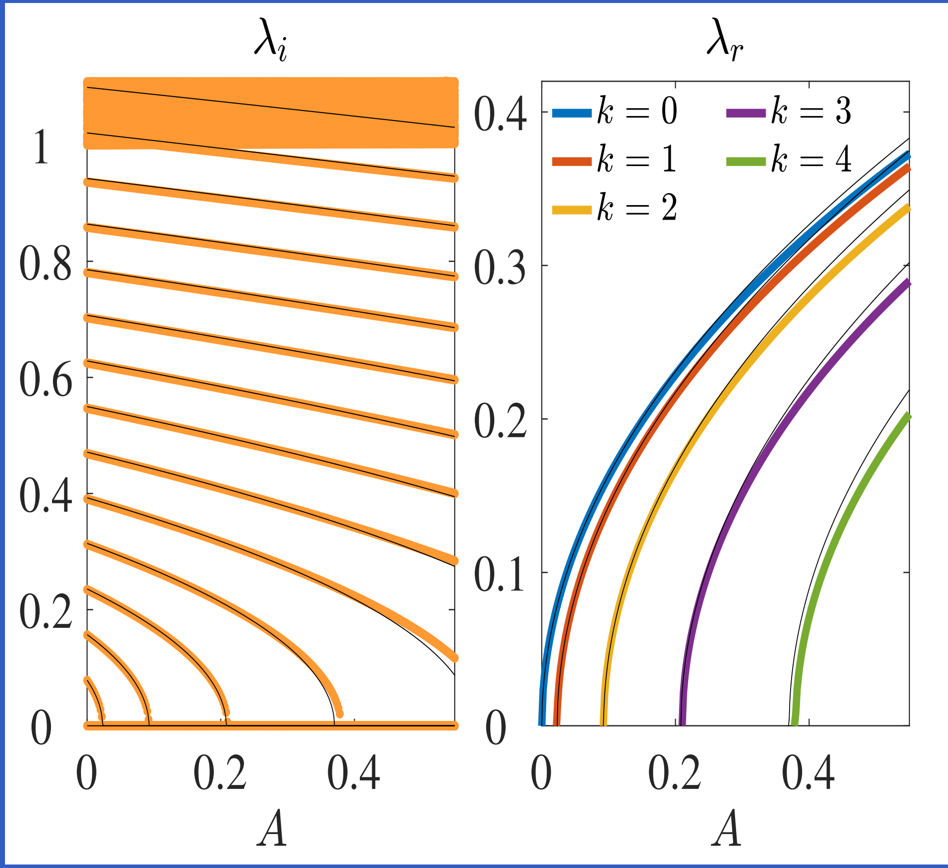
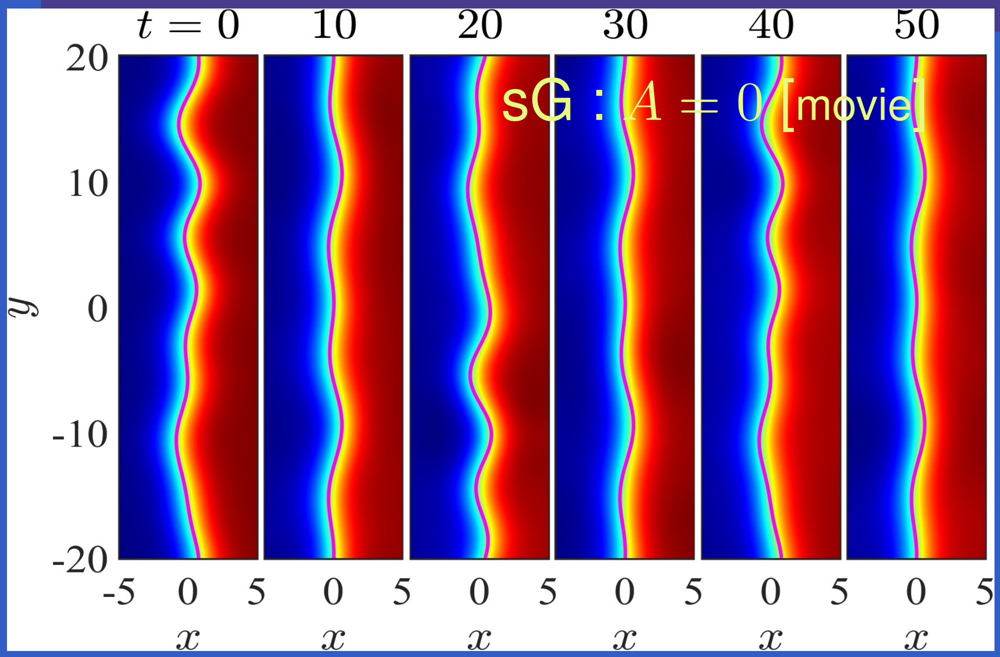
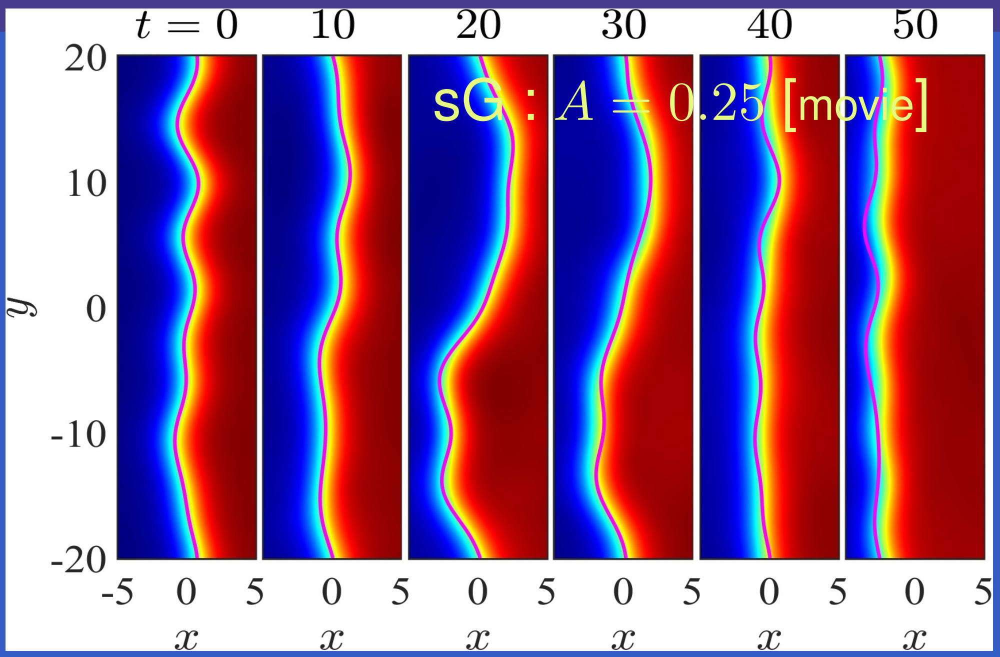
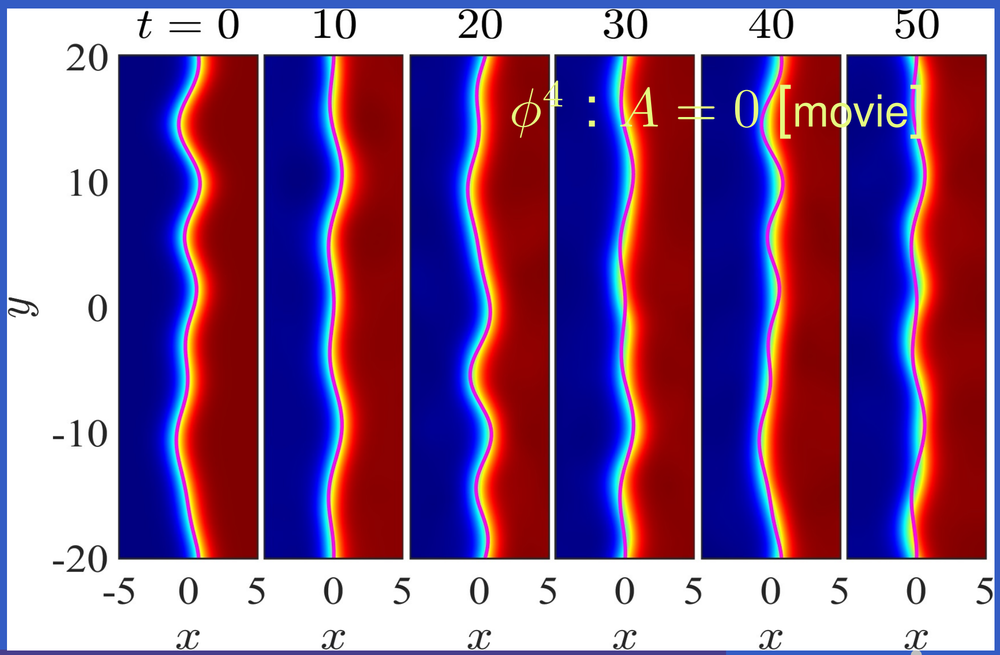
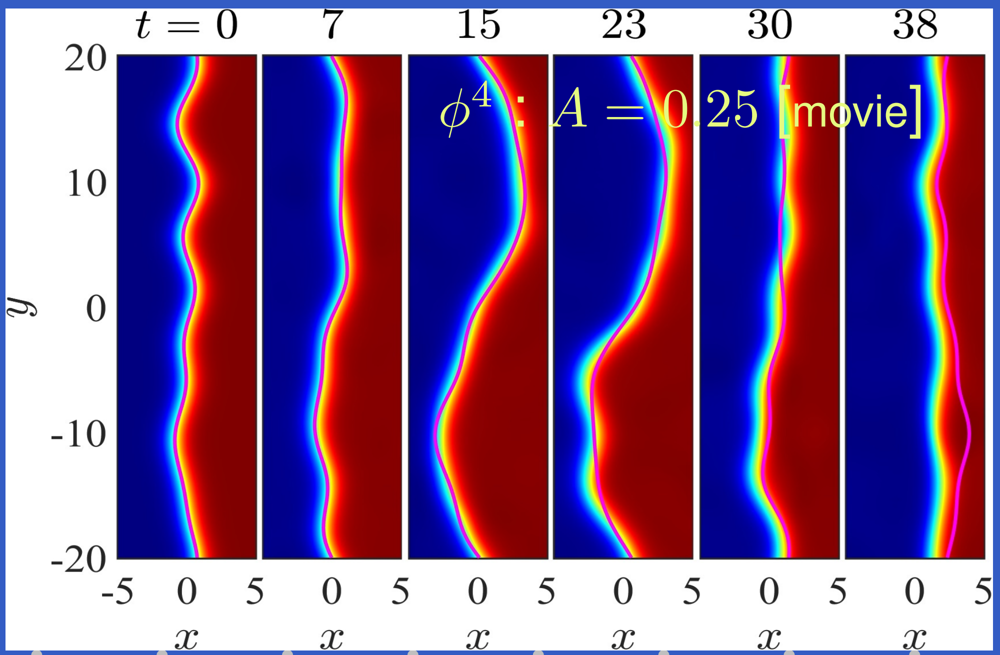


