# Topological solitons and their role in the long-term dynamics of classical field theories

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# Overview

Three classical field theories from *Manton–Sutcliffe, Topological Solitons, Cambridge 2004*:

1. **Scalar fields on the line** — (sine-Gordon, $\phi^4$) kinks and anti-kinks

   topological invariant: charge

2. **Wave maps** (energy critical) — harmonic maps from surfaces

   topological invariant: degree

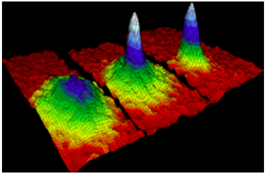
3. **Ginzburg-Landau / Yang-Mills-Higgs** — vortices in the plane

   topological invariant: degree

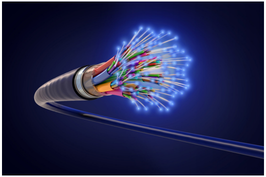
For each theory:

- Identify the *soliton* (stationary solution) and its moduli space
- Study *long-term dynamics*: stability, scattering, bubbling
- Tools: elliptic PDE, spectral theory, distorted Fourier transform, dispersive estimates, harmonic analysis

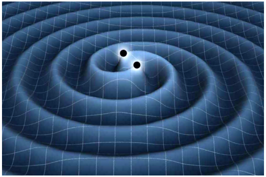
# Solitons and dispersive waves in nature



Bose-Einstein condensate
(NLS / Gross-Pitaevskii)



Fiber optic communication
(NLS envelope solitons)



Gravitational waves
(wave maps, Yang-Mills)

A common mathematical framework: *Hamiltonian PDEs* admitting localized, stable solutions — the **solitons**.

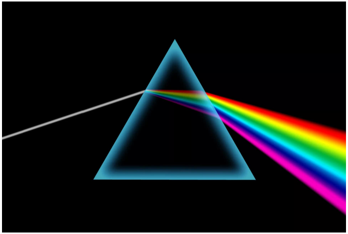
**Central question:** What is the long-term fate of a general solution?

# Dispersion

A linear PDE $i\partial_t u = \omega(-i\partial_x)u$ is *dispersive* if the plane wave $e^{i(kx-\omega(k)t)}$ satisfies a *nonlinear* dispersion relation $\omega = \omega(k)$.

Different frequencies travel at different group velocities $\omega'(k) \Rightarrow$ waves spread out, pointwise amplitudes decay.

**Klein-Gordon:** $\omega(k)^2 = k^2 + m^2$, group velocity $\omega'(k) = k/\omega < 1$.



Dispersion of white light: different frequencies, different speeds

**Free decay rates** ($d$ spatial dimensions): decay determined by Hessian $D^2\omega(k)$

$$\|e^{it\sqrt{1-\Delta}}f\|_{L^\infty} \lesssim t^{-d/2}\|\langle D\rangle^s f\|_{L^1}, \quad s > s(d)$$

**In 1D:** only $t^{-1/2}$ — this slow rate is a key difficulty.

# Common techniques

**Elliptic PDE**

- Variational structure, Pohozaev
- Moduli spaces of solitons
- Coercivity and Lyapunov functionals

**Spectral theory**

- asymptotic completeness, Mourre theory
- Agmon-Kato-Kuroda
- Weyl-Titchmarsh-Kodaira $m$-function
- Distorted Fourier transform
- Threshold resonances

**Harmonic analysis**

- Oscillatory integrals
- Paradifferential calculus
- Dispersive decay, Strichartz estimates

**Nonlinear dispersive PDE**

- Space-time resonances (Germain-Masmoudi-Shatah)
- Normal forms (Shatah)
- Vector fields: $x - 2it\nabla$, $t\nabla + x\partial_t$
- Fermi Golden Rule (Soffer, Weinstein)
- Concentration compactness
- Modulation theory

# Scalar fields

**Scalar field on the line** $\phi : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$:

$$\mathcal{L}[\phi, \phi_t] = \int_{\mathbb{R}^{1+1}} \Big( \tfrac{1}{2}(\partial_t \phi)^2 - \tfrac{1}{2}(\partial_x \phi)^2 - W(\phi) \Big) \, \mathrm{d}x \, \mathrm{d}t$$

Euler-Lagrange equation: $(\partial_t^2 - \partial_x^2)\phi + W'(\phi) = 0$, Hamiltonian

$$\mathcal{H}[\phi] = \int_{\mathbb{R}} \Big( \tfrac{1}{2}(\partial_t \phi)^2 + \tfrac{1}{2}(\partial_x \phi)^2 + W(\phi) \Big) \, \mathrm{d}x$$

Examples of self-interaction potentials:

$$W(\phi) = \begin{cases} 1 - \cos\phi & \text{(sine-Gordon equation)} \\ \frac{1}{4}(1-\phi^2)^2 & (\phi^4 \text{ model}) \\ \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 & \text{(cubic nonlinear Klein-Gordon)} \end{cases}$$

