Scalar and gravitational horizon hair as observable imprints of extremal black holes

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Extremal Black Holes and Black Hole Thermodynamics

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SIMULATING EXTREME SPACETIMES

LISA

Outline

Part-1 : Axisymmetric scalar perturbations in ERN

1. Motivation and setup
2. Horizon Hair
3. Signatures from radiation field
4. Numerical results

Part-2: Non-Axisymmetric gravitational perturbations in EK

1. Motivation and setup
2. The Beetle-Burko scalar
3. Ori's late time expansion
4. Numerical results

Motivation

- Can we find **distinguishing** features of the dynamics of perturbations of black holes?
- Eg. are there **signatures** that can help identify extremal from sub-extremal Black holes.
- If yes, can such signatures be observed at **null infinity**?

Setup of Physical problem

- We investigate the solution to

$$\Box_g \psi = 0$$

  on (extremal)Reissner-Nordstrom.

- Radiation field $\Psi(\tau, \vartheta)$ is defined as

$$\Psi(\tau, \vartheta) = \lim_{r \to \infty} (r\psi)(\tau, r, \vartheta)$$

- For numerics we often use the mode decomposition

$$\psi(\tau, r, \vartheta) = \sum_{\ell=0}^{\infty} \psi_\ell(\tau, r) Y_{\ell 0}(\vartheta)$$

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  - τ: the *retarded time* (on $\mathcal{I}^+$) or *advanced time* (on $\mathcal{H}^+$),
  - $\vartheta = (\theta^1, \theta^2)$: the *angle* which represents the direction of any given ideal observer along $\mathcal{I}^+$.

- For numerics we often use the mode decomposition

$$\psi(\tau, r, \vartheta) = \sum_{\ell=0}^{\infty} \psi_\ell(\tau, r) Y_{\ell 0}(\vartheta)$$

Horizon Hair

- For generic initial data intersecting the future event horizon $\mathscr{H}^+$, transverse derivatives of the field don't decay on $\mathscr{H}^+$ [1].
- For mode ell = 0, at $\mathscr{H}^+$

$$H_0[\psi] \equiv \frac{1}{M}[\partial_r(r\psi_0)]_{r=M}$$

- The field does decay at late times, then

$$(\partial_r \psi_0)_{r=M} \to H_0 \quad as \quad v \to \infty$$

[1]. Aretakis, S. Horizon instability of extremal black holes

Signatures from radiation field

- The asymptotic tail behavior of the radiation field is:

$$\Psi(\tau,\vartheta) = C_1(\vartheta)\frac{1}{\tau^2} + C_2(\vartheta)\frac{\log\tau}{\tau^3} + O\left(\frac{1}{\tau^3}\right)$$

- For the sub-extremal case [1]

$$C_1(\vartheta) = -2I^{(1)}[\psi], \quad C_2(\vartheta) = 8MI^{(1)}[\psi]$$

- For the extremal case[2]

$$C_1(\vartheta) = 4MH[\psi] - 2I^{(1)}[\psi], \quad C_2(\vartheta) = -32M^2H[\psi] + 8MI^{(1)}[\psi]$$

[Angelopolous, Aretakis, Gajic, 2019, 2020]

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Ansatz from numerics

$$C_1(\vartheta) = 4MH[\psi] - 2I^{(1)}[\psi], \quad C_2(\vartheta) = -32M^2H[\psi] + 8MI^{(1)}[\psi]$$

Signatures from radiation field

- From previous relations,

$$H[\psi] = -\frac{1}{16M^2}(C_2(\vartheta) + 4MC_1(\vartheta))$$

- This motivates a **signature**,

$$s_{\mathcal{I}^+}(\vartheta) = -\frac{1}{16M^2}(C_2(\vartheta) + 4MC_1(\vartheta))$$

- Knowledge of the radiation field along a fixed angle allows us to compute the horizon charge.

Numerical Methodology

- Solve the scalar wave equation in RN and ERN with l=m=0
- Use **compactified hyperboloidal coordinates** $(\tau, \rho, \theta, \phi)$ to access null infinity on the numerical domain.