Kink dynamics: PDE vs. AI









Filament-filament interactions

AI : Interacting DS stripes in NLS

- 1 DS: $u = e^{-i\mu t} \left[\sqrt{\mu - v^2} \tanh\left(\sqrt{\mu - v^2}(x - x_0) \right) + iv \right]$,
- 1 DS Hamiltonian: $H = \frac{4}{3} \int_{-\infty}^{\infty} B\, A^{3/2}\, dy$,
  with $A \equiv \mu - V(x_0) - x_{0t}^2$ and $B \equiv 1 + \frac{1}{2} x_{0y}^2$.

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- Introduce DS-DS interaction using perturbation theory ($\propto e^{-|x_2 - x_1|}$).
  Symmetric interaction: $x_1 = x_0 = -x_2$:

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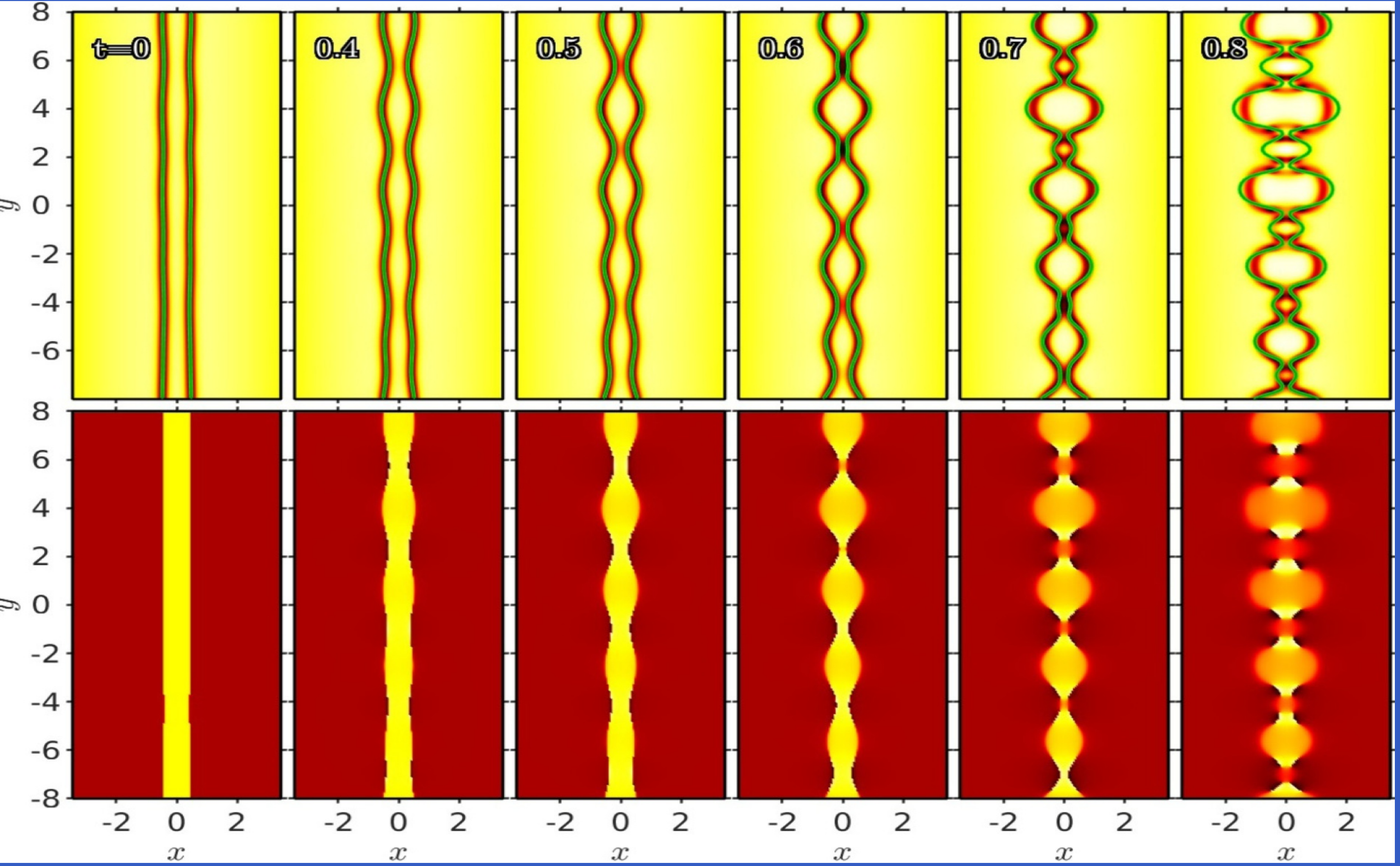
- Reduced PDE for x_0 (half DS separation) →