Topological solitons are kinks in the first two cases, for $NLKG_3$ it is a soliton

# Wave map

$u : \mathbb{R}_t \times \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$

Lagrangian $$\mathcal{L}[u] = \frac{1}{2} \int_{\mathbb{R}^{2+1}} (|\partial_t u|^2 - |\nabla u|^2) \, \mathrm{d}x \, \mathrm{d}t, \quad |u| = 1$$

PDE $$u_{tt} - \Delta u = A(u)(\partial_\alpha u, \partial^\alpha u) = u(-|u_t|^2 + |\nabla u|^2)$$

Hamiltonian $$\mathcal{H}[\phi] = \int_{\mathbb{R}^2} (|\partial_t u|^2 + |\nabla u|^2) \, \mathrm{d}x$$

Topological solitons are harmonic maps.

# Abelian Yang-Mills-Higgs

Vector potential: $A = A_0\,dt + A_1\,dx_1 + A_2\,dx_2$.

Gauge symmetry

$$(\phi, A) \;\mapsto\; \left(e^{i\chi}\phi,\; A + d\chi\right), \qquad \chi \in C^\infty(\mathbb{R}^{1+2})$$

Curvature: $dA = E_1\,dt \wedge dx_1 + E_2\,dt \wedge dx_2 + B\,dx_1 \wedge dx_2$ with magnetic field $B = \partial_1 A_2 - \partial_2 A_1$.

$$\mathcal{L}[u] = \frac{1}{2}\int_{\mathbb{R}^{2+1}} \left(-\,|E|^2 + B^2 - |D_0\phi|^2 + |D_1\phi|^2 + |D_2\phi|^2 + \frac{\lambda}{4}(1 - |\phi|^2)^2\right) \mathrm{d}x\,\mathrm{d}t,$$

Euler-Lagrange equations, with gauge-covariant derivatives $D_j = \partial_j - iA_j$:

$$-D_0^2\phi + D^j D_j\phi + \frac{\lambda}{2}\left(1 - |\phi|^2\right)\phi = 0, \quad \text{nonlinear Klein-Gordon}$$

$$\partial_0 E_1 + \partial_2 B - \operatorname{Im}\left(\overline{\phi}\,D_1\phi\right) = 0, \quad \text{Maxwell}$$

$$\partial_0 E_2 - \partial_1 B - \operatorname{Im}\left(\overline{\phi}\,D_2\phi\right) = 0, \quad \text{Maxwell}$$

$$\partial_1 E_1 + \partial_2 E_2 - \operatorname{Im}\left(\overline{\phi}\,D_0\phi\right) = 0. \quad \text{Gauss Law}$$

# Conservative and dissipative time evolutions

Each stationary energy functional $\mathcal{E}$ generates several natural dynamics:

| **Flow** | **Equation** | **Symmetry** |
|---|---|---|
| *Gradient / heat flow* | $\partial_t u = -\nabla\mathcal{E}(u)$ | dissipative |
| *Schrödinger flow* | $i\partial_t u = \nabla\mathcal{E}(u)$ | Galilei |
| *Wave / KG flow* | $\partial_t^2 u = -\nabla\mathcal{E}(u)$ | Lorentz |

- **Heat flow**: energy decreasing, selects canonical representatives in moduli space
- **Schrödinger flow**: conserves $\mathcal{E}$ and charge $\|u\|_{L^2}^2$; relevant for BEC, optics
- **Wave/KG flow**: conserves $\mathcal{E}$; relevant for relativistic field theory

In all three cases: *solitons* are stationary solutions, and the **moduli space** parametrizes their families.

# Part I

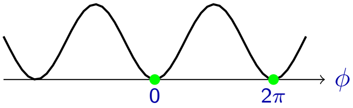
## Scalar fields and kinks

sine-Gordon, $\phi^4$, and cubic Klein-Gordon, asymptotic kink stability

# Two scalar field theories on the line

sine-Gordon equation (sG)

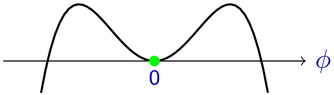
$$(\partial_t^2 - \partial_x^2)\phi + \sin\phi = 0$$




$$W(\phi) = 1 - \cos\phi$$

cubic KG equation ($KG_3$)

$$(\partial_t^2 - \partial_x^2 + 1)\phi - \phi^3 = 0$$



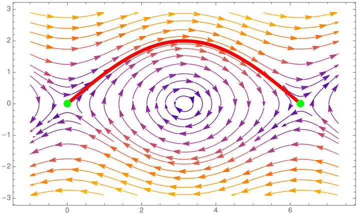

$$W(\phi) = \tfrac{1}{2}\phi^2 - \tfrac{1}{4}\phi^4$$

The *potential wells* determine the vacuum structure. Kinks interpolate between adjacent vacua (sG: $0 \to 2\pi$; $KG_3$: $0 \to 0$).