$$\Omega = 1 - \frac{\rho}{S}$$

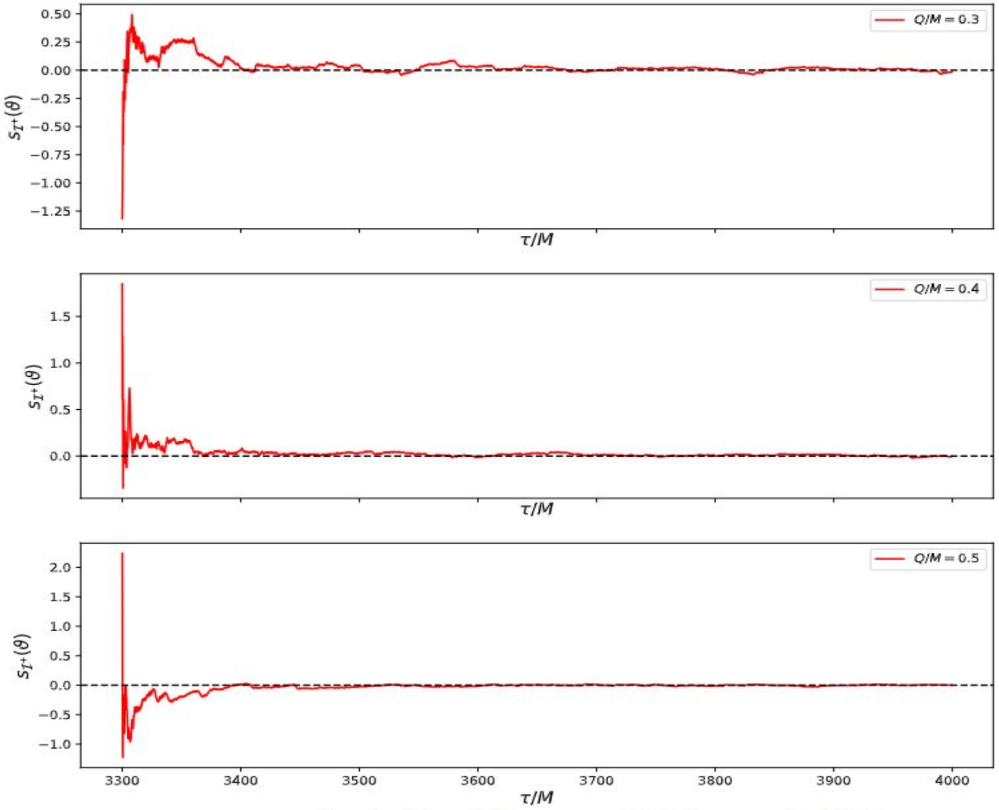
$$r = \frac{\rho}{\Omega(\rho)}$$

$$v \equiv t + r_* - r = \tau + \frac{\rho}{\Omega(\rho)} - \rho - 4M \log \Omega(\rho)$$

- We use a **fifth-order WENO** finite-difference scheme with a spectral **discontinuous Galerkin solver** in development.
- We solve a 1+1D system with S = 19 and horizon at 0.95. ID are Gaussian pulses with centers : (1.0,1.1,1.2) and widths (0.16,0.22, 0.32)

Vanishing signature on sub-extremal RN

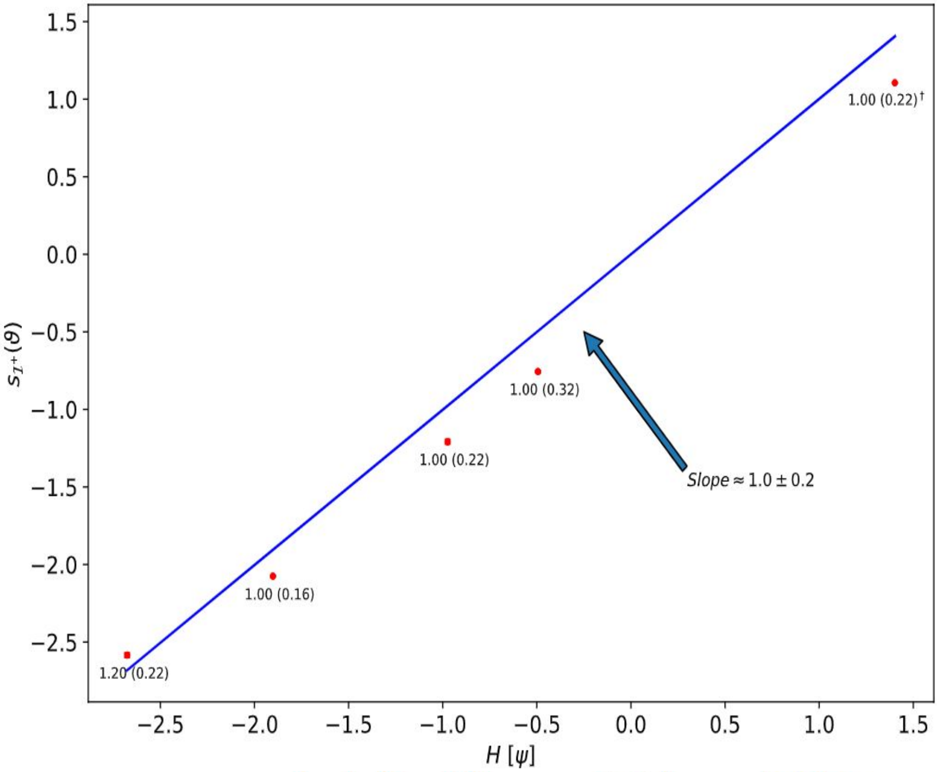
- Signature is computed from radiation field extracted at null infinity.
- All simulations use **compactly supported** initial data with support on the horizon.
- For sub-extremal BHs the signature **vanishes**.



Aretakis, Khanna, Sabharwal, 2025

Non-vanishing signature on extremal RN

- For the extremal case, we plot the signature vs horizon charge.
- Each data point is labeled by the initial data i.e. the **location and width** of the Gaussian pulse.
- This enables us to compute H from the signature.
- Extracting C1 and C2 accurately is challenging.



Aretakis, Khanna, Sabharwal, 2025

Key Takeaways

- There are signatures that can distinguish between extremal and sub-extremal black holes.
- In the extremal case, these signatures equal the horizon charge and can be computed at null infinity.
- Can similar signatures be obtained for the astrophysically relevant gravitational perturbations?
- Implementation using a **pseudo-spectral DG** scheme under construction to get higher accuracy.