$$\boxed{\begin{aligned} B\left(x_{0tt} + \frac{V'(x_0)}{2}\right) + \frac{A}{3}x_{0yy} &= \frac{V'(x_0)}{2}{x_0}_y^2 + x_{0y}x_{0t}x_{0ty} \\ &\quad - \left[(V'(x_0) + 2x_{0tt})(-3 + 4A^{1/2}x_0) - 8A^{3/2}\right] e^{-4A^{1/2}x_0} \end{aligned}}$$

Interacting DS stripes in NLS: PDE vs reduction



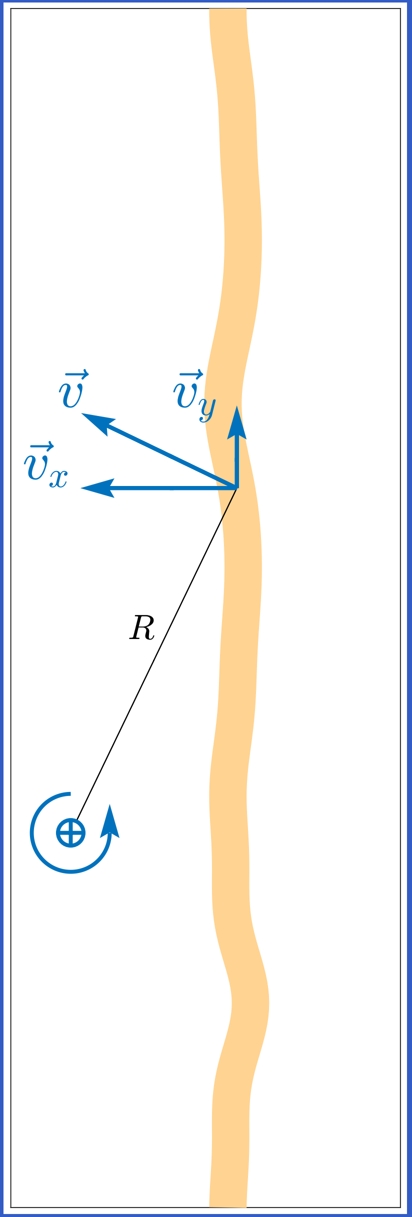
[movie]

Filament-vortex interactions

DSS-vortex interactions in NLS: Multiple scales

(A) VORTEX $\longrightarrow$ FILAMENT INTERACTION (PERT.)

- DSS: $u_{\rm dss} = B\tanh\left[B(x - X(y,t))\right] + iv_0,$
  width: B, const. horizontal vel.: v_0, with $v_0^2 + B^2 = \mu$.
- Point vortex: $u_{\rm v} = \sqrt{\mu}\, e^{iS\theta}$, $\theta = \tan^{-1}[(y - y_1), (x - x_1)]$.



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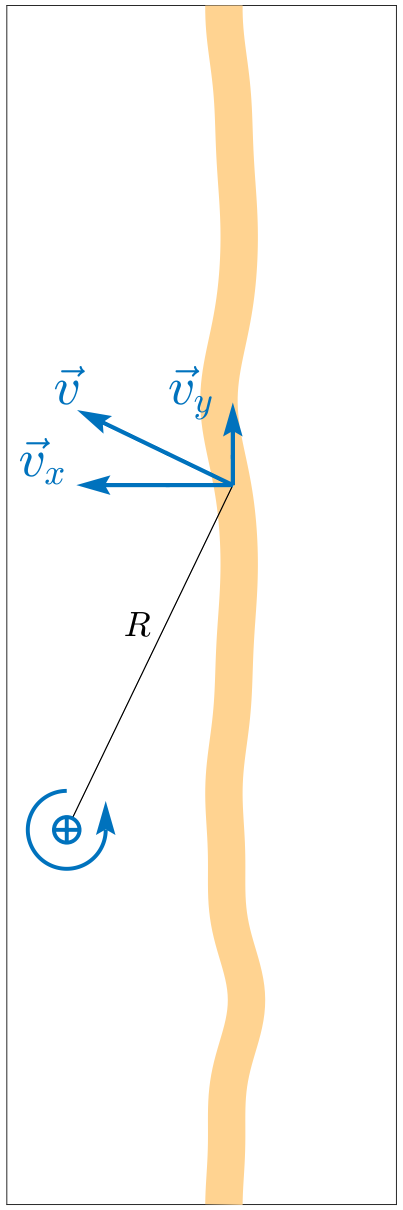
- Point vortex: $u_{\rm v} = \sqrt{\mu}\, e^{iS\theta}$, $\theta = \tan^{-1}[(y-y_1),(x-x_1)]$.
- Perturbation theory (multiple scales; only horizontal) [extension of NNK PRL 4 (2001) 043901]
  $X(y,t) = x_0(t) - h(y,t) = \xi_0 + v_0\, t - h(y,t)$.

$$\left\{ \begin{array}{rcl} h_t & = & \mathcal{V} + v^{(x)}, \\ \mathcal{V}_t & = & -\dfrac{\mu - v_0^2}{3}\, h_{yy}, \end{array} \right.$$

where

$$v^{(x)} = -S\,\frac{y-y_1}{R^2}.$$

- Note: vortex → filament interaction NOT included.