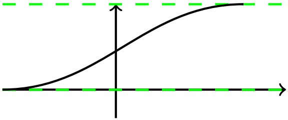
# Two static solitons

## sine-Gordon kink

$$\begin{cases} -\partial_x^2 K + \sin K = 0 \\ \lim_{x\to\pm\infty} K(x) = \pi \pm \pi \end{cases}$$

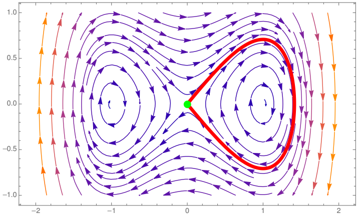



$$K(x) = 4\arctan(e^x)$$

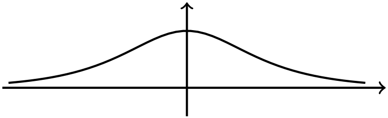


## cubic KG soliton (KG$_3$)

$$\begin{cases} -\partial_x^2 Q + Q - Q^3 = 0 \\ \lim_{x\to\pm\infty} Q(x) = 0 \end{cases}$$




$$Q(x) = \sqrt{2}\,\mathrm{sech}(x)$$



# Kinks, antikinks, and the moduli space

**Topological charge (sine-Gordon).**

$Q[\phi] = \frac{1}{2\pi} \int_{\mathbb{R}} \partial_x \phi \, \mathrm{d}x \in \mathbb{Z}$. Kink $K$: $+1$. Antikink $\bar{K}(x) = K(-x)$: $-1$.

**Lorentz invariance and moduli space.**

Boosted kink $K_{x_0,v}(x,t) = K\left(\frac{x-x_0-vt}{\sqrt{1-v^2}}\right)$, $|v| < 1$.

**Complete integrability (sine-Gordon).**

- Ablowitz–Kaup–Newell–Segur '73: Inverse Scattering
- Chen–Liu–Lu '20: soliton resolution via $\bar{\partial}$ and Riemann–Hilbert
- Multi-kink/antikink solutions explicit.
- Breathers: $\phi_b(x,t) = 4 \arctan\left(\frac{\sin(\omega t)}{\omega \cosh(x\sqrt{1-\omega^2})}\right)$.
- Wobbling kinks (soliton + breather): *not* asymptotically stable in energy space (Alejo–Muñoz–Palacios '20).

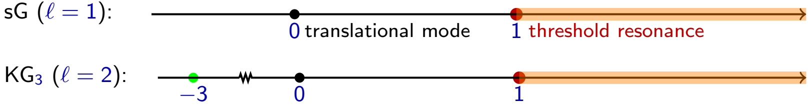
$\phi^4$ **model.** $W(\phi) = \frac{1}{4}(1-\phi^2)^2$; kink $K(x) = \tanh(x/\sqrt{2})$; rich kink–antikink collision dynamics, no exact integrability. Multi-kink dynamics: Jendrej, Lawrie '23, '24

# Asymptotic stability of solitons — setup

Write $\phi(t,x) = K(x) + u(t,x)$ (or $Q(x) + u$). Then $u$ solves:

$$(\partial_t^2 - \partial_x^2 + V(x) + 1)\, u = \alpha(x) u^2 + \beta_0 u^3$$

with *Pöschl-Teller potential* $V(x) = -\ell(\ell+1)\mathrm{sech}^2(x)$.




**Goal:** prove $\|u(t)\|_{L^\infty_x} \to 0$ as $t \to \infty$ for small initial data.

**Orbital stability** of kinks follows from coercivity of the energy modulo translations: the latter imposes orthogonality to the root mode. Henry, Perez, Wreszinksi '82 used energy as Lyapunov functional to prove orbital stability

**Asymptotic stability** is highly dependent on the topology, gives precise information: modulation equations on the finite-dimensional moduli space plus radiation (can be large in energy).

# Main difficulties for 1D nonlinear Klein-Gordon

The analysis of $(\partial_t^2 - \partial_x^2 + V + 1)u = \alpha(x)u^2 + \beta_0 u^3$ involves:

- **Slow dispersive decay:** $t^{-1/2}$ in $L^\infty$ in 1D — borderline for nonlinear analysis
- **Modified scattering** due to low-power nonlinearities:
  - $\beta_0 u^3 \rightsquigarrow$ logarithmic phase corrections
  - $\alpha(x)u^2 \rightsquigarrow$ amplitude corrections (potential resonances)
- **Non-trivial spectrum** of linearized operator: negative eigenvalues, threshold resonances, internal modes, zero modes
- **Distorted Fourier transform** (dFT) needed to diagonalize the operator; threshold resonance creates discontinuity at zero frequency
- **Resonant frequencies:** $\tilde{\mathcal{F}}[\alpha\varphi^2](\pm\sqrt{3})$ — vanishes for sine-Gordon (null structure), nonzero for $\mathrm{NLKG}_3$ (resonant source)

# Scattering matrix, threshold resonance

$H = -\partial_{xx} + V$, $\langle x \rangle V \in (L^1 \cap L^\infty)(\mathbb{R})$, $V$ real-valued.

