Intrinsic rigidity of extremal horizons and black hole uniqueness

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Extremal Black Holes and the Third Law of Black Hole Mechanics
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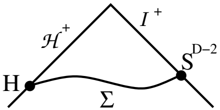
Outline

- Black holes & extremal horizons
- Intrinsic rigidity theorem for extremal horizons [Dunajski, JL '23]
  Matter fields: Maxwell [Colling, Katona, JL '24], general [Colling '25]
- Applications to black hole uniqueness theorems:
  extremal Schwarzschild de Sitter [Katona, JL '23]

Equilibrium black holes in General Relativity

Uniqueness theorem [Israel, Carter, Hawking, Robinson, Mazur, Bunting, '70...]

The DOC of asymptotically flat, stationary, vacuum, analytic, spacetime $(M, \mathbf{g})$, with connected black hole is a Kerr solution.



- Topology [Hawking '72; Chrusciel, Wald '94]: Cross-sections $H \cong S^2$.
  Rigidity [Hawking '72]: stationary $\implies$ axisymmetric or static
- $\mathcal{H}^+$ is Killing horizon: $\nabla_V V = \kappa V$ on $\mathcal{H}^+$, normal Killing V

  Extremal if $\kappa = 0$: uniqueness proven later [Meinel et al '08; Figueras, JL '09; Chrusciel, Nguyen '10]. Need *near-horizon geometry*...

Extremal horizons

- $\mathcal{H}^+$: null hypersurface with normal Killing vector V and $\kappa = 0$
  S: *cross-section*, $n := d - 2$-dimensional, transverse to V

  Intrinsic data on S: Riemannian (induced) metric g, 1-form X defined by $\nabla_W V = -\frac{1}{2} X(W) V$ on $\mathcal{H}^+$ for $W \in T\mathcal{H}^+$.

- $\mathrm{Ric}(\mathbf{g}) = \Lambda \mathbf{g}$ on $\mathcal{H}^+ \subset M \iff$ *quasi-Einstein equation* on S:

$$\mathrm{Ric}(g) = \frac{1}{2} X \otimes X - \frac{1}{2} \mathcal{L}_X g + \Lambda g$$

  Note: intrinsic data decouples from extrinsic data iff $\kappa = 0$!

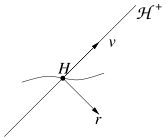
- Extremal isolated horizons also satisfy quasi-Einstein equation
  [Ashtekar, Beetle, Lewandowski, Pawlowski '02...]

Near-horizon geometry

- Isenberg-Moncrief (Gaussian-null) coords near an extremal $\mathcal{H}^+$

  $$\mathbf{g} = r^2 F(r,x)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + 2r\,X_a(r,x)\mathrm{d}v\mathrm{d}x^a + g_{ab}(r,x)\mathrm{d}x^a\mathrm{d}x^b$$

  $V = \partial_v$, ∂_r transverse, $\mathcal{H}^+ = \{r = 0\}$, x^a coords on S



- *Near-horizon geometry* [Reall '02]: spacetime with (g, X, F) replaced by intrinsic data $(g, X, F)_{r=0}$ (scaling limit).
  Einstein iff (S, g, X) quasi-Einstein, $F = \frac{1}{2}|X|^2 - \frac{1}{2}\mathrm{div}_g X + \Lambda$.
  [Chrusciel, Reall, Tod '05; Kunduri, JL '08...]

Static horizons

Static rigidity [Chrusciel, Tod, Reall '05]

Let (S, g, X) be a compact quasi-Einstein surface with $\mathrm{d}X = 0$. Then $X \equiv 0$ and $\mathrm{Ric}(g) = \Lambda g$ (trivial).

- Near-horizon geometry static $V \wedge \mathrm{d}V = 0 \iff \mathrm{d}X = 0$ on S.
- $\mathrm{AdS}_2 \times H^2$, $\mathrm{Mink}_2 \times T^2$ (flat), $\mathrm{dS}_2 \times S^2$ (Nariai)

$$\mathbf{g}_{\mathrm{NH}} = \Lambda r^2 \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + g$$

- $n \geq 3$: $\Lambda \leq 0$ trivial or product of NH BTZ if $\Lambda < 0$. [Bahuaud, Gunasekaran, Kunduri, Woolgar '22; Wylie '23; Kaminski, Lewandowski '24]

Axially symmetric two-horizons

Horizon uniqueness [Hajicek '74; Lewandowski, Pawlowski '02; Kunduri, JL '08]

Any axially-symmetric vacuum extremal horizon with an S^2 cross-section is an extremal Kerr or Kerr-(A)dS horizon.

