Quasilinear NLS models with intensity-dependent dispersion

Dmitry Pelinovsky
Department of Mathematics, McMaster University, Canada

ICERM Workshop on Nonlinear Waves

Joint work with:
P.G. Kevrekidis, Ryan Ross (Amherst, Massachusetts, USA);
Fabio Natali (Maringa, Brazil), Shuoyang Wang (McMaster, Canada)

I. NLS models with intensity-dependent dispersion

Generalized NLS models

The classical NLS equation realizes a balance between nonlinearity and dispersion for propagation of nonlinear dispersive waves.

$$i\psi_t + \alpha\psi_{xx} + \beta|\psi|^2\psi = 0. \tag{NLS}$$

Taking into account higher-order nonlinearity and dispersion gives an extended version of the NLS equation:

$$\begin{aligned} i\psi_t + \alpha\psi_{xx} + \beta|\psi|^2\psi + i\alpha_1\psi_{xxx} + \alpha_2\psi_{xxxx} \\ + i\beta_1|\psi|^2\psi_x + i\beta_2(|\psi|^2\psi)_x + \gamma|\psi|^4\psi = 0. \end{aligned} \tag{gNLS}$$

Well-posedness of initial-value problem, stability of nonlinear waves, global dynamics (scattering versus blowup in a finite time), ...

Quasilinear models with intensity-dependent dispersion

The dispersion coefficient α may depend on the wave intensity $|\psi|^2$:

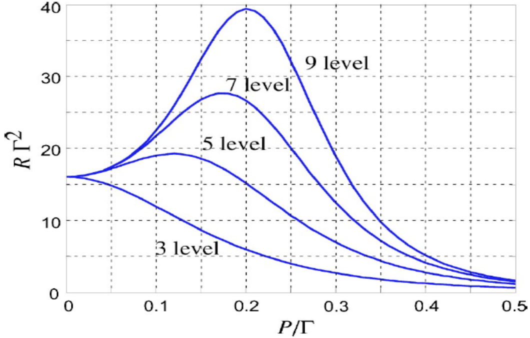


FIG. 3. Graphs showing dispersion (times linewidth squared), $R\Gamma^2$ as a function of P/Γ with $C/\Gamma = 0.25$ for Chain Λ systems of 3, 5, 7, and 9 states.

A.D. Greentree, D. Richards, J.A. Vaccaro, et al., Phys. Rev. A **67** (2003), 023818

Two NLS equations with intensity-dependent dispersion

The NLS equation where the dispersion vanishes at a selected intensity:

$$i\psi_t + \alpha(1 - |\psi|^2)\psi_{xx} + \beta|\psi|^2\psi = 0. \qquad \text{(NLS-IDD-1)}$$

Wave profiles are singular at the unit intensity.

C.Y. Lin, J.H. Chang, G. Kurizki, and R.K. Lee, Optics Letters **45** (2020) 1471–1474
R.M. Ross, P.G. Kevrekidis, and D.P., Quart. Appl. Math. **79** (2021) 641-665
D.P, R.M. Ross, and P.G. Kevrekidis, J. Phys. A: Math. Theor. **54** (2021) 445701
P.G. Kevrekidis, D.P, and R.M. Ross, IMA J. Appl. Math. **89** (2024) 989–1005
F. Natali, D.P., and S. Wang (2026), arXiv:2603.28514

The NLS equation where the dispersion diverges at a selected intensity:

$$i\psi_t + \alpha(1 - |\psi|^2)^{-1}\psi_{xx} + \beta|\psi|^2\psi = 0. \qquad \text{(NLS-IDD-2)}$$

Wave profiles are smooth but the time evolution is singular at $|\psi|^2 = 1$.

D.P. and M. Plum, SIMA **56** (2024) 2521-2568

Quasilinear NLS equation due to nonlocal Kerr effects

The NLS-IDD-1 model is a variant of the nonlocal Kerr model:

$$i\psi_t + \alpha\psi_{xx} + \beta\psi(\mathcal{K}_\epsilon * |\psi|^2) = 0,$$

with

$$\hat{K}_\epsilon(k) = 1 - \epsilon^2 k^2 + \mathcal{O}(\epsilon^4), \quad k \in \mathbb{R},$$

which yields the quasilinear NLS model

$$i\psi_t + \alpha\psi_{xx} + \beta\psi|\psi|^2 + \beta\epsilon^2\psi(|\psi|^2)_{xx} = 0. \tag{NLS-QL}$$

I. Iliev and K. Kirchev, Differential Integral Equations **6** (1993) 685–703
M. Colin, L. Jeanjean, and M. Squassina, Nonlinearity **23** (2010) 1353–1385
A. de Laire, and S. López-Martínez, Calc. Var. PDEs **64** (2025) 138

A. de Laire and E. Le Quiniou, Nonlinear Anal. **265** (2026) 114027

Connection to the Ablowitz–Ladik lattice model

A popular Salerno lattice model is an interpolation between the integrable Ablowitz–Ladik model and the non-integrable DNLS model:

$$i\partial_\tau \psi_n + (1 - |\psi_n|^2)(\psi_{n+1} + \psi_{n-1}) + \mu |\psi_n|^2 \psi_n = 0,$$

where $n \in \mathbb{Z}$ and $\psi_n = \psi_n(\tau)$.

If $\mu = 2 + h^2$ and $\psi_n(\tau) = e^{2i\tau}\psi(h^2\tau, hn)$ with a smooth $\psi = \psi(t, x)$, then a formal expansion in powers of the small stepsize h yields the NLS-IDD-1 model:

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} + |\psi|^2\psi = 0.$$

II. Stability of the bright soliton

NLS-IDD model

We consider the NLS equation with intensity-dependent dispersion

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} + |\psi|^2\psi = 0, \quad x \in \mathbb{R}, \quad t > 0.$$

It admits the following conserved quantities:

- energy for $\psi \in H^1(\mathbb{R})$ due to the time translation symmetry:

$$E(\psi) = \int_{\mathbb{R}} (|\psi_x|^2 + |\psi|^2) dx.$$

- mass for $\psi \in H^1(\mathbb{R})$ with small $\|\psi\|_{L^\infty} \leq C < 1$ due to the phase rotation symmetry:

$$Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx$$

- momentum for $\psi \in H^1(\mathbb{R})$ such that $\psi \neq 0$ due to the spatial translation symmetry.