**Jost solutions** $Hf_\pm = \xi f_\pm$ with $f_\pm(x,\xi) \sim e^{\pm ix\xi}$ as $x \to \pm\infty$.

**Wronskian** $W(\xi) = W[f_+(\cdot,\xi), f_-(\cdot,\xi)] \neq 0$ if $\xi \neq 0$.

**Reflection, transmission coefficients** $R_\pm$, $T$

$$f_+(\cdot,-\xi) + R_+(\xi)f_+(\cdot,\xi) = T(\xi)f_-(x,\xi),$$
$$f_-(\cdot,-\xi) + R_-(\xi)f_-(\cdot,\xi) = T(\xi)f_+(x,\xi)$$

The following are equivalent:

- $0$ is a **resonance**
- $W(0) = 0$
- $T(0) \neq 0$
- $\exists\, \varphi \neq 0$, bounded with $H\varphi = 0$ (then automatically $\varphi \notin L^2$)
- $(H - z^2)^{-1} = -z^{-1}\varphi \otimes \varphi + O(1)$ as $z \to 0$ (**small divisor**)

# Distorted Fourier transform

The distorted Fourier basis associated with $H$

$$e(x,\xi) := \frac{1}{\sqrt{2\pi}} \begin{cases} T(\xi) f_+(x,\xi), & \xi \geq 0, \\ T(-\xi) f_-(x,-\xi), & \xi < 0. \end{cases}$$

$\tilde{g}(\xi) = \langle g, e(\cdot,\xi)\rangle$, isometry on $L^2_{\mathrm{cont}}(\mathbb{R})$,

$$\int e(x,\xi) \otimes \overline{e(y,\xi)}\, d\xi = \delta_0(x-y)$$

With $\langle D\rangle P_{(0,\infty)}(H) = \sqrt{1+H} P_{(0,\infty)}(H)$ and all $t \geq 1$

$$\Big\| \langle x\rangle^{-4} \Big( e^{it\langle D\rangle} \chi_0(H) P_c\, g - c_0 \frac{e^{i\frac{\pi}{4}} e^{it}}{t^{\frac{1}{2}}} (\varphi \otimes \varphi) g \Big) \Big\|_{L^\infty_x} \leq \frac{C}{t^{\frac{3}{2}}} \| \langle x\rangle^4 g\|_{L^1_x}$$

$\chi_0(H)$ a small energy cut-off. Stationary phase argument, using the resolvent rather than distorted Fourier transform (Stone formula). Tensor structure present only at $0$ energy.

# Normal forms

For the model problem

$$(\partial_t^2 - \partial_x^2 + V(x) + m^2)u = \alpha(x)u^2 + \beta_0 u^3$$

Goal: Prove decay to zero of $u(t)$ for small initial data (ideally with sharp decay rates and asymptotics).

Remove the catastrophic quadratic term via **normal forms**, first for $V = 0$. Define a new dependent variable $v := u + \gamma(x)u^2$. Then

$$v_{tt} - v_{xx} + v = (\alpha - \gamma - \gamma_{xx})v^2 + 2\gamma(v_t^2 - v_x^2) - 4\gamma_x v v_x + O(v^3)$$

We expect better local decay for $v_x$ via linear estimate

$$\|\langle x\rangle^{-3}\partial_x e^{it\omega} f\|_2 \le C\langle t\rangle^{-\frac{3}{2}}\|\langle x\rangle^3 f\|_2$$

moreover, by asymptotics of free flow:

$$v_t(t,0) = \pm i v(t,0) + O(t^{-\frac{3}{2}})$$

So heuristically at least we arrive at **division problem**

$$\alpha - 3\gamma - \gamma_{xx} = 0, \qquad \hat{\alpha}(\xi) = (3 - \xi^2)\hat{\gamma}(\xi)$$

# Null structure and asymptotic stability of kinks

*sine-Gordon (sG)*

$$\tilde{\mathcal{F}}[\alpha\varphi^2](\xi) \sim \frac{(\xi^2-3)(\xi^2+1)}{\xi^2+i}\operatorname{sech}\Big(\frac{\pi\xi}{2}\Big)$$

$$\boxed{\tilde{\mathcal{F}}[\alpha\varphi^2](\pm\sqrt{3}) = 0}$$

(Lindblad-Lührmann-S-Soffer '20)

**Theorem (Lührmann-S '21)**

*Small odd perturbations of the sine-Gordon kink $K$ satisfy*

$$\|\phi(t) - K\|_{L^\infty} \leq \frac{C\varepsilon}{(1+t)^{1/2}}$$

*with modified scattering (log phase corrections).*

*cubic $KG_3$*

$$\boxed{\tilde{\mathcal{F}}[\alpha\varphi^2](\pm\sqrt{3}) \neq 0}$$

(Lührmann-S '23)

**Theorem (Lührmann-S '23)**

*Small even perturbations on the center-stable manifold satisfy*

$$\|\phi(t) - Q\|_{L^\infty} \leq \frac{C\varepsilon\log(2+t)}{(1+t)^{1/2}}$$

*for $0 \leq t \leq \exp(c\varepsilon^{-1/3})$.*

Alternative proofs for sine-Gordon: Chen-Liu-Lu '20 (Riemann-Hilbert), Koch-Yu '23 (Bäcklund), Chen-Lührmann '24 (no symmetry).