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Motivation

- Does the notion of horizon charge extend to **non-axisymmetric** gravitational perturbations.

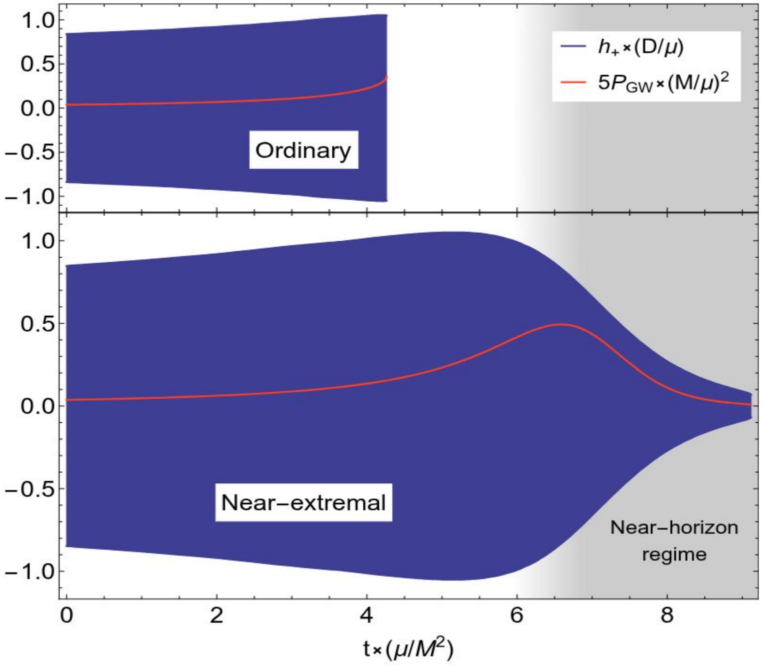
- If yes, can this be calculated in black hole's exterior?... away from horizon?

- Non-axisymmetric, radiative, gravitational field perturbations are precisely what **LVK, LISA** are designed to detect.

- We provide numerical evidence for such a quantity at late times.

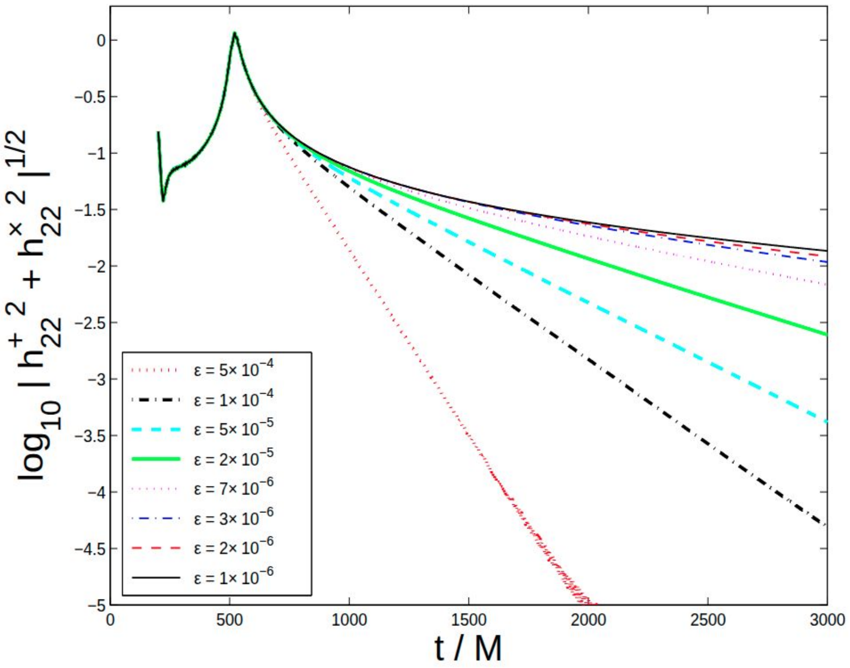
Search for a "Smoking gun"