DSS-vortex interactions in NLS: Variational approach

(B) VORTEX $\leftrightarrow$ FILAMENT INTERACTION (VA)

- DSS: $u_{\rm dss} = B(y,t)\tanh\left[B(y,t)(x - X(y,t))\right] + i\mathcal{V}(y,t)$, width: $B(y,t)$, horizontal vel.: $\mathcal{V}(y,t)$, with $\mathcal{V}^2 + B^2 = \mu$.
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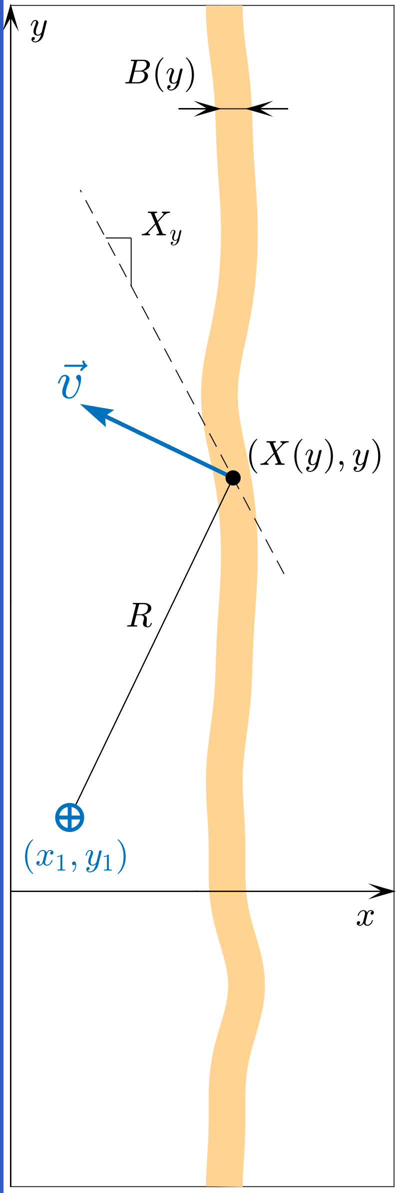

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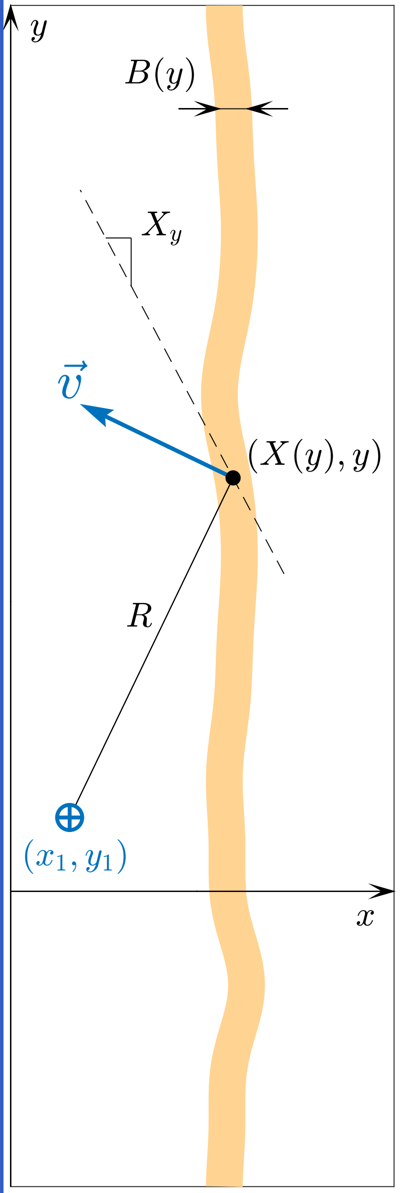
- Point vortex: $u_{\rm v} = \sqrt{\mu}\, e^{iS\theta}$, $\theta = \tan^{-1}[(y-y_1),(x-x_1)]$.
- VA (includes vortex → filament):

$$\left\{\begin{array}{rcl} X_t &=& \mathrm{PDE}_{\,\text{single DSS}}[X,\mathcal{V}] + v^{(x)} + v^{(y)}, \\ \mathcal{V}_t &=& \mathrm{PDE}_{\,\text{single DSS}}[X,\mathcal{V}], \end{array}\right.$$

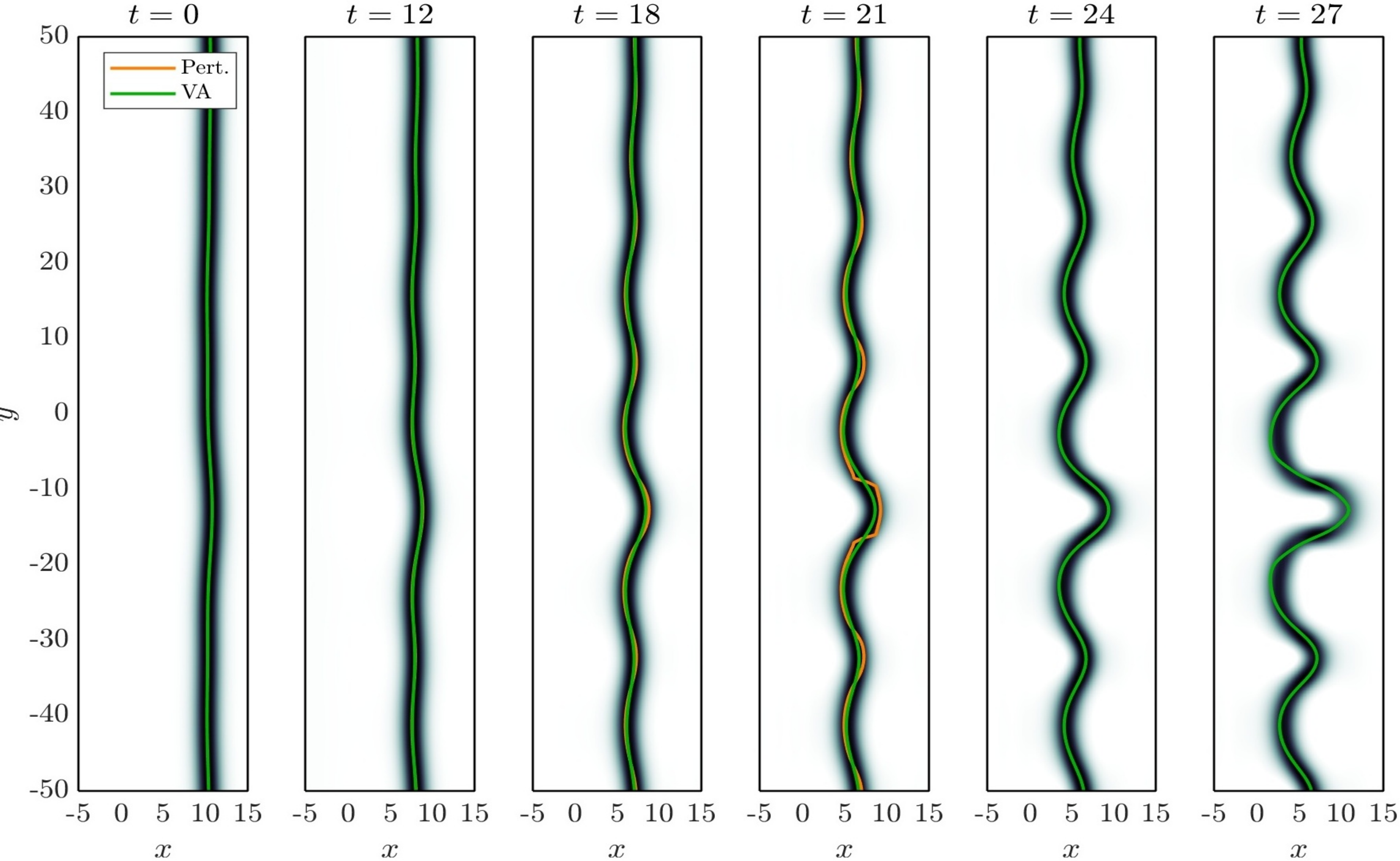
where now

$$v^{(x)} = -S\,\frac{y-y_1}{R^2}, \qquad v^{(y)} = -S\,\frac{(x-x_1)\,X_y}{R^2}.$$