# sine-Gordon kink asymp. stable under odd perturbations

**Theorem (Lührmann-S. '21)**

*$\exists\, \varepsilon_0 \in (0,1)$ so that data $(K + u_0, u_1)$ with $\|\langle x\rangle (u_0, u_1)\|_{H^3_x \times H^2_x} = \varepsilon \le \varepsilon_0$ and odd lead to sine-Gordon field $\phi$ where $u(t,x) := \phi(t,x) - K(x)$ decays:*

$$\|u(t,\cdot)\|_{L^\infty_x} \lesssim \varepsilon (1+t)^{-\frac{1}{2}}, \quad t \ge 0.$$

*Moreover, $\exists\, \widehat{W} \in L^\infty$ and $0 < \delta \ll 1$ s.t. for $t \ge 1$*

$$\left| u(t,x) + 2\,\mathrm{Re}\left( \frac{e^{i\frac{\pi}{4}}}{t^{\frac{1}{2}}} \int_0^x \frac{\cosh(y)}{\cosh(x)} e^{i\rho} e^{-i\psi(\frac{y}{\rho})\log(t)} \widehat{W}\big(\tfrac{y}{\rho}\big) \mathbb{1}_{(-1,1)}\big(\tfrac{y}{t}\big)\, \mathrm{d}y \right) \right| \lesssim \frac{\varepsilon}{t^{\frac{2}{3}-\delta}},$$

*where $\rho := \sqrt{t^2 - y^2}$ and $\psi(\xi) := \frac{1}{4}\langle \xi \rangle^{-7}(1+3\xi^2)\big|\widehat{W}(\xi)\big|^2$.*

- Asymptotic stability **false** in **local** $H^1 \times L^2$, even with a weighted condition $\langle x\rangle^s, 0 < s < \frac{1}{2}$: **wobbling kinks** are a counter example. Role of topology in asymptotic stability.
- $\phi^4$ does not have obstructions of this nature: local asymptotic stability of kink under odd perturbations with internal mode. Kowalczyk-Martel-Muñoz '17, '19. Radiation damping through Fermi Golden Rule, Soffer, Weinstein '99.

# Mini Bosons

Quote from Ionescu-Pausader, *"The Einstein–Klein–Gordon Coupled System Global Stability of the Minkowski Solution"*, Annals of Mathematics Studies Number 213, 2022:

> *A serious potential obstruction to small data global stability theorems is the presence of non-decaying "small" solutions, such as small solitons. A remarkable fact is that there are such small non-decaying solutions for the Einstein–Klein–Gordon system, namely the so-called mini-boson stars. These are time-periodic (therefore non-decaying) and spherically symmetric exact solutions of the Einstein–Klein–Gordon system.*
>
> *They were discovered numerically by physicists, such as Kaup [47], Friedberg–Lee–Pang [20] (see also [60]), and then constructed rigorously by Bizon–Wasserman [8]. These mini-bosons can be thought of as arbitrarily small (hence the name) in certain topologies, as explained in [8]. However, the mini-bosons (in particular the Klein–Gordon component) are not small in the stronger topology we use here, as described by (1.2.5), so we can thankfully avoid them in our analysis.*

# Part II

## Wave maps and the harmonic map heat flow

Energy critical dynamics in 2D, large data

# Harmonic map heat flow

Gradient flow of the Dirichlet energy

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u(x)|^2 \,\mathrm{d}x,$$
$$u : \mathbb{R}^2 \to \mathbb{S}^2$$

solves the heat equation (Eells, Sampson '64):

$$u_t = \Delta u + |\nabla u|^2 u = \mathcal{T}(u)$$
$$u(0,\cdot) = u_0(\cdot)$$

Tension: $\mathcal{T}(u) = \Pi_{T_u}\Delta u$ projection onto the tangent plane $T_u$
Energy monotone:

$$E(u(0)) - E(u(t)) = \int_0^t \|\partial_s u(s,\cdot)\|_2^2 \,\mathrm{d}s$$

Existence, regularity, energy concentration and singularities in finite time: (Struwe '85). Harmonic maps are stationary solutions to HMHF.

# Qing's bubbling theorem

Jie Qing '95 characterized singularity formation in Struwe's HMHF $\mathbb{R}^2 \to \mathbb{S}^2$ via a bubble decomposition along a carefully chosen sequence of times approaching a singular time $T_\ell$.