Local well-posedness of quasilinear NLS models

For general quasilinear NLS equations,

$$i\psi_t + f(|\psi|^2)\psi + g(|\psi|^2)\psi_{xx} = 0,$$

local well-posedness in Sobolev spaces has some history.

- $H^\infty(\mathbb{R})$: M. Poppenberg, Nonlinear Anal. **45** (2001) 723
- $H^s(\mathbb{R})$, for large $s \gg 1$:
  C. Kenig, G. Ponce, L. Vega, Inventiones Math. **158** (2004) 343
  J. Marzuola, J. Metcalfe, D. Tataru, Kyoto J. Math. **54** (2014) 529
- $H^s(\mathbb{R})$, for $s > 2$ and $s > 1$:
  J. Marzuola, J. Metcalfe, D. Tataru, ARMA **242** (2021) 1129
  M. Ifrim, D. Tataru, Inventiones Math. **242** (2025) 221
- $H^s_{\rm per}$, for $s > 2$:
  R. Feola, F. Indoli, Ann. I. H. Poincare **36** (2019) 119
  R. Feola, B. Grebert, F. Indoli, Analysis & PDE **16** (2023) 1133

Existence of bright solitons

These are the standing wave solutions of the form $\psi(x,t) = e^{i\omega t}\varphi_\omega(x)$ in

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} + |\psi|^2\psi = 0,$$

where φ_ω is a real-valued profile satisfying

$$\frac{d^2\varphi}{dx^2} = \frac{(\omega - \varphi^2)}{(1 - \varphi^2)}\varphi = -\frac{dV}{d\varphi},$$

which is integrable as

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad V(\varphi) := \frac{\omega - 1}{2}\log|1 - \varphi^2| - \frac{1}{2}\varphi^2.$$

Solitary waves with $\varphi_\omega(x) \to 0$ as $|x| \to \infty$ exist only if $\omega > 0$.

Existence of bright solitons

These are the standing wave solutions of the form $\psi(x,t) = e^{i\omega t}\varphi_\omega(x)$ in

$$i\psi_t + (1-|\psi|^2)\psi_{xx} + |\psi|^2\psi = 0,$$

where φ_ω is a real-valued profile satisfying

$$\frac{d^2\varphi}{dx^2} = \frac{(\omega-\varphi^2)}{(1-\varphi^2)}\varphi = -\frac{dV}{d\varphi},$$

which is integrable as

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad V(\varphi) := \frac{\omega-1}{2}\log|1-\varphi^2| - \frac{1}{2}\varphi^2.$$

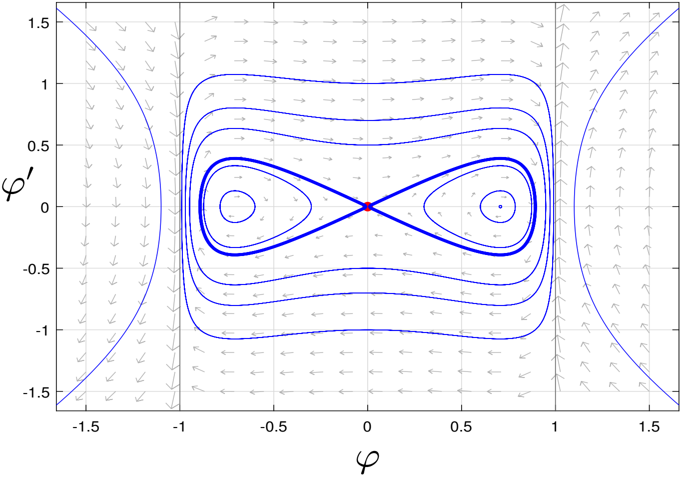
Solitary waves with $\varphi_\omega(x) \to 0$ as $|x| \to \infty$ exist only if $\omega > 0$.

Theorem

There exists a smooth profile φ_ω if and only if $\omega \in (0,1)$: $0 \leq \varphi_\omega(x) < 1$, $x \in \mathbb{R}$. Moreover, the family $\{\varphi_\omega\}_{\omega\in(0,1)}$ is smooth in ω.

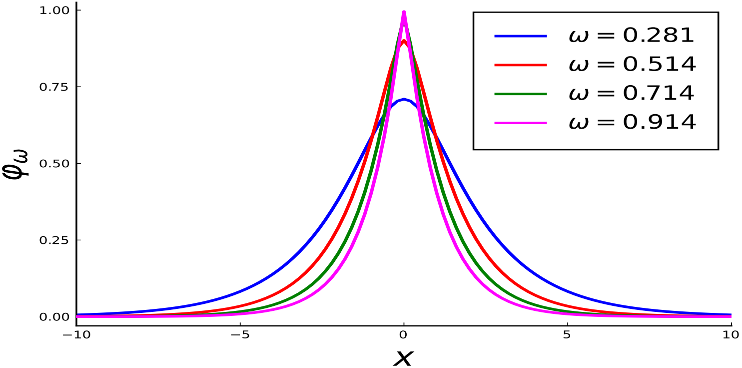
Existence of bright solitons

The phase portrait for $\omega \in (0,1)$ takes the form:



Existence of bright solitons

The smooth profile φ_ω is obtained from the numerical quadrature:



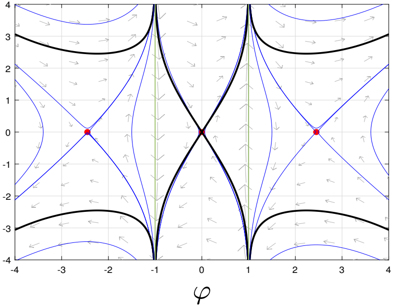
For $\omega \to 0$, the profile φ_ω is approximated by the sech-soliton:

$$\varphi_\omega(x) \sim \sqrt{2\omega}\,\mathrm{sech}(\sqrt{\omega}x)$$

For $\omega \to 1$, the profile is peaked as $\varphi_{\omega=1}(x) = e^{-|x|}$.