- Includes contribution from filament slope X_y !!!). :)))

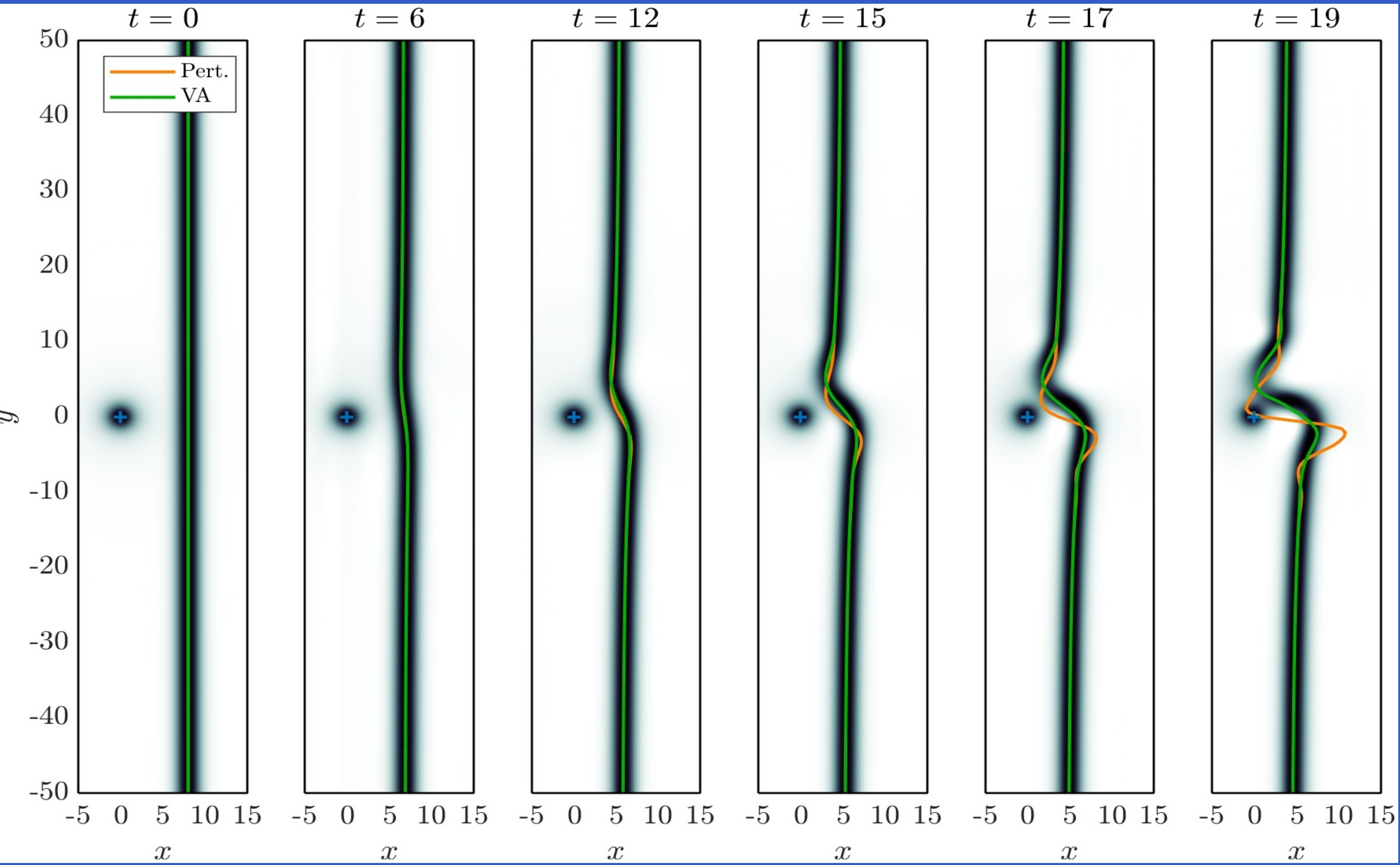


DSS snaking: Pert. theory vs. Variational approach



[movie]

vortex → DSS interactions: Pert. vs. VA



[movie]

DSS-vortex interactions in NLS: Variational approach

(C) VORTEX ↔ FILAMENT & VORTEX ↔ VORTEX

- Same methodology vortex ↔ filament and add vortex ↔ vortex
- Consider point vortices (NLS → inviscid Euler + quantum pressure)

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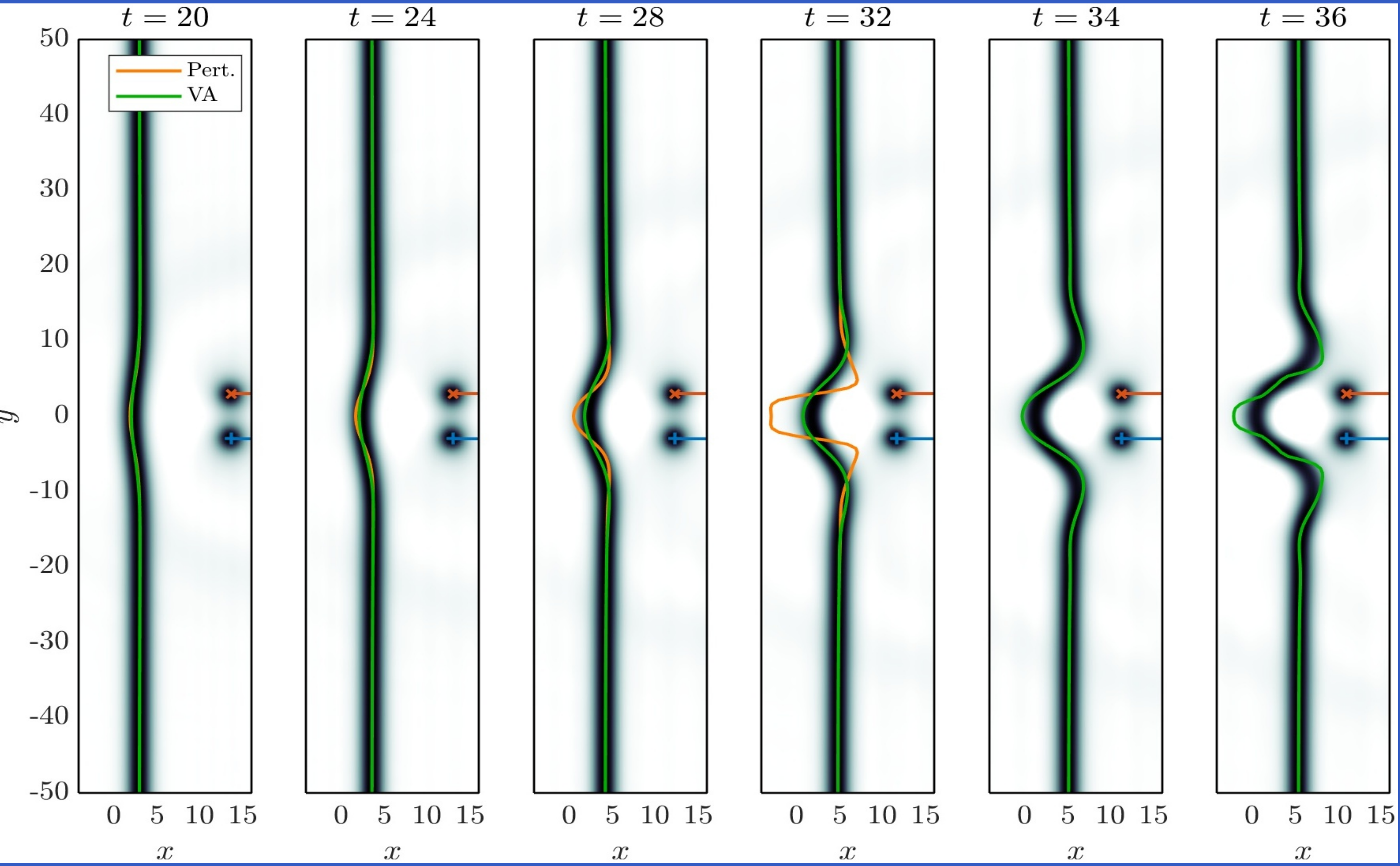
- Superposition of N vortices:

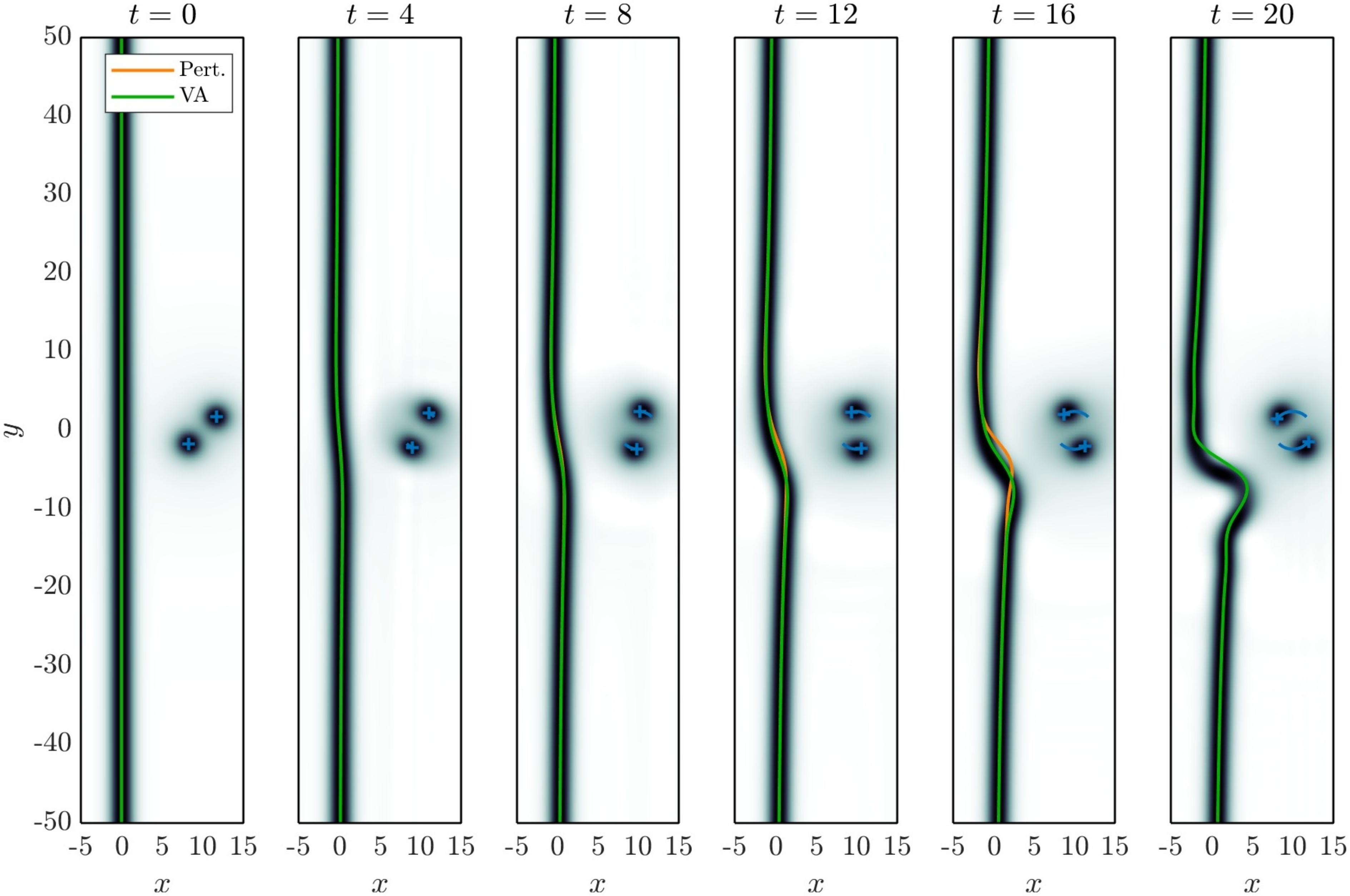
$$\begin{aligned} \dot{x}_m &= -\sum_{n\neq m}^{N} S_m \frac{y_m - y_n}{R_{mn}^2} \\ \dot{y}_m &= +\sum_{n\neq m}^{N} S_m \frac{x_m - x_n}{R_{mn}^2} \end{aligned}$$

vortex → DSS + vortex ↔ vortex: vortex dipole



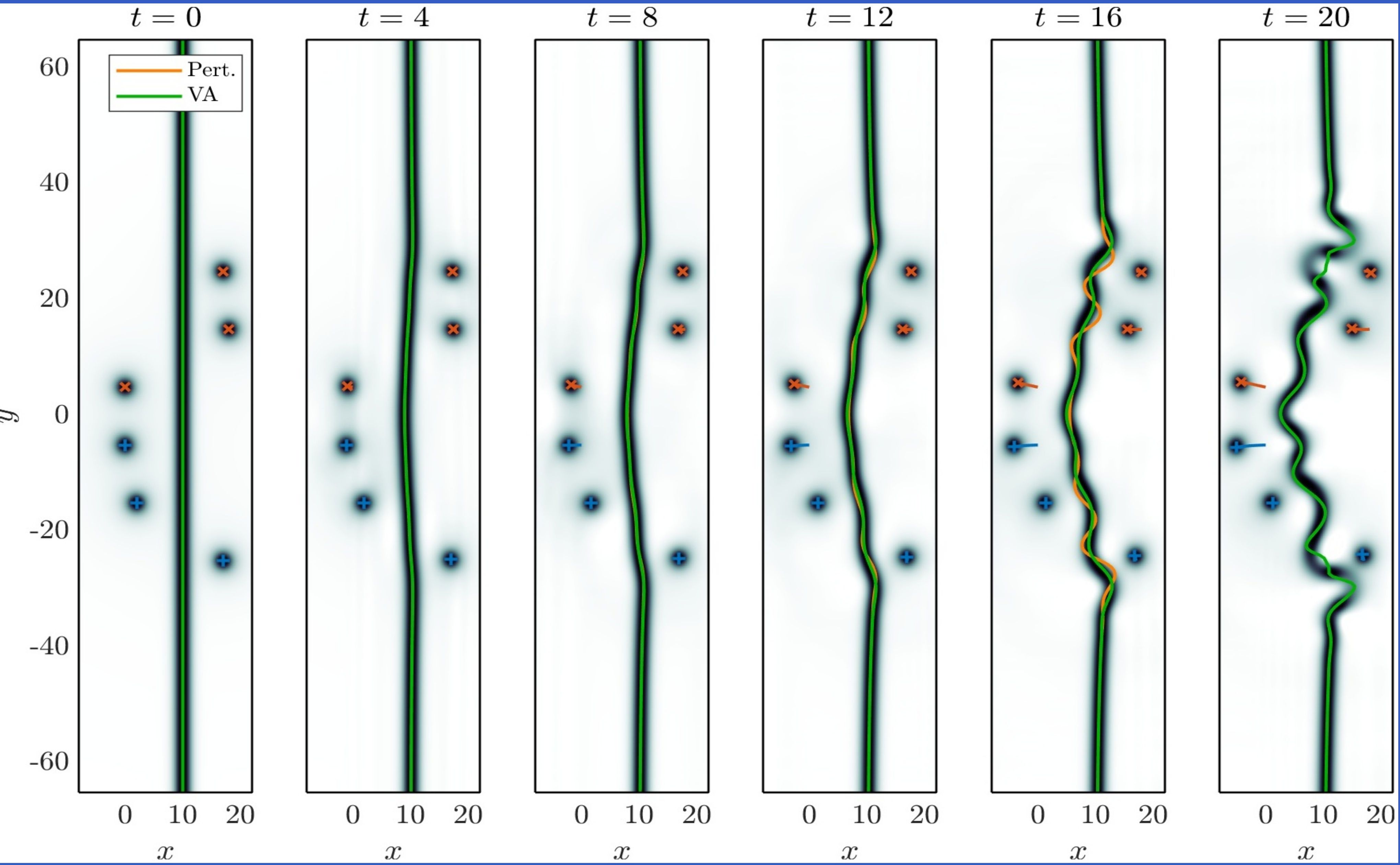
[movie]