- Axial symmetry reduces horizon equation to ODE system and general *local* solution can be found even for $\Lambda \neq 0$.
- Global analysis: metric that extends smoothly on S^2 ($\Lambda = 0$)

$$g = \frac{a^2(1+x^2)}{1-x^2}\mathrm{d}x^2 + \frac{4a^2(1-x^2)}{1+x^2}\mathrm{d}\phi^2 \ ,$$

$$X = \Gamma^{-1}K^\flat - \frac{\mathrm{d}\Gamma}{\Gamma}, \qquad K = \partial_\phi, \quad \Gamma = a^2(1+x^2)$$

$|x| < 1$, ∂_ϕ is 2π-periodic Killing field, $a > 0$ constant

Two-horizons

- Intrinsic topology theorem: $R_g = \frac{1}{2}|X|^2 - \text{div}_g X + 2\Lambda$.
  $\Lambda \geq 0$: $\int_S R_g \geq 0$ so $S = S^2, T^2$. T^2 case is trivial $X \equiv 0$.

- $\Lambda < 0$: genus $g > 0$ horizons all trivial $X \equiv 0$, $\text{Ric}(g) = \Lambda g$
  [Dobkowski-Rylko, Kaminski, Lewandowski, Szereszewski '18]

- General solution to horizon quasi-Einstein equation on S^2?

  - Axial symmetry motivated by rigidity theorem for black holes. But is there an intrinsic proof of axial symmetry?

  - Rigidity of linearised perturbs of the extremal Kerr horizon [Jezierski, Kaminski '12; Chrusciel, Szybka, Tod '17; Bauhaud, Gunasekaran, Kunduri, Woolgar '23]

Intrinsic rigidity of extremal horizons

Rigidity theorem [Dunajski, JL '23]

Let (S, g, X) be an n-dimensional compact Riemannian manifold with non-gradient X that satisfies horizon quasi-Einstein equation. Then there exists a Killing vector field K such that $[K, X] = 0$.

- $\mathrm{d}X \neq 0 \iff$ non-static. Complements static classification.
- More rigid than Einstein manifolds!
- $[K, X] = 0 \implies K$ a Killing vector of near-horizon geometry.

  $\Lambda > 0, n > 2$ proof more nontrivial [Colling, Dunajski, Kunduri, JL '24]
  Unified proof for all Λ [Kaminski, Lewandowski '24]

Corollaries

Horizon uniqueness theorem [Dunajski, JL '23]

The extremal Kerr horizon, possibly with cosmological constant, is unique solution to horizon quasi-Einstein eq on S^2.

- This completes classification of vacuum extremal horizons with a compact cross-section for any Λ.

Near-horizon symmetry enhancement theorem [Dunajski, JL '23]

A vacuum near-horizon geometry with compact cross-sections has an isometry algebra of a 2d maximally symmetric space (3d orbits).

- This is a significant generalisation of prior NH-symmetry theorem which *assumes* $U(1)^{n-1}$ symmetry [Kunduri, JL, Reall '07]

Rigidity proof - strategy

- Inspired by Kerr solution, for any function $\Gamma > 0$, let

$$K := \Gamma X + d\Gamma$$

  Idea: try to prove K is a Killing field for suitable choice of Γ.

- Given a 1-form X on a compact Riemannian mfd (S, g) there exists a smooth $\Gamma > 0$ so $\text{div}_g K = 0$. [Tod '92; JL, Reall '13]
  - Elliptic operator $L\psi := -\text{div}_g(d\psi + X\psi)$ on (S, g). Principal eigenfunction exists and is positive $L\psi = \mu\psi$.
  - Integrate over S implies $\mu = 0$. Take Γ to be solution $L\Gamma = 0$.

Rigidity proof – remarkable identity

- Precise form of quasi-Einstein equation implies, for *all* $\Gamma > 0$,

$$\begin{aligned}\tfrac{1}{4}|\mathcal{L}_K g|^2 &= \mathrm{div}_g Y + Z\mathrm{div}_g K \\ Y &:= \tfrac{1}{2}(\mathcal{L}_K g)(K,\cdot) - \tfrac{1}{2}K\Delta\Gamma - \tfrac{1}{2}K\mathrm{div}_g K - \Lambda\Gamma K \\ Z &:= -\frac{1}{2\Gamma}|K|^2 + \tfrac{1}{2}\Delta\Gamma + \tfrac{1}{2}\mathrm{div}_g K + \frac{1}{2\Gamma}\mathcal{L}_K\Gamma + \Lambda\Gamma\end{aligned}$$

- Now take Γ to be principal eigenfunction so $\mathrm{div}_g K = 0$:

$$\int_S |\mathcal{L}_K g|^2 \mathrm{dvol}_g = 0 \qquad \Longrightarrow \qquad \mathcal{L}_K g = 0$$

- Identity reduces to $\mathrm{div}_g Y = 0 \iff -\nabla^2(\mathcal{L}_K\Gamma) = 2\Lambda\mathcal{L}_K\Gamma$; $\Lambda \leq 0 \implies \mathcal{L}_K\Gamma = 0$ so $[K, X] = 0$. $\Lambda > 0$ a bit more work.

  Generalised identity unifies proof $\forall\Lambda$ [Kaminski, Lewandowski '24]

Electrovacuum extremal horizons

Rigidity theorem [Colling, Katona, JL '24]

Let (S, g, X) be a 2d compact cross-section of an electrovacuum extremal horizon with non-gradient X. Then there exists a Killing vector field K such that $[K, X] = \mathcal{L}_K \psi = \mathcal{L}_K \beta = 0$.