Existence of bright solitons

Nonsmooth (cusped) solitons exist formally for $\omega \in (1, \infty)$:



Existence and stability of cusped solitons for $i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0$ in

R.M. Ross, P.G. Kevrekidis, and D.P., Quart. Appl. Math. **79** (2021) 641-665

D.P, R.M. Ross, and P.G. Kevrekidis, J. Phys. A: Math. Theor. **54** (2021) 445701

Stability of the bright soliton via conserved quantities

Theorem (P.G. Kevrekidis, D.P, and R.M. Ross (2024))

For every $\omega \in (0,1)$*, the spatial profile* φ_ω *is a local nondegenerate (up to two symmetries) minimizer of the energy* $E(\psi)$ *subject to fixed mass* $Q(\psi)$ *in* $H^1(\mathbb{R})$ *if and only if the mapping* $\omega \mapsto Q(\varphi_\omega)$ *is strictly increasing.*

Stability of the bright soliton via conserved quantities

Theorem (P.G. Kevrekidis, D.P, and R.M. Ross (2024))

For every $\omega \in (0,1)$, the spatial profile φ_ω is a local nondegenerate (up to two symmetries) minimizer of the energy $E(\psi)$ subject to fixed mass $Q(\psi)$ in $H^1(\mathbb{R})$ if and only if the mapping $\omega \mapsto Q(\varphi_\omega)$ is strictly increasing.

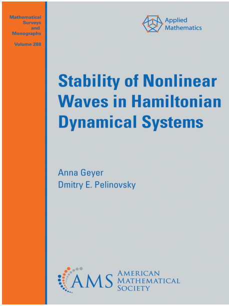
Remark:

This result yields spectral stability of bright solitons. It does not imply the orbital stability along the orbit $\{\varphi_\omega(\cdot - \xi)e^{i\theta}\}_{\xi,\theta\in\mathbb{R}}$ because of the lack of local well-posedness in $H^1(\mathbb{R})$. We call it the energetic stability.

Stability of the bright soliton via conserved quantities

Theorem (P.G. Kevrekidis, D.P, and R.M. Ross (2024))

For every $\omega \in (0,1)$, the spatial profile φ_ω is a local nondegenerate (up to two symmetries) minimizer of the energy $E(\psi)$ subject to fixed mass $Q(\psi)$ in $H^1(\mathbb{R})$ if and only if the mapping $\omega \mapsto Q(\varphi_\omega)$ is strictly increasing.



Mathematical
Surveys
and
Monographs

Volume 288

Applied
Mathematics

Stability of Nonlinear Waves in Hamiltonian Dynamical Systems

Anna Geyer
Dmitry E. Pelinovsky

AMS AMERICAN MATHEMATICAL SOCIETY

The proof is an exercise from the book.

Analysis of energetic stability I

Recall the conserved quantities

$$E(\psi) = \int_{\mathbb{R}} (|\psi_x|^2 + |\psi|^2) dx, \quad Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx.$$

The Hamiltonian structure of the NLS equation is non-standard:

$$i\partial_t \psi = (1 - |\psi|^2) \frac{\delta}{\delta \bar{\psi}} (E - Q), \quad \frac{\delta}{\delta \bar{\psi}} (E - Q) = -\partial_x^2 \psi - \frac{|\psi|^2 \psi}{1 - |\psi|^2}.$$

Conservation of Q is related to the phase rotation symmetry due to

$$\psi = (1 - |\psi|^2) \frac{\delta Q}{\delta \bar{\psi}}.$$

The standing wave $\psi(x,t) = e^{i\omega t} \varphi_\omega$ is defined by a critical point of $\Lambda_\omega := E + (\omega - 1) Q$ with

$$0 = (1 - |\varphi|^2) \frac{\delta}{\delta \bar{\varphi}} (E + (\omega - 1) Q).$$

Analysis of energetic stability II

Expansion of Λ_ω at φ_ω yields

$$\Lambda_\omega(\varphi_\omega + u + iv) - \Lambda_\omega(\varphi_\omega) = \langle \mathcal{S}_+ u, u \rangle_{L^2} + \langle \mathcal{S}_- v, v \rangle_{L^2} + \mathcal{O}(\|u + iv\|_{H^1}^3),$$

where $\mathcal{S}_\pm : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are given by

$$\mathcal{S}_- := -\partial_x^2 + \frac{\omega - \varphi_\omega^2}{1 - \varphi_\omega^2},$$
$$\mathcal{S}_+ := -\partial_x^2 + \frac{\omega + 2\varphi_\omega \partial_x^2 \varphi_\omega - 3\varphi_\omega^2}{1 - \varphi_\omega^2}.$$

- Since $\mathcal{S}_- \varphi_\omega = 0$ and $\varphi_\omega > 0$, the spectrum of $\mathcal{S}_-$ in $L^2(\mathbb{R})$ is positive with a simple zero eigenvalue.
- Since $\mathcal{S}_+ \partial_x \varphi_\omega = 0$ and $\partial_x \varphi_\omega$ has a simpe zero on $\mathbb{R}$, the spectrum of $\mathcal{S}_+$ in $L^2(\mathbb{R})$ has a simple negative and a simple zero eigenvalue.

Analysis of energetic stability III

The profile φ_ω is a saddle point of $\Lambda_\omega = E + (\omega - 1)Q$ with a simple negative eigenvalue and a double zero eigenvalue due to two symmetries. Due to the non-degeneracy of the kernel, the mapping $\omega \to \varphi_\omega \in H^1(\mathbb{R})$ is C^1 with

$$\mathcal{S}_+ \partial_\omega \varphi_\omega = -v_\omega.$$

The profile φ_ω is a strict (local) minimizer of E subject to fixed Q if and only if

$$\langle \mathcal{S}_+^{-1} v_\omega, v_\omega \rangle_{L^2} = -\langle \partial_\omega \varphi_\omega, v_\omega \rangle_{L^2} = -\frac{1}{2} \partial_\omega Q(\varphi_\omega) < 0,$$

where v_ω is given by the constraint of fixed Q:

$$v_\omega := \frac{\delta Q}{\delta \bar{\varphi}_\omega} = \frac{\varphi_\omega}{1 - \varphi_\omega^2}.$$

This yields the sharp energetic stability criterion of a monotonic increase of the mapping $\omega \mapsto Q(\varphi_\omega)$.

