Numerical Simulations of Black Hole Interiors

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Outline

Singularities

BKL conjecture

Tetrad methods

Spacelike singularities in spacetimes with T^3 Cauchy surfaces

Changes for the asymptotically flat case

Can harmonic time slicing foliate the interior?

Conclusion

The Penrose singularity theorem (1965)

Does angular momentum halt the collapse process?

No. Anything close to the Oppenheimer-Snyder toy model has a trapped surface: a surface from which even the outgoing light rays are drawn closer to each other.

The attractive property of gravity makes this convergence of the light rays ever larger, until it becomes infinite.

Something catastrophic happens (though Penrose doesn't tell us what)

BKL picture of singularities

What are singularities like? Belinskii, Khalatnikov and Lifschitz (1970) conjecture that (1) matter doesn't matter and (2) time derivatives are more important than space derivatives.

FLRW cosmologies

$$\begin{aligned} ds^2 &= -dt^2 + a^2(dx^2 + dy^2 + dz^2) \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi\rho}{3} \\ P &= w\rho \\ \rho &\propto a^{-3(1+w)} \end{aligned}$$

Homogeneous, anisotropic cosmology

$$
\begin{aligned}
ds^2 &= -dt^2 + \alpha^2 dx^2 + \beta^2 dy^2 + \gamma^2 dz^2 \\
P &= w\rho \\
a^3 &= \alpha\beta\gamma \\
\rho &\propto a^{-3(1+w)}
\end{aligned}
$$

Impose Einstein field equations

$$
\begin{aligned}
\frac{\dot\alpha}{\alpha} &= \frac{\dot a}{a} + c_1 a^{-3} \\
\frac{\dot\beta}{\beta} &= \frac{\dot a}{a} + c_2 a^{-3} \\
\frac{\dot\gamma}{\gamma} &= \frac{\dot a}{a} + c_3 a^{-3} \\
c_1 + c_2 + c_3 &= 0 \\
\left(\frac{\dot a}{a}\right)^2 &= \frac{8\pi\rho}{3} + k^2 a^{-6} \\
k^2 &= (c_1^2 + c_2^2 + c_3^2)/6
\end{aligned}
$$

So matter doesn't matter (unless it has $w = 1$ like a scalar field).

The idea for the conjecture that time derivatives are more important comes from the notion that cosmological horizons get small near the singularity.

Is this conjecture right? (See Rodnianski and Speck, also Oliynyk and Beyer) But also do numerical simulations and see.

D. Garfinkle, PRL **93**, 161101 (2004)

D. Garfinkle and F. Pretorius, Phys. Rev. D **102**, 124067 (2020) and arXiv:2010.01399

Tetrad methods

The spacetime is described in terms of a coordinate system (t, x^i) and a tetrad $(\mathbf{e}_0, \mathbf{e}_\alpha)$ where both the spatial coordinate index i and the spatial tetrad index α go from 1 to 3.

The commutators of the tetrad components are decomposed as follows:

$$
\begin{aligned}
[\mathbf{e}_0, \mathbf{e}_\alpha] &= \dot{u}_\alpha \mathbf{e}_0 - (H\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta)\mathbf{e}_\beta \\
[\mathbf{e}_\alpha, \mathbf{e}_\beta] &= (2a_{[\alpha}\delta_{\beta]}{}^\gamma + \epsilon_{\alpha\beta\delta} n^{\delta\gamma})\mathbf{e}_\gamma ,
\end{aligned}
$$

where $n^{\alpha\beta}$ is symmetric, and $\sigma^{\alpha\beta}$ is symmetric and trace free.

Scale invariant variables

The scale invariant tetrad variables are defined by $\boldsymbol{\partial}_0 \equiv \mathbf{e}_0/H$ and $\boldsymbol{\partial}_\alpha \equiv \mathbf{e}_\alpha/H$ while scale invariant versions of the other gravitational variables are given by

$$\{E_\alpha{}^i, \Sigma_{\alpha\beta}, A^\alpha, N_{\alpha\beta}\} \equiv \{e_\alpha{}^i, \sigma_{\alpha\beta}, a^\alpha, n_{\alpha\beta}\}/H.$$

Note that the relation between the scale invariant tetrad variables and the coordinate derivatives is

$$\boldsymbol{\partial}_0 = \mathcal{N}^{-1}\partial_t \quad (1)$$
$$\boldsymbol{\partial}_\alpha = E_\alpha{}^i\partial_i, \quad (2)$$

where $\mathcal{N} = NH$ is the scale invariant lapse.