vortex → DSS + vortex ↔ vortex: corotating vortices



[movie]

vortex $\to$ DSS + vortex $\leftrightarrow$ vortex: vortex soup



[movie]

Recap (inc. “not shown here”) and Outlook

- Solitonic filament dynamics (inc. *nonzero* external potentials):
  - Dark soliton stripes (NLS; AI)
  - Ring dark solitons (NLS; AI)
  - Spherical shells in 3D (NLS; AI)
  - Dark-Bright solitons (NLS; AI)
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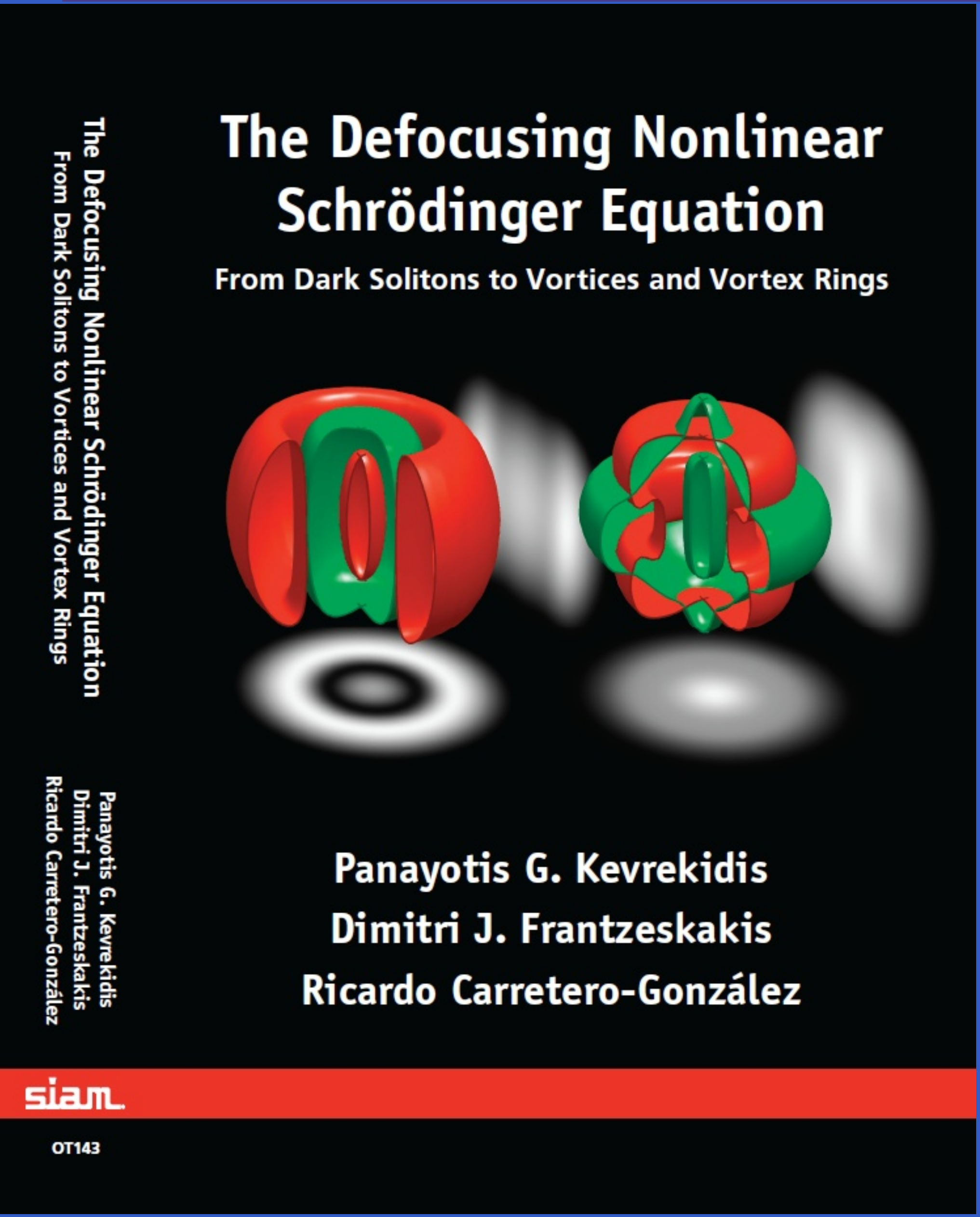
- Current Work and Outlook
  - Asymmetric filament-filament interactions ($x_1 \neq -x_2$)
  - Diff. Geometry → filaments on arclength/curvature/normal vel.
  - Transition between dark solitons and vortices
  - Include gain/loss → polariton BECs

Details of these filament reductions:

- Kink-antikink interactions in the one- and two-dimensional sine-Gordon equation. Comm. Nonlinear Sci. Numer. Simulat. **109** (2022) 106123.
- Pairwise Interactions of Ring Dark Solitons with Vortices and other Rings: Stationary States, Stability Features and Nonlinear Dynamics. Phys. Rev. A **104** (2021) 023314
- Breather stripes and radial breathers of the two-dimensional sine-Gordon equation. Comm. Nonlinear Sci. Numer. Simulat. **94** (2021) 105596.
- Reduced dynamics for one and two dark soliton stripes in the defocusing nonlinear Schrödinger equation: a variational approach. Phys. Rev. Research **1** (2019) 033043.
- Dynamics of interacting dark soliton stripes. Phys. Rev. A **100** (2019) 033607.
- Dynamics and stabilization of bright soliton stripes in the hyperbolic-dispersion nonlinear Schrödinger equation. Comm. Nonlinear Sci. Numer. Simulat. **74** (2019) 268-281.
- Planar and Radial Kinks in Nonlinear Klein-Gordon Models: Existence, Stability and Dynamics. Phys. Rev. E **98** (2018) 052217.
- Adiabatic invariant analysis of dark and dark-bright soliton stripes in two-dimensional Bose-Einstein condensates. Phys. Rev. A **97** (2018) 063604.
- Adiabatic Invariant Approach to Transverse Instability: Landau Dynamics of Soliton Filaments. Phys. Rev. Lett. **118** (2017) 244101.