h_+ component of a quasicircular inspiral



Gralla, Hughes, Warburton, 2016

amplitude of strain h_{22} vs time t



Burko, Khanna, 2016

Inspiration for the Beetle-Burko Scalar

- The perturbed Weyl Scalars satisfy

$$\mathcal{T}_{-2}\Psi_4^{(1)} = 0$$

$$\mathcal{T}_{2}\Psi_0^{(1)} = 0$$

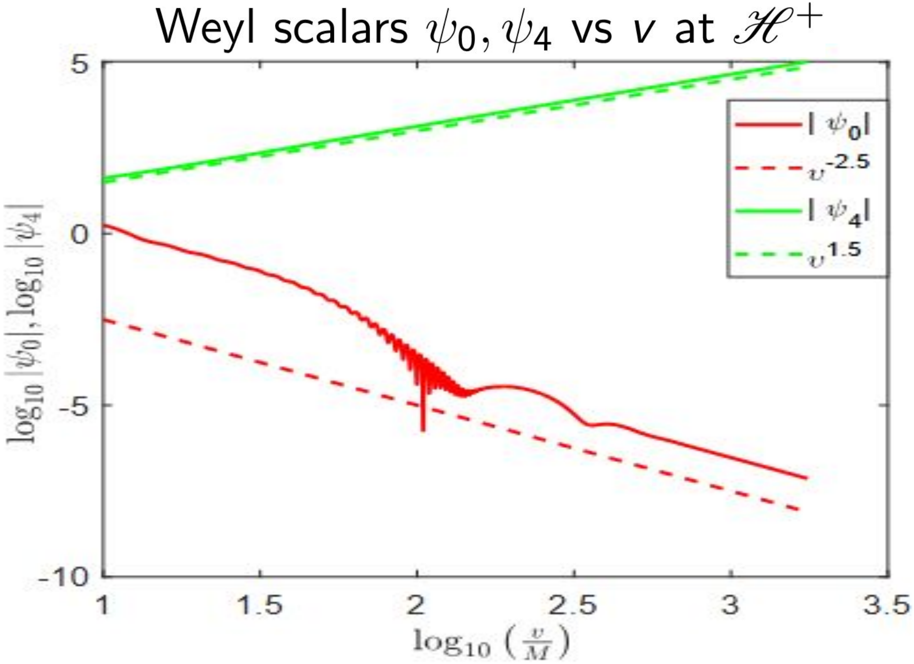
where the differential operators depend on the spin-weight s.

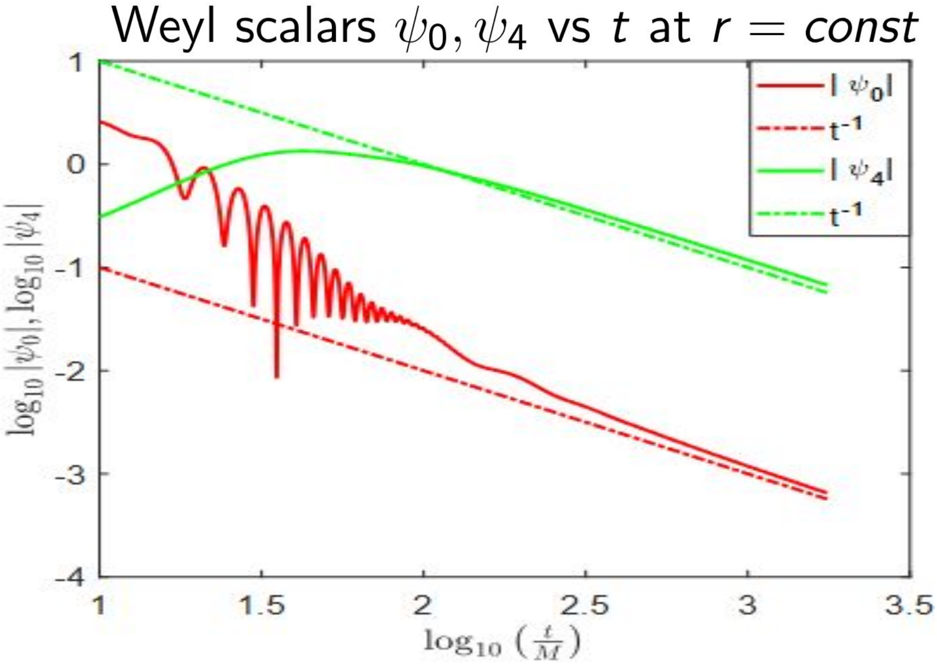
- Kretschmann scalar can be written in terms of the Weyl scalars as

$$K = 8(\Psi_0\Psi_4 + 3\Psi_2^2 - 4\Psi_1\Psi_3) + c.c$$

The Beetle-Burko scalar

- Late-time power law decay rates for (2,2) mode in extremal Kerr. Predicted By Casals-Gralla-Zimerman '18.





The Beetle-Burko scalar

- The Beetle-Burko scalar is the combination

$$\xi = \Psi_0 \Psi_4$$

- Retains scalar behavior and invariance under coordinate and tetrad transformations.