A twist for the spectral stability argument I

The energetic stability implies that the linear spectral problem

$$\begin{pmatrix} 0 & \mathcal{S}_- \\ -\mathcal{S}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

admits no eigenvalues $\lambda \in \mathbb{C}\backslash\{i\mathbb{R}\}$ with $(u, v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$.

However, the correct spectral stability problem is

$$\begin{pmatrix} 0 & (1-\varphi_\omega^2)\mathcal{S}_- \\ -(1-\varphi_\omega^2)\mathcal{S}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

defined in the weighted Hilbert space $\mathcal{H} \times \mathcal{H}$, $\mathcal{H} := L^2(\mathbb{R}, (1-\varphi_\omega^2)^{-1}\, dx)$.

However, smooth profiles satisfy $0 \leq \varphi_\omega(x) < 1$, $x \in \mathbb{R}$ and hence $(1-\varphi_\omega^2)^{1/2}$ is bounded and invertible.

A twist for the spectral stability argument II

By using

$$u = (1-\varphi_\omega^2)^{1/2}\tilde{u}, \quad v = (1-\varphi_\omega^2)^{1/2}\tilde{v},$$

the correct spectral stability problem

$$\begin{pmatrix} 0 & (1-\varphi_\omega^2)\mathcal{S}_- \\ -(1-\varphi_\omega^2)\mathcal{S}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

becomes

$$\begin{pmatrix} 0 & (1-\varphi_\omega^2)^{1/2}\mathcal{S}_-(1-\varphi_\omega^2)^{1/2} \\ -(1-\varphi_\omega^2)^{1/2}\mathcal{S}_+(1-\varphi_\omega^2)^{1/2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix},$$

where $(1-\varphi_\omega^2)^{1/2}\mathcal{S}_\pm(1-\varphi_\omega^2)^{1/2}$ have the same number of negative and zero eigenvalues as $S_\pm$ in $L^2(\mathbb{R})$ by Sylvester's inertia law.

Theorem (P.G. Kevrekidis, D.P, and R.M. Ross (2024))

For every $\omega \in (0,1)$ for which $\omega \to Q(\varphi_\omega)$ is strictly increasing, the stabiity problem admits no eigenvalues $\lambda \in \mathbb{C}\backslash\{i\mathbb{R}\}$ with $(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$.

Numerical explorations from Kevrekidis–P–Ross (2024)

A bubble of instability is detected for $\omega \in (\omega_1, \omega_2)$, where $0 < \omega_1 < \omega_2 < 1$.

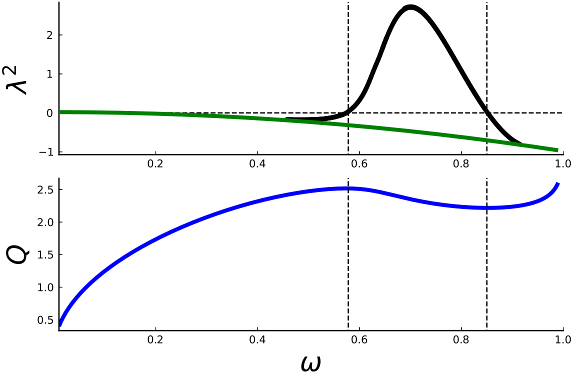
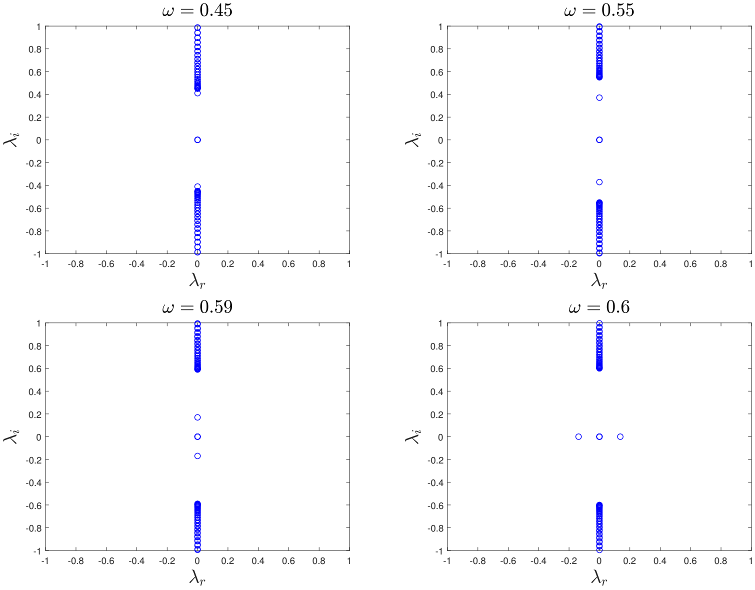


Figure: Top: Squared eigenvalue λ^2 and the map $\omega \mapsto Q(\varphi_\omega)$. The dashed vertical lines are drawn at ω_1 and ω_2.

Numerical explorations from Kevrekidis–P–Ross (2024)



Numerical explorations from Kevrekidis–P–Ross (2024)

The transition at ω_1 and ω_2 between the unstable branch for $\omega_1 < \omega < \omega_2$ and stable branches for $\omega < \omega_1$ and $\omega > \omega_2$ is consistent between the monotonicity of $\omega \to Q(\varphi_\omega)$ and eigenvalues λ of the spectral problem.

Numerical explorations from Kevrekidis–P–Ross (2024)

The transition at ω_1 and ω_2 between the unstable branch for $\omega_1 < \omega < \omega_2$ and stable branches for $\omega < \omega_1$ and $\omega > \omega_2$ is consistent between the monotonicity of $\omega \to Q(\varphi_\omega)$ and eigenvalues λ of the spectral problem.

...Except that the second transition at $\omega = \omega_2$ is a numerical artefact of discretization found in the context of the periodic problem...