Time coordinate

The time coordinate t is chosen so that

$$e^{-t} = 3H.$$

Here t and H are dimensionless quantities. Surfaces of constant time are constant mean curvature surfaces and the singularity is approached as $t \to -\infty$.

$$C^{abcd}C_{abcd} = e^{-4t}F(\Sigma_{\alpha\beta}, N_{\alpha\beta}, \dots)$$

Vacuum Einstein equations yield evolution equations for the scale invariant variables

$$
\begin{aligned}
\partial_t E_\alpha{}^i &= E_\alpha{}^i - \mathcal{N}(E_\alpha{}^i + \Sigma_\alpha{}^\beta E_\beta{}^i), \\
\partial_t A_\alpha &= A_\alpha + \tfrac{1}{2}\Sigma_\alpha{}^\beta \boldsymbol{\partial}_\beta \mathcal{N} - \boldsymbol{\partial}_\alpha \mathcal{N} \\
&\quad + \mathcal{N}\left(\tfrac{1}{2}\boldsymbol{\partial}_\beta \Sigma_\alpha{}^\beta - A_\alpha - \Sigma_\alpha{}^\beta A_\beta\right) \\
\partial_t N^{\alpha\beta} &= N^{\alpha\beta} - \epsilon^{\gamma\delta(\alpha}\Sigma_\delta{}^{\beta)}\boldsymbol{\partial}_\gamma \mathcal{N} \\
&\quad + \mathcal{N}\left(-N^{\alpha\beta} + 2N^{(\alpha}{}_\gamma \Sigma^{\beta)\gamma} - \epsilon^{\gamma\delta(\alpha}\boldsymbol{\partial}_\gamma \Sigma_\delta{}^{\beta)}\right) \\
\partial_t \Sigma_{\alpha\beta} &= \Sigma_{\alpha\beta} + \boldsymbol{\partial}_{<\alpha}\boldsymbol{\partial}_{\beta>}\mathcal{N} + A_{<\alpha}\boldsymbol{\partial}_{\beta>}\mathcal{N} + \epsilon_{\gamma\delta(\alpha}N_{\beta)}{}^\delta \boldsymbol{\partial}^\gamma \mathcal{N} \\
&\quad + \mathcal{N}\Big[-3\Sigma_{\alpha\beta} - \boldsymbol{\partial}_{<\alpha}A_{\beta>} - 2N_{<\alpha}{}^\gamma N_{\beta>\gamma} + N^\gamma{}_\gamma N_{<\alpha\beta>} \\
&\quad + \epsilon_{\gamma\delta(\alpha}(\boldsymbol{\partial}^\gamma N_{\beta)}{}^\delta - 2A^\gamma N_{\beta)}{}^\delta)\Big].
\end{aligned}
$$

as well as constraint equations

$$
\begin{aligned}
0 &= \epsilon^{\alpha\beta\lambda}[\boldsymbol{\partial}_\alpha E_\beta{}^i - A_\alpha E_\beta{}^i] - N^{\lambda\gamma} E_\gamma{}^i, \\
0 &= \boldsymbol{\partial}_\alpha N^{\alpha\gamma} + \epsilon^{\alpha\beta\gamma} \partial_\alpha A_\beta - 2A_\alpha N^{\alpha\gamma}, \\
0 &= \boldsymbol{\partial}_\beta \Sigma_\alpha{}^\beta - 3\Sigma_\alpha{}^\beta A_\beta - \epsilon_{\alpha\beta\gamma} N^{\beta\delta} \Sigma_\delta{}^\gamma, \\
0 &= 1 + \tfrac{2}{3}\boldsymbol{\partial}_\alpha A^\alpha - A^\alpha A_\alpha - \tfrac{1}{6} N^{\alpha\beta} N_{\alpha\beta} + \tfrac{1}{12}(N^\gamma{}_\gamma)^2 - \tfrac{1}{6}\Sigma^{\alpha\beta}\Sigma_{\alpha\beta}.
\end{aligned}
$$

and an elliptic equation for $\mathcal{N}$

$$
-\boldsymbol{\partial}^\alpha \boldsymbol{\partial}_\alpha \mathcal{N} + 2A^\alpha \boldsymbol{\partial}_\alpha \mathcal{N} + \mathcal{N}(3 + \Sigma_{\alpha\beta}\Sigma^{\alpha\beta}) = 3
$$

results of the simulations

Spatial derivatives become negligible compared to time derivatives in the simulations, not because they are small but because they are multiplied by ${E_\alpha}^i$ which is becoming small as the singularity is approached.

In the vacuum case there is a series of epochs of approximately constant $\Sigma_{\alpha\beta}$ and approximately zero $N_{\alpha\beta}$. These epochs are punctuated by short transitions in which $N_{\alpha\beta}$ grows and then decays all while $\Sigma_{\alpha\beta}$ takes on new constant values. This is just what one would expect from the properties of general homogeneous, anisotropic cosmologies.