**Theorem (Qing '95)**

*Let $(x_0, T_0)$ be a singularity of a HMHF solution $u : [0,\infty) \times \mathbb{R}^2 \to \mathbb{S}^2$. Then there exist $t_n \to T_0-$ and harmonic spheres $\omega_k : \mathbb{R}^2 \to \mathbb{S}^2$ such that*

$$\lim_{t\to T_0-} E_R(u(t,\cdot), x_0) = E_R(u(T_0,\cdot), x_0) + \sum_{k=1}^{p} E(\omega_k),$$

$$u(t_n,\cdot) = u(T_0,\cdot) + \sum_{k=1}^{p} \Big(\omega_k\big(\tfrac{\cdot - a_n^k}{\lambda_n^k}\big) - \omega_k(\infty)\Big) + o_{W^{1,2}(B_R)}(1).$$

*for $R > 0$ small, with $\lambda_n^k \to 0$ and $a_n^k \to x_0$. Bubbles are* asymptotically orthogonal*.*

Proved via bubbling for a **Palais–Smale** sequence.

# Harmonic maps

**Theorem**

*$u : \mathbb{R}^2 \to \mathbb{S}^2$ weak non-constant solution of $\Delta u + u|\nabla u|^2 = 0$ of finite energy. Then $u : \mathbb{S}^2 \to \mathbb{S}^2$ smooth harmonic map (Hélein, Sacks-Uhlenbeck), nonzero degree. Conformal modulo orientation (Eells-Wood). Cauchy-Riemann system*

$$\partial_1 u \mp u \times \partial_2 u = 0 \Longleftrightarrow \partial_2 u \pm u \times \partial_1 u = 0$$

*holds, $u$ unique minimizer of energy in its homotopy class, $E(u) = 4\pi|\deg(u)|$. There exist $P, Q \in \mathbb{C}[z]$ without common linear factor satisfying*

$$\max(\deg(P), \deg(Q)) = |\deg(u)| \geq 1$$

*and such that $u = \frac{P}{Q}$ for $deg(u) > 0$, or $\bar{u} = \frac{P}{Q}$ for $\deg(u) < 0$.*

**Bogomolnyi identity:**

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_1 u - u \times \partial_2 u|^2 + \int_{\mathbb{R}^2} \partial_1 u \cdot u \times \partial_2 u$$

# Soliton resolution in the equivariant case

Consider $k$-equivariant maps $u : \mathbb{R}^2 \to \mathbb{S}^2$, i.e.,

$$u(t, re^{i\theta}) = (\sin\psi(t,r)\cos k\theta,\ \sin\psi(t,r)\sin k\theta,\ \cos\psi(t,r)).$$

Harmonic maps given by $\psi(t,r) = m\pi \pm Q(r/\lambda)$ for $m \in \mathbb{Z}$, $\lambda > 0$, and $Q(r) = 2\arctan(r^k)$.

**Theorem (Jendrej–Lawrie '22)**

*Let $\psi(t,r)$ solve the HMHF. Suppose $T = \infty$. Then, there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}$ and $C^1$ functions $0 < \lambda_1(t) < \cdots < \lambda_N(t)$ such that,*

$$\lim_{t\to T} \|\psi(t,\cdot) - m\pi - \sum_{j=1}^{N} \pm\big(Q(\cdot/\lambda_j(t)) - \pi\big)\|_{\mathcal{E}} = 0,$$

*and $\lim_{t\to T} \sum \lambda_j(t)/\lambda_{j+1}(t) = 0$. Similar when $T < \infty$.*

Note: $\lambda_{N+1}(t) := \sqrt{t}$, and subsequent equivariant bubbles always have opposite orientations as maps $\mathbb{R}^2 \to \mathbb{S}^2$.

# Continuous time bubbling

Relative to a suitable metric $\delta$ measuring distance to **multi-bubble configurations**, we have:

**Theorem (Jendrej, Lawrie, S. '23)**

*$u(t) : [0, T_+) \times \mathbb{R}^2 \to \mathbb{S}^2$ smooth HMHF solution, maximal $T_+ = T_+(u_0) \in (0, \infty]$. If $T_+ < \infty$, then $\forall\, y \in \mathbb{R}^2$,*

$$\lim_{t \to T_+} \boldsymbol{\delta}\big(u(t); D(y, \sqrt{T_+ - t})\big) = 0.$$

*Arbitrary $t_n \to T_+$ and $D(y_n, R_n\rho_n) \subset D(y, \sqrt{T_+ - t})$, $R_n \to \infty$, assume energy evacuates from necks of disks. Then,*

$$\lim_{n \to \infty} \boldsymbol{\delta}\big(u(t_n); D(y_n, \rho_n)\big) = 0.$$

*Analogous statement on $D(y, \sqrt{t})$ if $T_+ = \infty$.*

Solution remains close to multi-bubble configurations at parabolic scales, and on all smaller disks whose boundaries do not intersect bubbles, for all times up to $T_+$.

# Wave Maps (WM)

Lorentzian analogue of harmonic maps: $u : \mathbb{R}^{1+d}_{t,x} \to N, \quad \Box u \perp T_u N$.
Geodesics as WM: $\gamma : \mathbb{R} \to N$, $\Box \phi = 0$, $u(t) = \gamma(\phi(t))$.
**Energy critical** in $d = 2$.