III. Stability of the periodic waves

NLS-IDD model in the periodic domain

We consider the NLS equation with intensity-dependent dispersion

$$i\psi_t + (1-|\psi|^2)\psi_{xx} + |\psi|^2\psi = 0, \quad x \in [0,L], \quad t > 0.$$

It admits the following conserved quantities:

- energy for $\psi \in H^1_{\rm per}$ due to the time translation symmetry:

$$E(\psi) = \oint (|\psi_x|^2 + |\psi|^2) dx.$$

- mass for $\psi \in H^1_{\rm per}$ with small $\|\psi\|_{L^\infty} \leq C < 1$ due to the phase rotation symmetry:

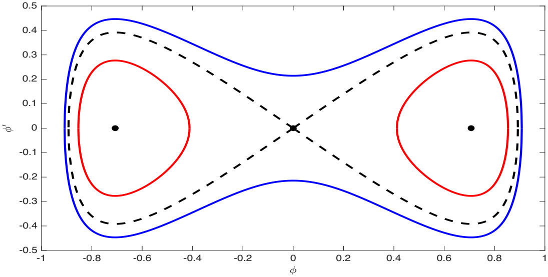
$$Q(\psi) = -\oint \log|1-|\psi|^2| dx$$

- momentum for $\psi \in H^1_{\rm per}$ such that $\psi \neq 0$ due to the spatial translation symmetry.

Existence of periodic waves

The same first-order invariant for the standing waves $\psi(x,t) = e^{i\omega t}\varphi(x)$:

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad V(\varphi) := \frac{\omega-1}{2}\log|1-\varphi^2| - \frac{1}{2}\varphi^2.$$



The phase portrait corresponds to $\omega \in (0,1)$. Periodic orbits have different periods for fixed ω. For stability problem (and applications), we need to find the periodic orbits for fixed $L > 0$ versus parameter ω.

Existence of periodic waves

Theorem (Natali–P–Wang, 2026)

Fix the spatial period $L > 0$ and define

$$\omega_L = \frac{2\pi^2}{L^2 + 2\pi^2}, \quad \Omega_L = -\frac{4\pi^2}{L^2}.$$

For any $\omega \in (\omega_L, 1)$, there exists a L-periodic even wave satisfying

$$\begin{cases} 0 < \varphi(x) < 1, & \forall\, x \in \mathbb{T}_L, \\ \varphi(x) = \varphi(-x), & \forall\, x \in \mathbb{T}_L. \end{cases}$$

For any $\omega \in (\Omega_L, 1)$, there exists a L-periodic odd wave satisfying

$$\begin{cases} -1 < \varphi(x) < 1, & \forall x \in \mathbb{T}_L, \\ \varphi(x) = -\varphi(-x) = \varphi\left(\frac{L}{2} - x\right), & \forall\, x \in \mathbb{T}_L. \end{cases}$$

Each generates a 2-parameter orbit $\{\varphi(\cdot - s)e^{-i\theta}\}_{\xi,\theta\in\mathbb{R}}$ due to symmetries.

Existence of periodic waves

Even wave is constructed in

$$H^2_{\rm per,e} = \left\{ f \in H^2_{\rm per} : \quad f(-x) = f(x) \;\; x \in \mathbb{T}_L \right\},$$

by using local bifurcation for zeros of

$$\mathrm{F} : H^2_{\rm per,e} \times \mathbb{R} \to L^2_{\rm per,e}, \quad \mathrm{F}(\varphi,\omega) = -(1-\varphi^2)\varphi'' + \omega\varphi - \varphi^3.$$

near $(\varphi,\omega) = (\sqrt{\omega_L}, \omega_L)$ with $\omega_L = \frac{2\pi^2}{L^2+2\pi^2}$ for $\omega > \omega_L$.

Odd wave is constructed in

$$H^2_{\rm per,o} = \left\{ f \in H^2_{\rm per} : \quad f(-x) = -f(x) \;\; x \in \mathbb{T}_L \right\},$$

by using local bifurcation for zeros of $\mathrm{F} : H^2_{\rm per,o} \times \mathbb{R} \to L^2_{\rm per,o}$ near $(\varphi,\omega) = (0, \Omega_L)$ with $\Omega_L = -\frac{4\pi^2}{L^2}$ for $\omega > \Omega_L$.

Existence of periodic waves

To show that the even wave exists for all $\omega \in (\omega_L, 1)$ and the odd wave exists for all $\omega \in (\Omega_L, 1)$, one can use the global bifurcation theory. It also yields continuity of the pairs $(\varphi, \omega) \in H^2_{\rm per,e} \times (\omega_L, 1)$ and $(\varphi, \omega) \in H^2_{\rm per,o} \times (\Omega_L, 1)$ with respect to ω.

Existence of periodic waves

To show that the even wave exists for all $\omega \in (\omega_L, 1)$ and the odd wave exists for all $\omega \in (\Omega_L, 1)$, one can use the global bifurcation theory. It also yields continuity of the pairs $(\varphi, \omega) \in H^2_{\rm per,e} \times (\omega_L, 1)$ and $(\varphi, \omega) \in H^2_{\rm per,o} \times (\Omega_L, 1)$ with respect to ω.

However, there is an alternative way, which also provides more information about the periodic orbits and their stability. It relies on the period function:

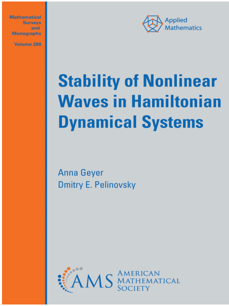
$$T(\mathcal{E}, \omega) = \oint \frac{d\varphi}{\sqrt{2(\mathcal{E} - V(\varphi))}},$$

for each first-order invariant

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad V(\varphi) := \frac{\omega - 1}{2} \log|1 - \varphi^2| - \frac{1}{2}\varphi^2.$$

Existence of periodic waves

To show that the even wave exists for all $\omega \in (\omega_L, 1)$ and the odd wave exists for all $\omega \in (\Omega_L, 1)$, one can use the global bifurcation theory. It also yields continuity of the pairs $(\varphi, \omega) \in H^2_{\rm per,e} \times (\omega_L, 1)$ and $(\varphi, \omega) \in H^2_{\rm per,o} \times (\Omega_L, 1)$ with respect to ω.