In the case of free scalar field matter, things are similar, except that there is a last transition after which the singularity is approached with $\Sigma_{\alpha\beta}$ constant and $N_{\alpha\beta}$ zero.

What changes are needed in the asymptotically flat case?

There is a null singularity (see Israel and Poisson, Ori, Dafermos and Luk)

No CMC slicing and no scale invariant variables: try harmonic time slicing and unrescaled tetrad variables.

No symmetry so we need parallel code (PAMR)

The evolution equations for the tetrad quantities are as follows:

$$
\begin{aligned}
\mathbf{e}_0(e_\alpha{}^i) &= -(H\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta)e_\beta{}^i \\
\mathbf{e}_0(N) &= 3HN \\
\mathbf{e}_0(\dot{u}_\alpha) &= 3\mathbf{e}_\alpha(H) + 2H\dot{u}_\alpha - \sigma_\alpha{}^\beta \dot{u}_\beta \\
\mathbf{e}_0(H) &= \tfrac{1}{3}\mathbf{e}_\alpha(\dot{u}^\alpha) - H^2 + \tfrac{1}{3}\dot{u}_\alpha(\dot{u}^\alpha - 2a^\alpha) - \tfrac{1}{3}\sigma_{\alpha\beta}\sigma^{\alpha\beta} \\
\mathbf{e}_0(a_\alpha) &= 3\mathbf{e}_\alpha(H) - \tfrac{3}{2}\mathbf{e}_\beta(\sigma_\alpha{}^\beta) - H(\dot{u}_\alpha + a_\alpha) \\
&+ \sigma_\alpha{}^\beta\left(\tfrac{1}{2}\dot{u}_\beta + 5a_\beta\right) + 2\epsilon_{\alpha\beta\gamma}n^{\beta\delta}\sigma_\delta{}^\gamma \\
\mathbf{e}_0(n^{\alpha\beta}) &= -\epsilon^{\gamma\delta(\alpha}\mathbf{e}_\gamma(\sigma_\delta{}^{\beta)}) - Hn^{\alpha\beta} + 2n^{(\alpha}{}_\lambda\sigma^{\beta)\lambda} - \epsilon^{\gamma\delta(\alpha}\dot{u}_\gamma\sigma_\delta{}^{\beta)} \\
\mathbf{e}_0(\sigma_{\alpha\beta}) &= \mathbf{e}_{<\alpha}(\dot{u}_{\beta>}) - \mathbf{e}_{<\alpha}(a_{\beta>}) - 3H\sigma_{\alpha\beta} + \dot{u}_{<\alpha}\dot{u}_{\beta>} \\
&+ a_{<\alpha}\dot{u}_{\beta>} + \epsilon_{\gamma\delta(\alpha}\mathbf{e}^\gamma(n_{\beta)}{}^\delta) + \epsilon_{\gamma\delta(\alpha}n_{\beta)}{}^\delta(\dot{u}^\gamma - 2a^\gamma) \\
&- 2n_{<\alpha}{}^\gamma n_{\beta>\gamma} + nn_{<\alpha\beta>}
\end{aligned}
$$

The constraint quantities are

$$
\begin{aligned}
\mathcal{C}_{u1} &= \dot{u}_\alpha - N^{-1} e_\alpha(N) \\
\mathcal{C}_{\mathrm{com}} &= \epsilon^{\alpha\beta\lambda} \left(e_\alpha({e_\beta}^i) - a_\alpha {e_\beta}^i \right) - n^{\lambda\gamma} {e_\gamma}^i \\
\mathcal{C}_{u2} &= \epsilon^{\alpha\beta\lambda} \left(e_\beta(\dot{u}_\alpha) + a_\alpha \dot{u}_\beta \right) + n^{\lambda\gamma} \dot{u}_\gamma \\
\mathcal{C}_J &= e_\alpha(n^{\alpha\delta}) + \epsilon^{\alpha\beta\delta} e_\alpha(a_\beta) - 2 a_\alpha n^{\alpha\delta} \\
\mathcal{C}_C &= e_\beta({\sigma_\alpha}^\beta) - 2 e_\alpha(H) - 3 {\sigma_\alpha}^\beta a_\beta - \epsilon_{\alpha\beta\gamma} n^{\beta\delta} {\sigma_\delta}^\gamma \\
\mathcal{C}_G &= 4 e^\alpha(a_\alpha) + 6H^2 - 6 a^\alpha a_\alpha - n^{\alpha\beta} n_{\alpha\beta} + \tfrac{1}{2} n^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta}
\end{aligned}
$$