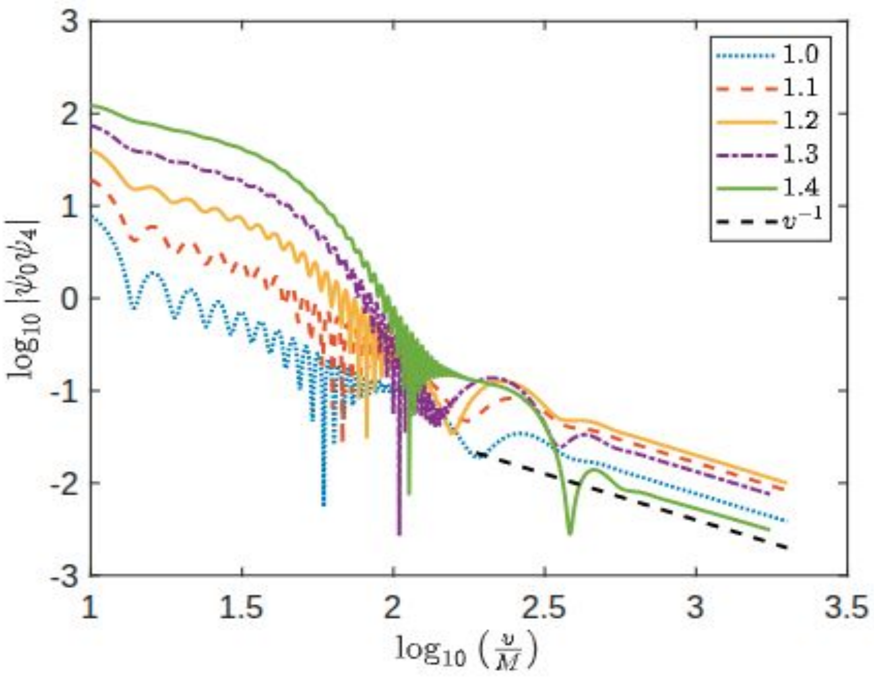


Figure 1. Plots of the Beetle-Burko scalar $|\xi_{2,2}|$ as function of advanced time v on $\mathscr{H}^+$. $|\xi_{2,2}|$ decays with an inverse power of v. We show this for various initial conditions that are labeled by the centers of the respective Gaussians.

The Beetle-Burko scalar

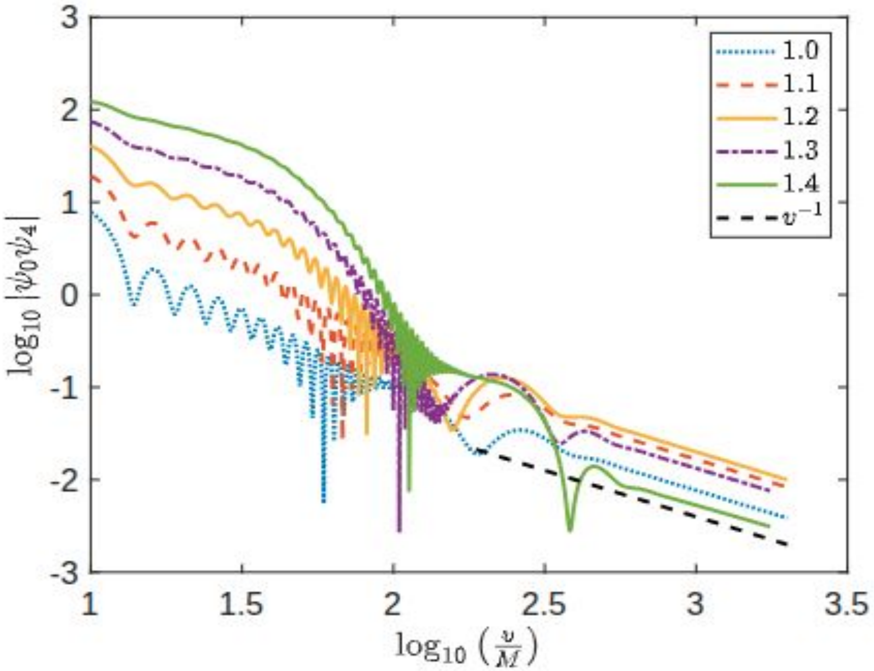
- The Beetle-Burko scalar is the combination

$$\xi = \Psi_0 \Psi_4$$

- Retains scalar behavior and invariance under coordinate and tetrad transformations.
- Numerically we compute

$$\xi_{\ell,m}(t,r) = \Psi_{2,\ell,m}(t,r)\Psi_{-2,\ell,m}(t,r)$$

as the dominant multipole.



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Non-axisymmetric hair

- The proposed hair is the transverse derivative of the Beetle-Burko scalar.
- This is only conserved at late times.