**Equivariant WM.**
Christodoulou, Tahvildar-Zadeh, Shatah (1990s): LWP/GWP.
Struwe '03: for $d = 2$, $N = S^2$, singularities via bubbling off a harmonic sphere.
Krieger, S., Tataru '05: blowup examples via a harmonic sphere.
Raphael, Rodnianski '09: different blowup mechanism
Jendrej–Lawrie '21: **Full soliton resolution**
Duyckaerts–Kenig–Martel–Merle: same result for $k = 1$.

**No symmetry. Curvature of $N$ decides dynamics for large data**
Klainerman, Selberg '97: LWP $s > \frac{d}{2}$ (null forms, Lorentzian spaces $X^{s,b}$).
Tataru, Tao '00: small energy GWP.
Sterbenz, Tataru '09: GWP for $E < E_{\min}$.
Krieger–S. '09: GWP $N = \mathbb{H}^2$ (Bahouri–Gérard '98, Kenig–Merle '05)
Krieger–Miao–S. '26: rigidity of KST blowup WM under non-equivariant perturbations.

# Part III

## Ginzburg–Landau vortices

### Orbital stability, asymptotic stability, and the distorted Fourier transform

*Ongoing work with J. Lührmann (Texas A&M, Cologne), J. Palacios (Toronto, EPFL), F. Pusateri (Toronto), S. Shahshahani (UMass, Amherst).*

# Ginzburg–Landau vortices (abelian Yang–Mills–Higgs)

**Fields.** $\phi: \mathbb{R}^{1+2} \to \mathbb{C}, \qquad A = A_\mu \, dx^\mu, \quad A_\mu: \mathbb{R}^{1+2} \to \mathbb{R}.$

**Action.** $F = dA, \qquad D_\mu = \nabla_\mu - iA_\mu.$

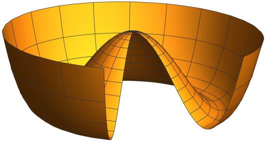
$$\mathcal{L} = \int_{\mathbb{R}^{1+2}} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi \, \overline{D^\mu \phi} + \frac{\lambda}{8} (1 - |\phi|^2)^2 \right) dx \, dt, \qquad \lambda > 0,$$

**Higgs potential.**

$$V(\phi) = \frac{\lambda}{8} (1 - |\phi|^2)^2.$$

**Euler–Lagrange equations (YMH).**

$$\begin{cases} D^\mu D_\mu \phi + \dfrac{\lambda}{2} (1 - |\phi|^2) \, \phi = 0, \\ \nabla^\mu F_{\mu\nu} + \operatorname{Im} \left( \overline{\phi} \, D_\nu \phi \right) = 0. \end{cases}$$



**Gauge symmetry.**

$$(\phi, A) \mapsto \left( e^{i\chi} \phi, \ A + d\chi \right), \qquad \chi \in C^\infty(\mathbb{R}^{1+2})$$

# Energy, degree, and vortices

**Static energy.**

$$E(\phi, A) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |B|^2 + \frac{1}{2} |D_j \phi|^2 + \frac{\lambda}{8} (1 - |\phi|^2)^2 \right) dx, \qquad B := F_{12}.$$

**Finite energy fields.** $E(\phi, A) < \infty \;\Rightarrow\; |\phi(x)| \to 1$ as $|x| \to \infty$.

Then the winding number $\deg(\phi) \in \mathbb{Z}$ gauge-invariant, and

$$\deg(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} B \, dx \qquad \text{(magnetic flux)}.$$

**Static finite energy solutions.**

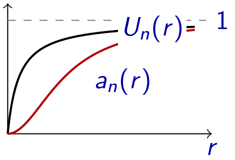
**(i) Vacuum states:**

$$(\phi, A) = \left( e^{i\chi}, 0 \right), \qquad \chi \in [0, 2\pi).$$

Asymptotic stability: Tsutsumi '03.

**(ii) Vortices (degree $n$):**

$$(\phi, A) = \left( e^{in\theta} U_n(r),\; a_n(r)\, d\theta \right).$$




Orbital stability: Gustafson '02; Gustafson–Sigal '00.

$\lambda < 1$: all $n \in \mathbb{Z}$;

$\lambda > 1$: $n = \pm 1$.

# Self-dual case $\lambda = 1$

**Bogomol'nyi factorization.**

$$E(\phi, A) = \frac{1}{2}\int_{\mathbb{R}^2}\left(\left(B - \tfrac{1}{2}(1-|\phi|^2)\right)^2 + \left|(D_1 + iD_2)\phi\right|^2\right) dx \; + \; \frac{1}{2}\int_{\mathbb{R}^2} B\, dx.$$

In particular, $\frac{1}{2}\int_{\mathbb{R}^2} B\, dx = \pi \ \deg(\phi)$.