spacelike singularities and null singularities

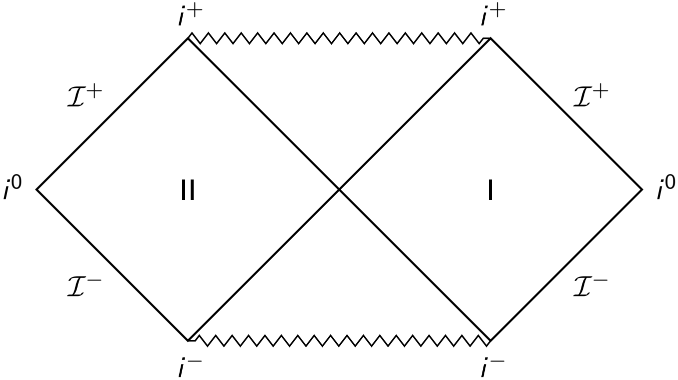

Figure: Extended Schwarzschild spacetime

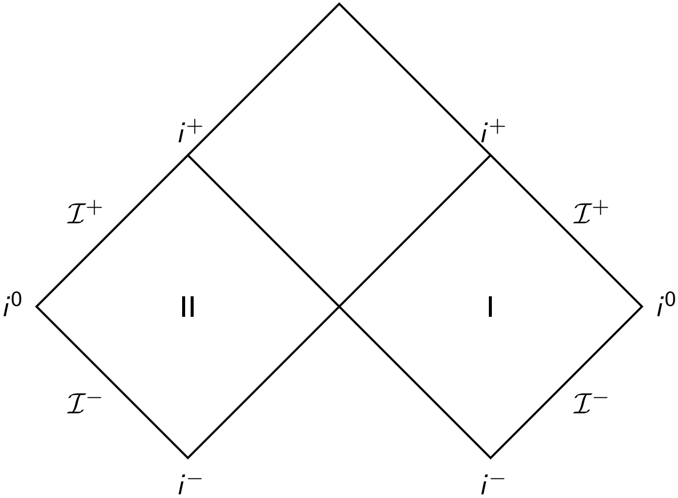

Figure: (part of) Reissner-Nordstrom spacetime

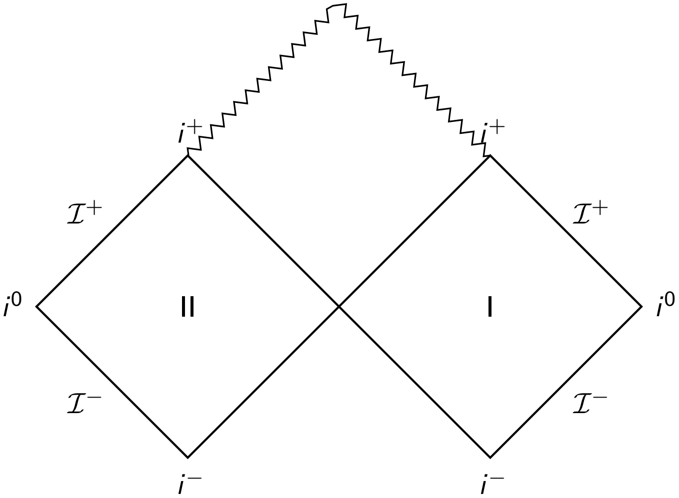

Figure: perturbed Reissner-Nordstrom spacetime

Reissner-Nordstrom: appropriate foliation

(DG, CQG **42**, 195005 (2025) and arxiv:2503.20969)

What is a foliation appropriate for numerical simulations and that asymptotically approaches the inner horizon?

$$ds^2 = -F dt^2 + F^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3}$$

$$F = (r - r_+)(r - r_-)/r^2 \tag{4}$$

null coordinate v

$$v = t + \int F^{-1} dr \tag{5}$$

$$ds^2 = -F dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6}$$

Harmonic time function T

$$T = v - r - 2M \ln(r - r_-) \tag{7}$$

Inner horizon approached as $T \to \infty$.

How do we characterize the limiting behavior as the inner horizon is approached?

By using a rescaled tetrad.

Tetrad (e_0, e_1, e_2, e_3)

e_0 is normal to constant T hypersurfaces.

e_1 is in radial direction orthogonal to e_0

e_2 is in θ direction

e_3 is in ϕ direction

Define H to be 1/3 of mean curvature

Define rescaled tetrad

$(E_0, E_1, E_2, E_3) = (e_0, e_1, e_2, e_3)/H$

For spacelike singularities, E_0 tends to a nonzero quantity as $T \to \infty$ but E_1, E_2 and E_3 all vanish in this limit

In contrast, for our tetrad, E_0 and E_1 tend to nonzero limits which are null as $T \to \infty$ while E_2 and E_3 vanish in this limit

Conclusions

We have a well tested numerical method for simulating spacelike singularities in the compact Cauchy surface case.

This method has been adapted and appropriately modified to tackle the case of black hole interiors.

Now we need to do the simulations and see what happens.