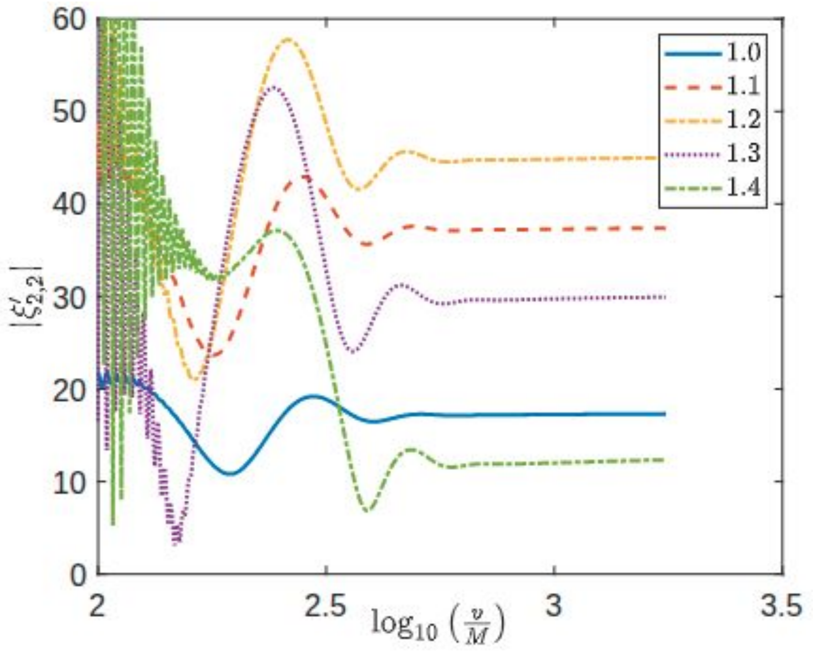
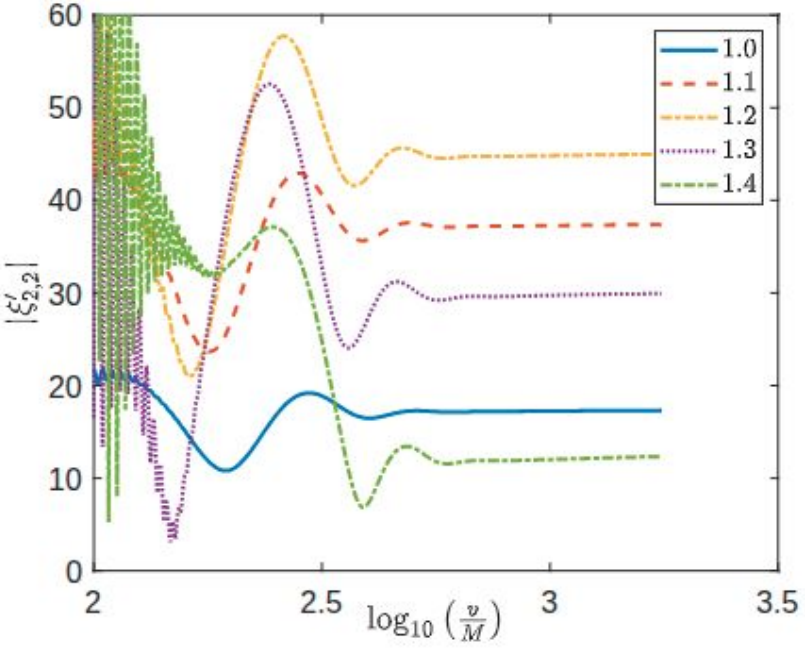


Figure 2. Plots of the proposed non-axisymmetric charge $|\xi'_{2,2}|$ on $\mathscr{H}^+$ as function of advanced time v. While $|\xi_{2,2}|$ decays with an inverse power of v, the proposed quadrupolar charge $|\xi'_{2,2}|$ is a constant at late times.

Non-axisymmetric hair

- The proposed hair is the transverse derivative of the Beetle-Burko scalar.
- This is only conserved at late times.
- The proposed quantity is only the **first term** for a potential conserved charge in an inverse-time expansion.



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Ori late-time expansion

- The late time expansion of a field Ψ in a black hole space-time can be written as

$$\Psi_{s,\ell,m}(t,r) = e_{s,\ell,m} r^{-q_{s,\ell,m}} (r-M)^{-p_{s,\ell,m}} t^{-n_{s,\ell,m}} + \mathcal{O}(t^{-n_{s,\ell,m}-k_{s,\ell,m}})$$

where $e_{s,\ell,m}$ are the Ori-coefficients.

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where $e_{s,\ell,m}$ are the **Ori-coefficients**.

- This is expected to be valid for $t >> r_*$
- We compare the Ori-coefficient with the charge like quantity on $\mathscr{H}^+$

Numerical Methodology

Requires ingoing kerr coordinates

- Remove azimuthal dependence to reduce to 2+1D form
- Define **auxiliary variables** to convert into a 1+1D system
- Challenging since onset of tails requires long simulations
- Quad precision required to resolve **mode-coupling**.

$$-\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\partial_{tt}\Psi - \frac{4Mar}{\Delta}\partial_{t\phi}\Psi$$
$$-2s\left[r - \frac{M(r^2-a^2)}{\Delta} + ia\cos\theta\right]\partial_t\Psi$$
$$+\Delta^{-s}\partial_r\left(\Delta^{s+1}\partial_r\Psi\right) + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Psi\right) +$$
$$\left[\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right]\partial_{\phi\phi}\Psi + 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\partial_\phi\Psi$$
$$-\left(s^2\cot^2\theta - s\right)\Psi = -4\pi\left(r^2+a^2\cos^2\theta\right)T,$$

$$\Delta = r^2 - 2Mr + a^2$$

Numerical Methodology

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A GPU-accelerated mixed-precision WENO method for extremal black hole and gravitational wave physics computations

Scott E. Field · Sigal Gottlieb · Zachary J. Grant · Leah F. Isherwood · Gaurav Khanna

$$A^{\tau\tau}\partial_\tau^2\Psi + A^{\tau\rho}\partial_\tau\partial_\rho\Psi + A^{\rho\rho}\partial_\rho^2\Psi + A^{\theta\theta}\partial_\theta^2\Psi + B^\tau\partial_\tau\Psi + B^\rho\partial_\rho\Psi + B^\theta\partial_\theta\Psi + C\Psi = 0$$

⇓

$$\pi \equiv \partial_\tau\Psi + b\,\partial_\rho\Psi\ ,$$
$$b \equiv -\left(\tilde{A}^{\tau\rho} + \sqrt{(\tilde{A}^{\tau\rho})^2 + 4\tilde{A}^{\rho\rho}}\right)/2$$

⇓

$$\partial_\tau \boldsymbol{u} + \boldsymbol{M}\partial_\rho\boldsymbol{u} + \boldsymbol{L}\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = 0$$

Estimates of exponents

- To get the Ori-coefficients, we use the expression

$$|\xi_{2,2}(t,r)| = e_{\xi_{2,2}} r^{-q_{\xi_{2,2}}} (r-M)^{-p_{\xi_{2,2}}} t^{-n_{\xi_{2,2}}} + O(t^{-n_{\xi_{2,2}}-k_{\xi_{2,2}}})$$

- We construct the time series data for $\xi_{\ell m}(t,r)$ by using solutions of the Teukolsky equation.