SIAM, 2015:



The Defocusing Nonlinear Schrödinger Equation

From Dark Solitons to Vortices and Vortex Rings

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siam

OT143

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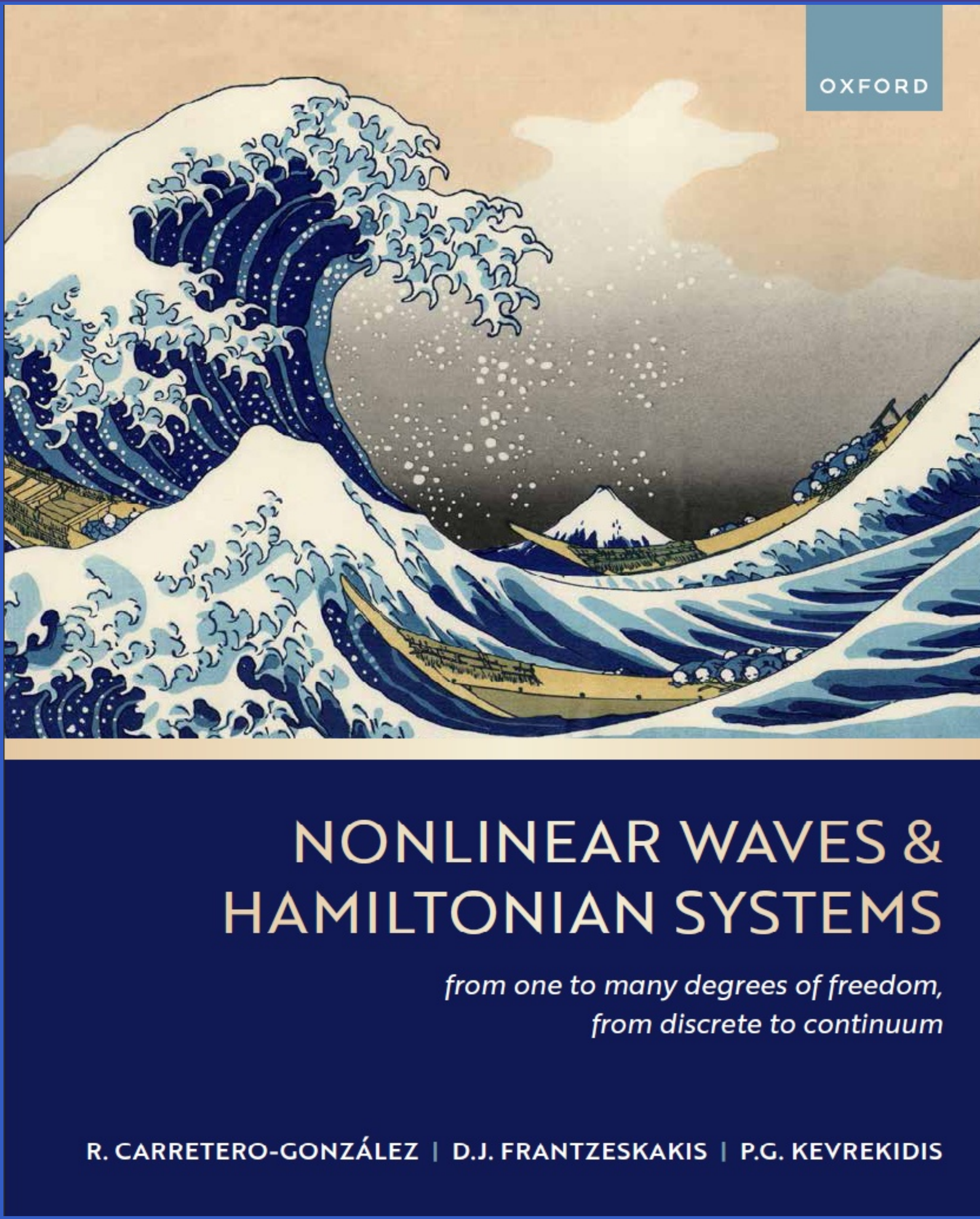
OUP, 2025:



OXFORD

NONLINEAR WAVES & HAMILTONIAN SYSTEMS

from one to many degrees of freedom,
from discrete to continuum

R. CARRETERO-GONZÁLEZ | D.J. FRANTZESKAKIS | P.G. KEVREKIDIS

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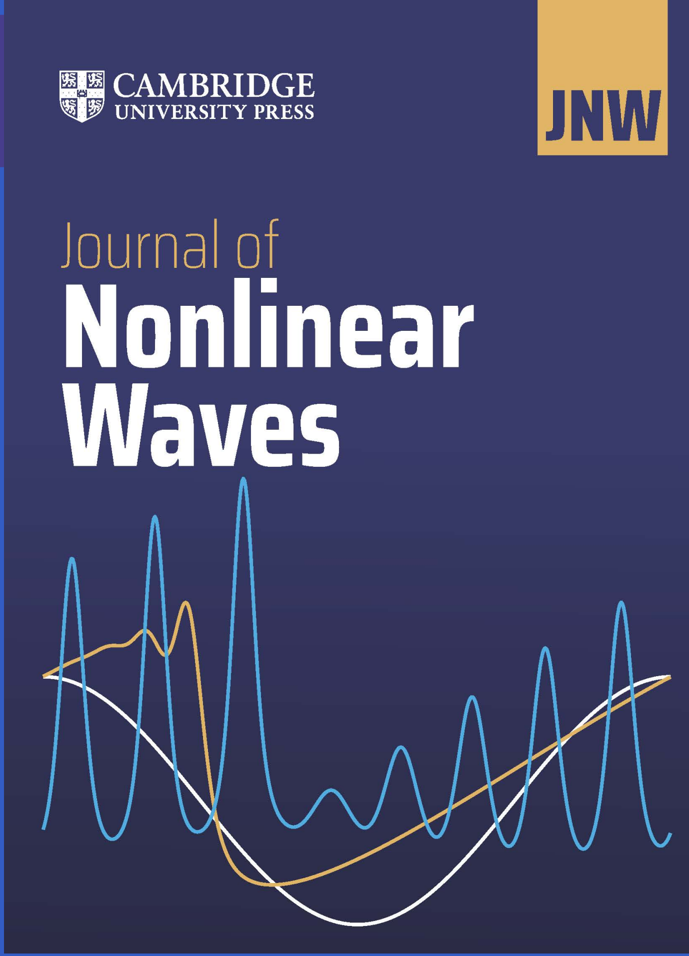

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  - MATH-636 : Mathematical Modeling
  - MATH-693A : Numerical Optimization
- Spring Year 1:
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  - MATH-638 : Dynamical Systems & Chaos II
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  - MATH-635 : Pattern Formation
  - MATH-639 : Nonlinear Waves
  - MATH-797 : Research
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