**Bogomol'nyi equations.**

$$\begin{cases} B - \tfrac{1}{2}(1-|\phi|^2) = 0, \\ (D_1 + iD_2)\phi = 0. \end{cases}$$

**Degree-one vortex ansatz.** $\phi = e^{i\theta}U(r), \quad A = a(r)\, d\theta.$

This reduces the Bogomoln'yi system to

$$\begin{cases} U'(r) = \dfrac{1-a(r)}{r}\, U(r), & U(0) = 0,\ U(\infty) = 1,\ U'(r) > 0, \\ a'(r) = \dfrac{r}{2}\,(1 - U(r)^2), & a(0) = 0,\ a(\infty) = 1,\ a'(r) > 0. \end{cases}$$

References: Taubes (1980); Jaffe–Taubes (1980).

# Asymptotic stability (equivariant perturbations)

**Theorem (LPPSS '26)**

*The degree-one vortex $(\phi, A)$ is asymptotically stable under the self-dual abelian Yang–Mills–Higgs evolution with respect to equivariant perturbations that are small in a weighted Sobolev norm.*

What goes into the proof:

- Fix Stuart's gauge: perturbations are orthogonal to all infinitesimal "pure gauge" directions.
- Reduce dynamics (in the equivariant sector) to a dispersive system driven by a self-adjoint matrix Schrödinger operator.
- Rely on supersymmetry to compute the distorted Fourier transform, bound linear flow.
- Show no unstable spectrum; isolate a 2D internal mode below the threshold and prove that it is dissipative via a FGR.
- Show decay of vortex perturbation via a bootstrap: normal forms, local energy decay, multilinear bounds.

# Linearized operator around the 1-vortex

Equivariant perturbations:

$$\phi = e^{i\theta}\big(U(r) + \varepsilon(t,r)\big), \qquad \varepsilon = \alpha + i\beta,$$
$$A = a(r)\,d\theta + \eta_\theta(t,r)\,d\theta + \eta_r(t,r)\,dr + \eta_0(t,r)\,dt.$$

Orthogonal gauge condition (Stuart '94):

$$\Big(\partial_r + \frac{1}{r}\Big)\eta_r = U\,\beta.$$

This yields a linear system of the schematic form

$$\big(\partial_t^2 + \mathcal{L}\big)\begin{pmatrix} \varepsilon \\ \frac{\eta_\theta}{r} - i\eta_r \end{pmatrix} = \mathcal{N}, \qquad (-\Delta + U^2)\eta_0 = \mathcal{N}_0,$$

with

$$\mathcal{L} = \begin{pmatrix} L_1 & -2U' \\ -2U' & L_2 \end{pmatrix}$$
$$L_1 := -\Delta + \frac{(1-a_\theta)^2}{r^2} - \frac{a_\theta'}{r} + U^2, \qquad L_2 = -\Delta + \frac{1}{r^2} + U^2$$

# Spectrum of $\mathcal{L}$ (equivariant)

Supersymmetric factorization:

$$\mathcal{L} = \mathcal{B}^*\mathcal{B}, \qquad \mathcal{B} = \begin{pmatrix} \partial_r - \frac{1-a}{r} & U \\ U & \partial_r + \frac{1}{r} \end{pmatrix}, \quad \ker(\mathcal{B}) = \ker(\mathcal{B}^*) = \{0\}$$

Hence

$$\mathcal{B}\mathcal{B}^* = \begin{pmatrix} -\Delta + V_1 & 0 \\ 0 & -\Delta + U^2 \end{pmatrix}, \quad V_1 = \frac{(2-a_\theta)^2}{r^2} + \frac{1}{2}(1+U^2) > 1$$

- Continuous spectrum starts at the threshold 1.
- There is a discrete internal mode with eigenvalue $\lambda^2 \approx 0.78$ (numerical) and 2D eigenspace.
- We compute the distorted FT for $\mathcal{B}\mathcal{B}^*$ via Sturm-Liouville theory and solution of connection problem via perturbative asymptotic analysis of Krieger-S-Tataru '05. Pull it back to $\mathcal{L}$ by applying $\mathcal{B}^*$.

# Proof ideas (very rough sketch)

Spectral decomposition:

$$\begin{pmatrix} \alpha + i\beta \\ \frac{\eta_\theta}{r} - i\eta_r \end{pmatrix} = \vec{w}(t) + b_1(t)\,\vec{Y}_1 + b_2(t)\,\vec{Y}_2.$$

Coupled evolution (quadratic and cubic nonlinearities):

$$\begin{aligned} &\left(\partial_t^2 + \mathcal{L}\right)\vec{w} = \mathcal{Q}(\vec{w}, b, \eta_0) + \mathcal{C}(\vec{w}, b, \eta_0), \\ &\left(\partial_t^2 + \lambda^2\right) b_j = \langle \vec{Y}_j, \mathcal{N} \rangle. \end{aligned}$$

- Good–bad decomposition (Léger–Pusateri '22 style): separate a localized "bad" piece.
- Flat–sharp decomposition: solve a simpler forced system for the flat part, then perturb.