Estimates of exponents

- By fitting the time series data for $\xi_{\ell m}(t,r)$, we get:

$$|\xi_{2,2}(t,r)| = e_{\xi_{2,2}} r^{-q_{\xi_{2,2}}} (r-M)^{-p_{\xi_{2,2}}} t^{-n_{\xi_{2,2}}} + O(t^{-n_{\xi_{2,2}} - k_{\xi_{2,2}}})$$

ID	$q^{\xi}_{2,2}$	$p^{\xi}_{2,2}$	$n^{\xi}_{2,2}$	$\lvert\xi'_{2,2}\rvert$	$e^{\xi}_{2,2}$
1.0	-1.072	0.9016	1.981	17.3263	0.0359
1.1	-1.072	0.9013	2.019	37.3826	0.0785
1.2	-1.071	0.9010	2.052	44.9732	0.0958
1.3	-1.071	0.9006	2.101	29.9314	0.0652
1.4	-1.070	0.8997	2.102	12.3420	0.0273

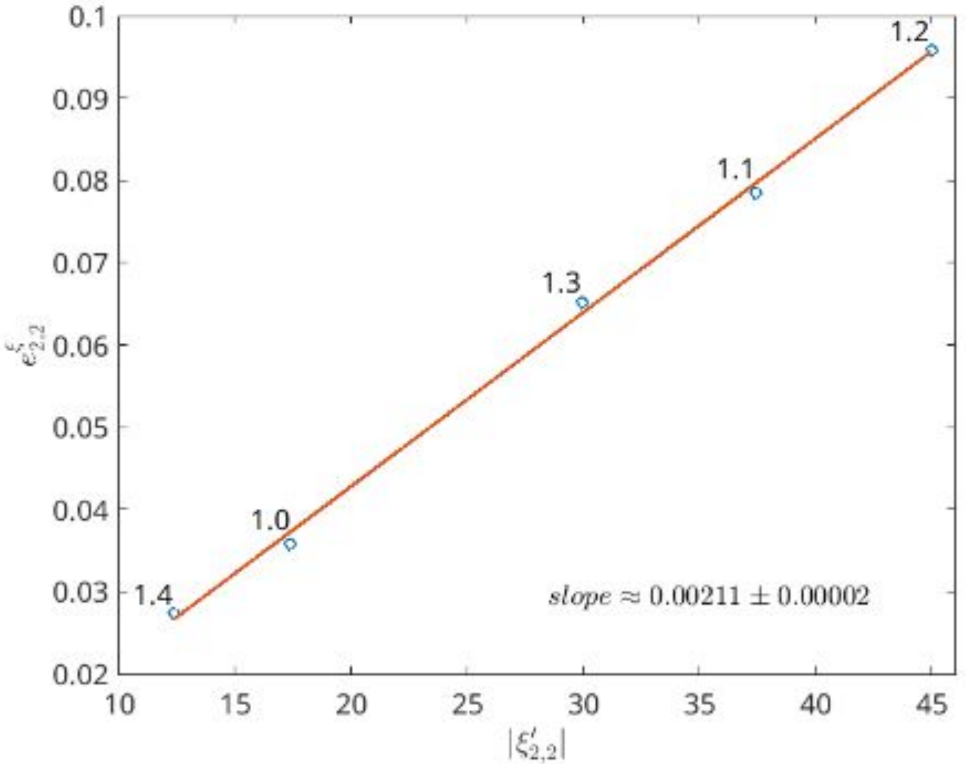
Table I: The parameters used in the expansion (5), and the values of $|\xi'_{2,2}|$ and $e^{\xi}_{2,2}$ for multiple initial data configurations. The table depicts values of those quantities for which the initial Gaussian's center is at $\rho/M = \{1.0, 1.1, 1.2, 1.3, 1.4\}$.

Behavior of Ori-coefficients vs hair

- We plot the Ori-coefficients vs the charge for different ID.
- Data is well described by a linear relationship.

$$e^{\xi}_{2,2} = \alpha|\xi'_{2,2}| + \beta$$

- Demonstration that the charge can be computed through quantities in the exterior.



Key Takeaways

- Attempt to extend the notion of horizon hair to non-axisymmetric gravitational perturbations for extremal Kerr.
- So called only conserved at late-times, so it should interpreted as the first term in the inverse-time expansion of a conjectured conserved charge.
- Can we expect such relationship to hold for higher modes?
- Would the linear trend continue for initial data farther away?