- Maxwell data induced on S: electric/magnetic potentials ψ, β
- Remarkable identity generalises: if $K = \Gamma X + \mathrm{d}\Gamma$ for any $\Gamma > 0$

$$\tfrac{1}{4}|\mathcal{L}_K g|^2 + |\nabla(\Gamma\sqrt{\psi^2+\beta^2})|^2 = \mathrm{div}_g \tilde{Y} + \tilde{Z}\,\mathrm{div}_g K$$

- S^2 and axial sym $\implies$ extremal Kerr-Newman$-\Lambda$ horizon [Lewandowski, Pawlowski '02; Kunduri, JL '08]

  Completes classification (higher genus can be ruled out)

Extremal horizons & matter fields

Rigidity theorem [Colling '25]

Let (S, g, X, T, U) be a compact cross-section of extremal horizon with (null) dominant energy condition and X non-gradient. There exists a Killing field K and $[K, X] = \mathcal{L}_K T = \mathcal{L}_K U = 0$.

- Quasi-Einstein on cross-section S with source: T (pull-back of stress tensor $\mathcal{T}_{\mu\nu}$) and U defined by $\mathcal{T}_{\mu\nu} V^\mu = U V_\nu$ on $\mathcal{H}^+$
- General remarkable identity: $\frac{1}{4}|\mathcal{L}_K g|^2 + \gamma = \mathrm{div}_g \tilde{Y} + \tilde{Z} \mathrm{div}_g K$ where $K = \Gamma X + \mathrm{d}\Gamma$ and $\mathcal{T}_{NH}(\ell, \ell) = r^2 \gamma$ for some null ℓ.
- Near-horizon symmetry enhancement theorem again holds. Matter: p-forms, YM, (charged) scalars, all inherit symmetry.

Classification of extremal black holes?

- Given a near-horizon geometry a corresponding black hole solution may not exist or be unique.
- *Transverse derivatives* at extremal horizon: $g^{(n)}_{ab} := \partial^n_r g_{ab}|_{r=0}$. Einstein $\implies$ elliptic problem on S at each order n [Li, JL '15]
- Vacuum axisymmetric $n = 1$ transverse derivatives of extremal Kerr(-Λ) horizon are unique. [Li, JL '15 '18]
- Unique at every order? Don't expect so, no asymptotic input! E.g. Reissner-Nordstrom/Kerr-Newman 1st order not unique [Li, JL '18, Kolanowski '21]

  But in vacuum? Uniqueness of extremal Kerr black hole?

Example: extremal Schwarzschild de Sitter

Uniqueness theorem [Katona, JL '23]

Any analytic Einstein spacetime with $\Lambda > 0$, that contains a static extremal Killing horizon with compact cross-section S, is locally isometric to extremal Schwarzschild-dS or $\mathrm{dS}_2 \times S^2$ (Nariai).

- Inductive proof: if tracefree $\hat{g}_{ab}^{(k)} = 0$ for $k \leq n-1$ then $-\nabla^2 \hat{g}_{ab}^{(n)} = -\Lambda \left(n^2 + n + 2\right) \hat{g}_{ab}^{(n)}$, so $\Lambda > 0 \implies \hat{g}_{ab}^{(n)} = 0$.
- No global assumptions such as asymptotics or topology! Hence rules out (analytic) multi-black holes.
- Classification of 4d static black holes with $\Lambda \neq 0$ open!
    - Static black hole binaries in dS (non-extremal) [Dias et al '23]
    - Uniqueness of Schwarzschild-AdS? [Chrusciel, Delay '05 '17]

Comments

- Constructive uniqueness proof for spacetimes with prescribed near-horizon geometry.

  Generalises to Reissner-Norstrom-dS if $A_H \Lambda \geq 2\pi$ [Katona '24]

- $\Lambda = 0$: unique analytic spacetime containing static extremal horizon, flat T^{d-2} cross-section, $\mathbf{g}_{\text{plane wave}} = 2\mathrm{d}v\mathrm{d}r + g(r)$.
  [Isenberg, Moncrief '82; Katona '24]

- $\Lambda < 0$ extremal hyperbolic Schwarzschild-AdS: uniqueness if $\lambda_n = (n^2 + n - 2)|\Lambda| \notin \mathrm{spec}(\Delta)$ on hyperbolic surface

- Argument that generic extremal black holes with $\Lambda \neq 0$ do not have smooth horizons [Horowitz, Kolanowski, Santos '22]

Summary & Outlook

- Geometry of extremal black holes is a quasi-Einstein manifold. Study horizon topology/geometry independently to black hole.

- Extremal horizons enjoy remarkable intrinsic rigidity. Even for $\Lambda \neq 0$ (no BH no-hair theorems here!).

  - Intrinsic proof of uniqueness of extremal Kerr horizon!

  - Axially symmetric in all dimensions!

  - Near-horizon symmetry enhancement generic

- Uniqueness of (analytic) $\Lambda > 0$ spacetimes containing a *static* extremal horizon. Rotation?

  Does extremal Kerr (Λ) black hole satisfy such uniqueness?