Mathematical
Surveys
and
Monographs

Volume 288

Applied
Mathematics

Stability of Nonlinear Waves in Hamiltonian Dynamical Systems

Anna Geyer
Dmitry E. Pelinovsky

AMS AMERICAN MATHEMATICAL SOCIETY

See Chapter 3 for the period function for orbital stability of periodic waves.

Period function and its applications

For orbits on the phase plane, we consider the period function

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad T(\mathcal{E},\omega) = \oint \frac{d\varphi}{\sqrt{2(\mathcal{E} - V(\varphi))}}.$$

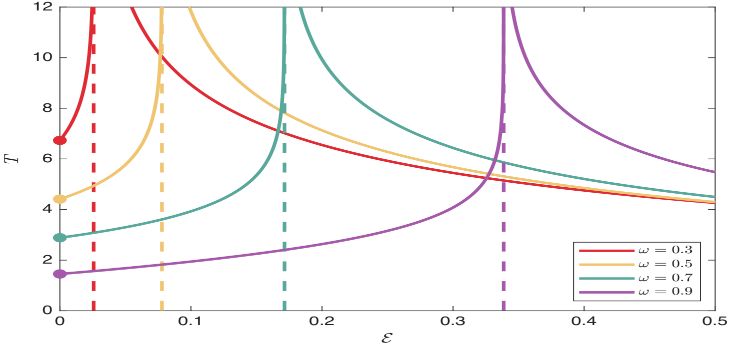


Figure: The period function $T(\mathcal{E},\omega)$ versus $\mathcal{E}$ for fixed values of ω.

Period function and its applications

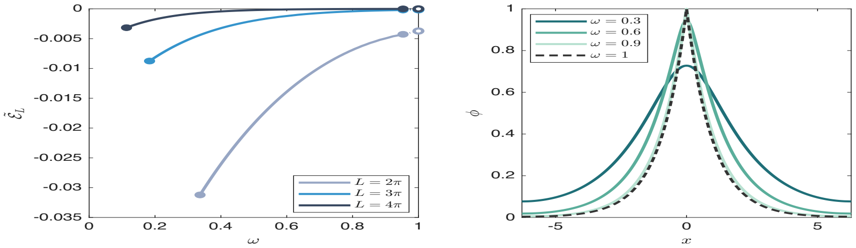
Theorem (Natali–P–Wang, 2026)

The period function $T = T(\mathcal{E}, \omega)$ *is a* C^1 *function of* $\mathcal{E} \in (0,\infty) \backslash \mathcal{E}_\omega$ *if* $\omega \in (0,1)$ *and* $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$ *if* $\omega \in (-\infty, 0)$, *where* $\mathcal{E}_\omega = V(0)$.

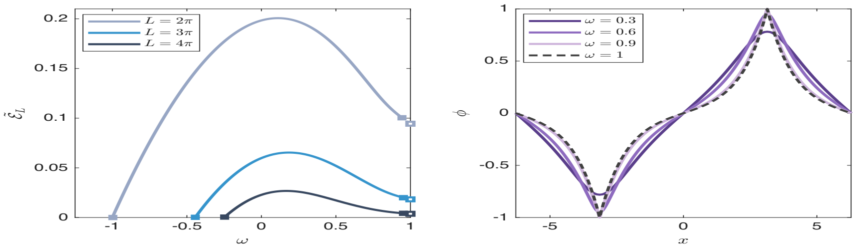
- *For any* $\omega \in (0,1)$, *the mapping* $(0, \mathcal{E}_\omega) \ni \mathcal{E} \to T(\mathcal{E}, \omega)$ *is monotonically increasing.*
- *For any* $\omega \in (-\infty, 1)$, *the mapping* $(\mathcal{E}_\omega, \infty) \ni \mathcal{E} \to T(\mathcal{E}, \omega)$ *is monotonically decreasing.*

- For any fixed $L > 0$ and $\omega \in (\omega_L, 1)$, there exists a unique root $\mathcal{E}_L(\omega) \in (0, \mathcal{E}_\omega)$ of $T(\mathcal{E}, \omega) = L$ for the even wave.
- For any fixed $L > 0$ and $\omega \in (\Omega_L, 1)$, there exists a unique root $\mathcal{E}_L(\omega) \in (\mathcal{E}_\omega, \infty)$ of $T(\mathcal{E}, \omega) = L$ for the odd wave.

Even waves:



Odd waves:



From smooth profiles to peaked profiles

Both families of solutions terminate at $\omega = 1$, for which

$$\frac{d^2\varphi}{dx^2} = \frac{(\omega - \varphi^2)}{(1 - \varphi^2)}\varphi$$

becomes the linear equation $\varphi''(x) = \varphi(x)$ with the exact solution:

$$\omega = 1: \quad \varphi(x) = \frac{\cosh\left(\frac{L}{2} - |x|\right)}{\cosh\left(\frac{L}{2}\right)}, \quad \mathcal{E}_L = -\frac{1}{2\cosh^2\left(\frac{L}{2}\right)}$$

and a similar peaked solution for the odd wave with

$$\omega = 1: \quad \mathcal{E}_L = \frac{1}{2\sinh^2\left(\frac{L}{2}\right)}.$$

Remark: Numerical detection of roots is a stiff problem near $\omega = 1$. The numerical data is missing in a tiny gap near $\omega = 1$.