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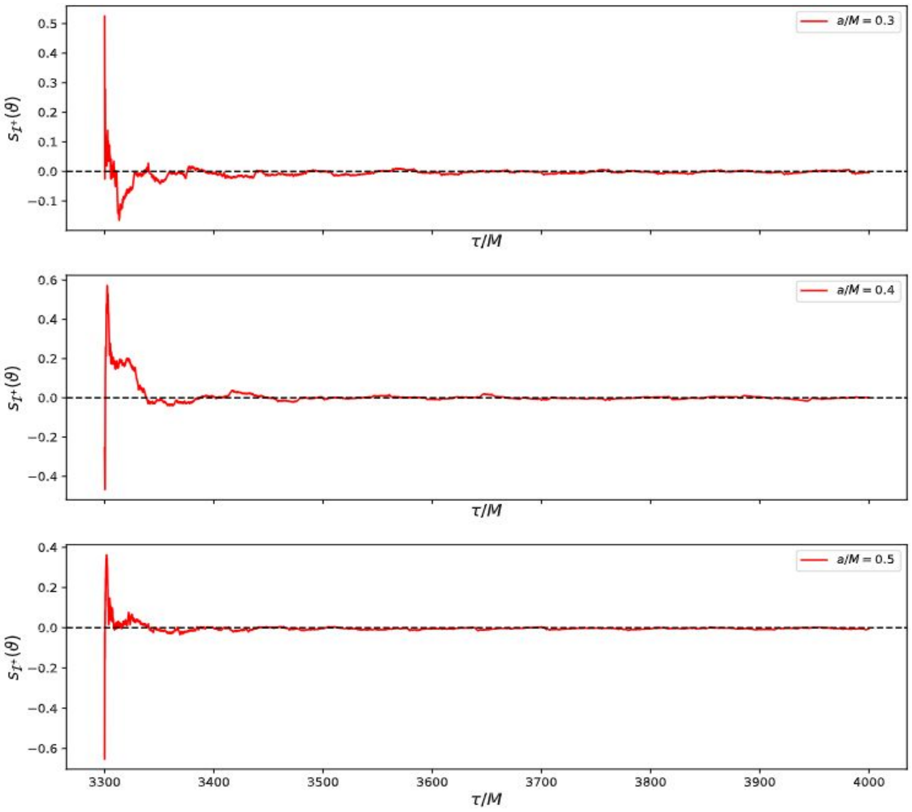
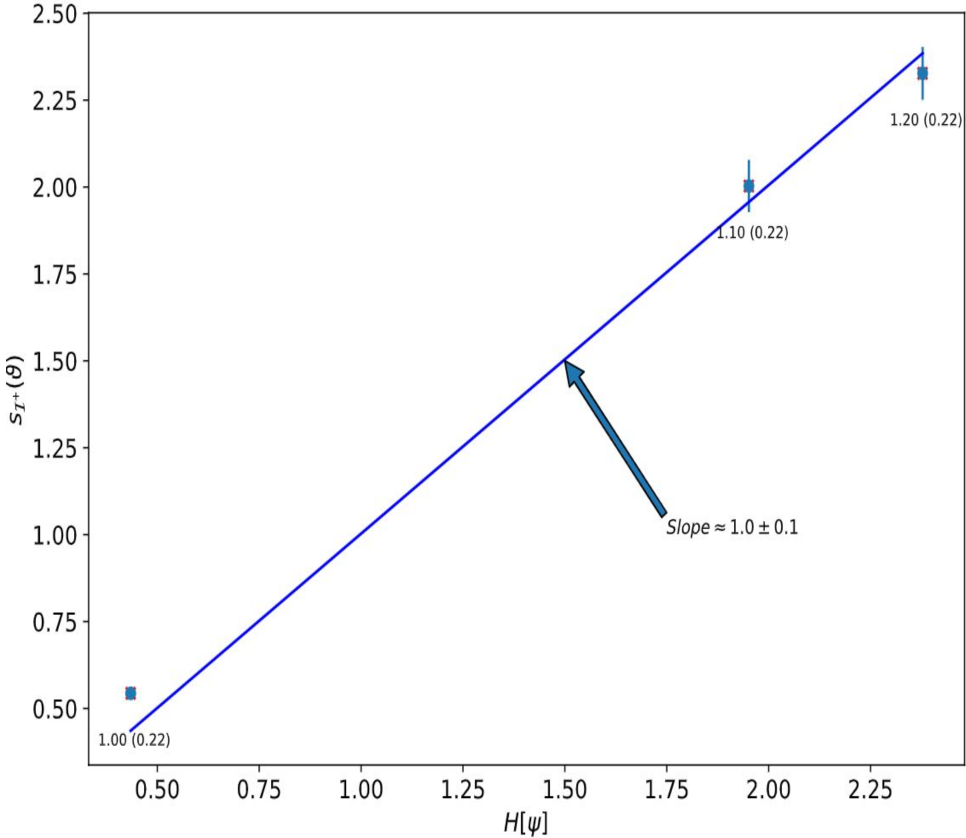
Thank You!

Kerr Plots



a/M = 0.3
a/M = 0.4
a/M = 0.5
$s_{\mathcal{I}^+}(\vartheta)$
τ/M
3300
3400
3500
3600
3700
3800
3900
4000



$s_{\mathcal{I}^+}(\vartheta)$
$H[\psi]$
1.00 (0.22)
1.10 (0.22)
1.20 (0.22)
Slope ≈ 1.0 ± 0.1