Monotonicity of the period function

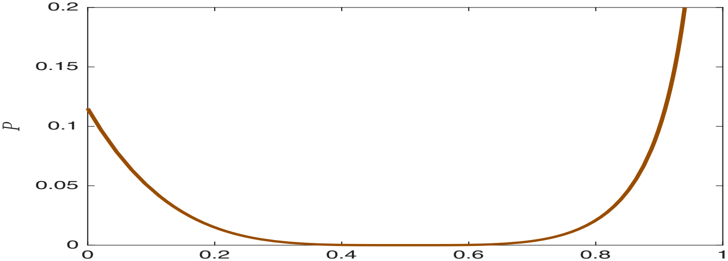
Even waves: By a theorem of C. Chiconi (1987), the period function

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad T(\mathcal{E},\omega) = 2\int_m^M \frac{d\varphi}{\sqrt{2(\mathcal{E} - V(\varphi))}}, \quad 0 < m < M < 1,$$

is monotonically increasing in $\mathcal{E}$ if $I''(\varphi) > 0$, $\varphi \in (0,1)$, where

$$I(\varphi) = \frac{V(\varphi)}{[V'(\varphi)]^2} \quad \Rightarrow \quad I''(\varphi) = \frac{P(\varphi)}{(V'(\varphi))^4(1-\varphi^2)^4},$$

where $P(\varphi)$ is a combination of polynomials and logarithmic functions.



Monotonicity of the period function

Odd waves: The period function can be transformed at $\mathcal{E} = V(M)$:

$$\begin{aligned} T(\mathcal{E},\omega) &= 4\int_0^M \frac{d\varphi}{\sqrt{2[V(M)-V(\varphi)]}} \\ &= \sqrt{2}\int_0^q \frac{dt}{\sqrt{t[W(q)-W(t)]}} \\ &= \int_0^1 \frac{\sqrt{2q}}{\sqrt{u[W(q)-W(qu)]}}\,du, \end{aligned}$$

where $q = M^2$, $t = \varphi^2 = qu$, and

$$W(t) := \frac{1}{2}(\omega - t) + \frac{1}{2}(1-\omega)\log\frac{1-\omega}{1-t}, \quad t \in (0,1).$$

Equipped with the analysis of convexity, this yields

$$\frac{\partial T}{\partial q} = \frac{1}{\sqrt{2q}}\int_0^1 \frac{du}{\sqrt{u[W(q)-W(qu)]}} - \frac{\sqrt{q}}{\sqrt{2}}\int_0^1 \frac{W'(q)-uW'(qu)}{\sqrt{u[W(q)-W(qu)]^3}}\,du < 0.$$

Energetic stability of the even wave

Expansion of $\Lambda_\omega = E + (\omega - 1)Q$ near the periodic profile φ yields

$$\Lambda_\omega(\varphi + u + iv) - \Lambda_\omega(\varphi) = \langle \mathcal{S}_+ u, u \rangle_{L^2} + \langle \mathcal{S}_- v, v \rangle_{L^2} + \mathcal{O}(\|u + iv\|^3_{H^1_{\rm per}}),$$

where $\mathcal{S}_\pm : H^2_{\rm per} \subset L^2_{\rm per} \to L^2_{\rm per}$ given by

$$\mathcal{S}_- = -\partial_x^2 + \frac{\omega - \varphi^2}{1 - \varphi^2}, \quad \mathcal{S}_+ = -\partial_x^2 + \frac{\omega + 2\varphi\partial_x^2\varphi - 3\varphi^2}{1 - \varphi^2}.$$

Theorem (Natali–P–Wang, 2026)

Fix the spatial period $L > 0$. The profile $\varphi \in H^1_{\rm per}$ is a C^1 function of ω for the even wave in $(\omega_L, 1)$. For any $\omega \in (\omega_L, 1)$, the even wave is a local minimizer of energy E for a fixed mass Q in $H^1_{\rm per}$, which is degenerate only due to translational and rotational symmetries, if and only if the mapping $\omega \to Q(\varphi)$ is monotonically increasing.

Sturm's theory for periodic Schrödinger operators

Spectrum of either $\mathcal{S}_+$ or $\mathcal{S}_-$ in $L^2_{\rm per}$ consists of an unbounded sequence of real eigenvalues satisfying

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 \;\dots\; < \lambda_{2n-1} \leq \lambda_{2n} \;\cdots,$$

where the eigenfunction for λ_{2n-1} and λ_{2n} has exactly $2n$ zeroes in the periodic domain.

Let $n(\mathcal{S}_\pm)$ denote the Morse index (number of negative eigenvalues) and $z(\mathcal{S}_\pm)$ denote multiplicity of the zero eigenvalue.

- Since $\mathcal{S}_-\varphi = 0$ and $\varphi > 0$, then $\lambda_0 = 0$ and $n(\mathcal{S}_-) = 0$, $z(\mathcal{S}_-) = 1$.
- Since $\mathcal{S}_+\partial_x\varphi = 0$ and $\partial_x\varphi$ has exactly two nodes, then either $\lambda_1 = 0 < \lambda_2$, or $\lambda_1 < \lambda_2 = 0$, or $\lambda_1 = \lambda_2 = 0$. However, this question is completely solved with $\partial_{\mathcal{E}} T > 0$, or $\partial_{\mathcal{E}} T < 0$, or $\partial_{\mathcal{E}} T = 0$. Due to strict increase of the period function, we have $n(\mathcal{S}_+) = z(\mathcal{S}_+) = 1$.

Constrained minimization of $\Lambda_\omega = E + (\omega - 1)Q$

Due to non-degeneracy

$$\mathrm{Ker}(\mathcal{S}_+) = \mathrm{span}(\partial_x \varphi), \quad \mathrm{Ker}(\mathcal{S}_-) = \mathrm{span}(\varphi),$$

the mapping $\omega \to \varphi \in H^1_{\mathrm{per}}$ is C^1 and

$$\mathcal{S}_+ \partial_\omega \varphi = -\frac{\varphi}{1 - \varphi^2} = \frac{\delta Q}{\delta \bar{\varphi}}.$$

The saddle point of $\Lambda_\omega = E + (\omega - 1)Q$ is a strict (local) minimizer of E subject to fixed Q and the two symmetries if and only if

$$\langle \mathcal{S}_+^{-1} \frac{\varphi}{1 - \varphi^2}, \frac{\varphi}{1 - \varphi^2} \rangle_{L^2_{\mathrm{per}}} = -\langle \frac{\varphi}{1 - \varphi^2}, \partial_\omega \varphi \rangle_{L^2_{\mathrm{per}}} = -\frac{1}{2} \frac{d}{d\omega} Q(\varphi) < 0.$$

Energetic stability of the odd wave

Theorem (Natali–P–Wang, 2026)

Fix the spatial period $L > 0$. The profile $\varphi \in H^1_{\rm per}$ is a C^1 function of ω for the odd wave in $(\Omega_L, 1)$. For any $\omega \in (\Omega_L, 1)$, the odd wave is a local minimizer of energy E for a fixed mass Q in $\mathcal{Y} \subset H^1_{\rm per}$, which is degenerate only due to translational and rotational symmetries, if and only if the mapping $\omega \to Q(\varphi)$ is monotonically increasing, where

$$\mathcal{Y} = \left\{ u \in H^1_{\rm per} : \quad u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \quad \forall\, x \in \mathbb{T}_L \right\}.$$

- Since $\mathcal{S}_-\varphi = 0$ and φ has two nodes, then $n(\mathcal{S}_-) = z(\mathcal{S}_-) = 1$.
- Since $\mathcal{S}_+\partial_x\varphi = 0$ and $\partial_x\varphi$ has also two nodes, then the strict decrease of the period function implies $n(\mathcal{S}_+) = 2$ and $z(\mathcal{S}_+) = 1$.

The extra symmetry was used before, e.g. in T.Gallay–M.Haragus (2007).

Criterion for the energetic stability

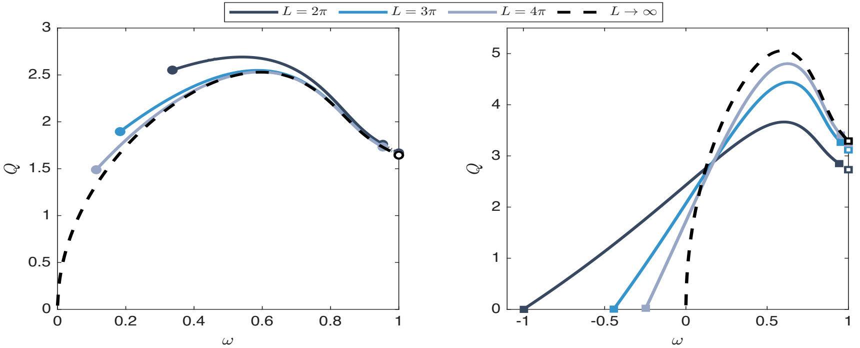


Figure: Dependence of $Q(\varphi)$ versus ω for $L = 2\pi, 3\pi, 4\pi$ and in the limit $L \to \infty$.

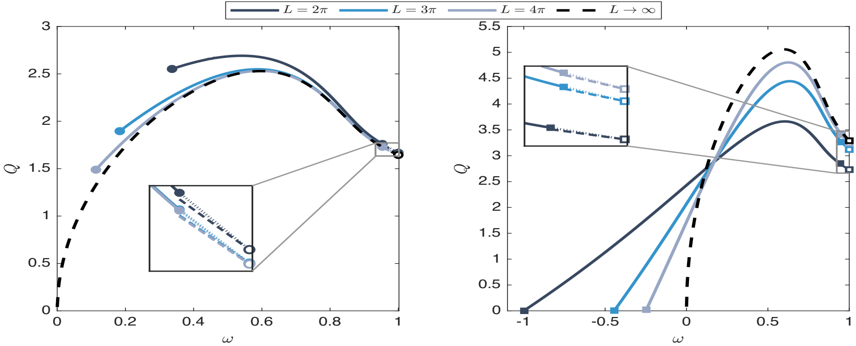
Conjecture: There exist $\omega_*, \Omega_* \in (0,1)$ such that the even wave is energetically stable for $\omega \in (\omega_L, \omega_*)$ and unstable for $\omega \in (\omega_*, 1)$, whereas the odd wave is stable for $\omega \in (\Omega_L, \Omega_*)$ and unstable for $\omega \in (\Omega_*, 1)$.

IV. Conclusive Remarks

Very recent addition to the proof of the conjecture

Conjecture: There exist $\omega_*, \Omega_* \in (0,1)$ such that the even wave is energetically stable for $\omega \in (\omega_L, \omega_*)$ and unstable for $\omega \in (\omega_*, 1)$, whereas the odd wave is stable for $\omega \in (\Omega_L, \Omega_*)$ and unstable for $\omega \in (\Omega_*, 1)$.

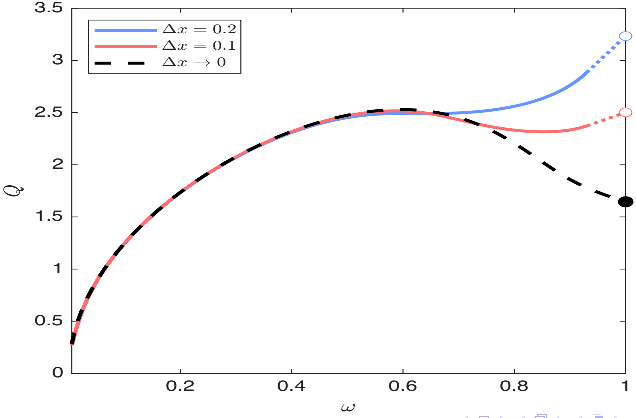
S. Wang computed very precise asymptotics of $Q(\varphi)$ near the local bifurcation at $\omega = \omega_L, \Omega_L$ and near the peaked limit $\omega = 1$.



Numerical artefact from Kevrekidis–P–Ross (2024)

By using the finite-difference approximation, one can recover the numerical artefact with "stable slope" of the mapping $\omega \to Q(\varphi)$.

$$(1-\varphi_j^2)(D^2\varphi)_j - (\omega - \varphi_j^2)\varphi_j = 0, \quad (D^2\varphi)_j = \frac{\varphi_{j-1} - 2\varphi_j + \varphi_{j+1}}{(\Delta x)^2}.$$



Conclusion

- We considered new variations of the cubic NLS model with the intensity-dependent dispersion.
- We proved spectral and energetic stability of the bright solitons as twisted versions of the stability problem for the cubic NLS equation.
- We extended analysis of energetic stability to the periodic waves with precise proofs of monotonicity of the period function and a nearly complete proof of the slope of the mass function.
- Local well-posedness of the quasilinear NLS models with the intensity-dependent dispersion in the energy space is still open.

THE END!