Third law of black hole mechanics - supersymmetry and numerics

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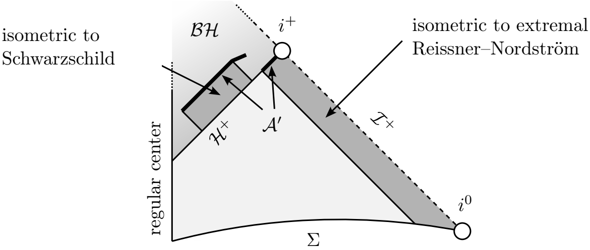
HSR arxiv:2410.11956 (PRD)
A. McSharry & HSR arxiv:2507.06870 (PRD)
M. Gadioux, J. Santos & HSR arxiv:2512.10008
J. Crump, M. Gadioux, J. Santos & HSR, to appear

Kehle-Unger third-law violating solutions (2022)

Einstein-Maxwell theory coupled to a massless charged scalar field. Spherically symmetric gravitational collapse of the scalar field can result in formation of an exactly extremal Reissner-Nordström black hole in finite advanced time, with an intermediate phase in which the spacetime is exactly Schwarzschild at the horizon:



(image credit: Kehle and Unger)

Bounded charge to mass ratio

The third-law violating solutions of Kehle and Unger involve a massless charged scalar field or massless Vlasov matter: matter with large charge to mass ratio.

What happens if the charge to mass ratio of matter is bounded?

Local mass-charge inequality

We'll consider Einstein-Maxwell theory coupled to matter satisfying the local mass-charge inequality of Gibbons & Hull (1981):

$$T_{00}^{(m)} \geq \sqrt{T_{0i}^{(m)} T_{0i}^{(m)} + J_0^2 + \tilde{J}_0^2}$$

where indices $(0, i)$ refer to an arbitrary orthonormal frame, $T_{ab}^{(m)}$ is the energy-momentum tensor of matter (excluding the Maxwell field) and J_a, $\tilde{J}_a$ are the electric and magnetic currents of matter.

This is a strengthened version of the dominant energy condition.

For a scalar field of mass m and charge q it is equivalent to $m \geq |q|$.

Global mass-charge inequality

If matter satisfies the local mass-charge inequality then a strengthened version of the positive mass theorem applies.

If Σ is a complete asymptotically flat hypersurface then the ADM mass M and electric and magnetic charges Q, P measured at spatial infinity satisfy the “BPS bound” (Gibbons & Hull 1981)

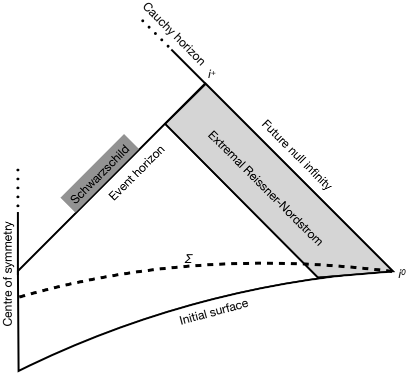
$$M \geq \sqrt{Q^2 + P^2}$$

The proof involves spinors. If the inequality is saturated then there exists a “supercovariantly constant” spinor ϵ in $D(\Sigma)$

$$\hat{\nabla}_a \epsilon \equiv \nabla_a \epsilon + \frac{1}{4} F_{bc} \gamma^b \gamma^c \gamma_a \epsilon = 0$$

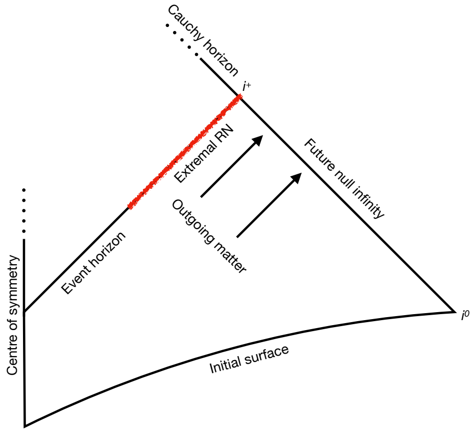
A spacetime admitting such a spinor globally is said to be supersymmetric. Example: extremal RN.

This immediately excludes spacetimes of the form constructed by Kehle and Unger:



Spacetime has $M = |Q|$ (and $P = 0$) so saturates BPS bound. Hence there exists a supercovariantly constant spinor ϵ. But from ϵ we can construct a causal vector $X^a \equiv \bar{\epsilon}\gamma^a\epsilon$ which is *Killing*. So spacetime is time-independent: contradiction!

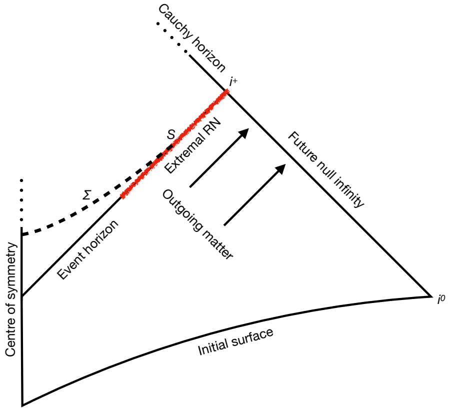
But: method of Kehle and Unger can also be used to construct larger class of solutions that are exactly extremal RN near the horizon, but have $M > \sqrt{Q^2 + P^2}$ at spatial infinity.



The argument just described does not exclude the existence of such solutions in theories satisfying the local mass-charge inequality.

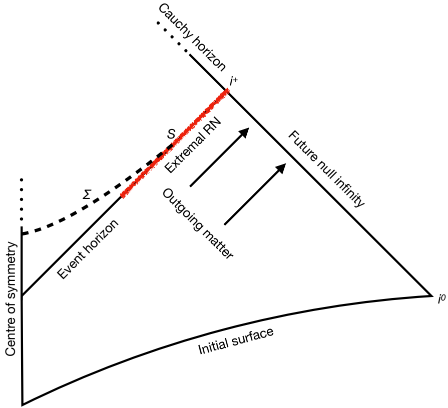
Compact interior

We are considering the possibility of forming an extremal RN black hole in gravitational collapse. In such a situation, the black hole would have *compact interior*, i.e., a horizon cross-section S would have $S = \partial\Sigma$ for a compact spacelike surface Σ. (The maximal analytic extension of extremal RN does *not* have compact interior because of the singularity at $r = 0$.)



The main results

Theorem (third law). If matter satisfies the local mass-charge inequality and S has the same metric, Maxwell field and extrinsic curvature as a horizon cross-section of extremal RN then one *cannot* write $S = \partial\Sigma$ with Σ a compact spacelike surface, i.e., S does not have compact interior.



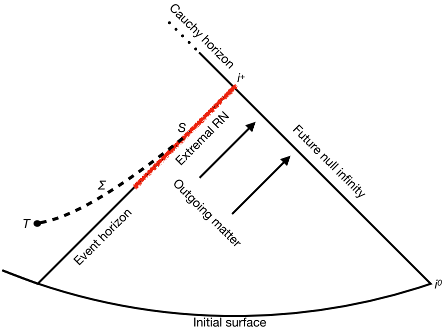
If matter satisfies the local mass-charge inequality then an extremal RN black hole cannot form in gravitational collapse.

Hence a non-extremal black hole cannot become extremal *if the initial black hole was formed from collapse*.

What if the initial black hole was not formed in collapse e.g. if we start with a a *two-sided* non-extremal black hole? Could we make it extremal by throwing in charged matter?

Israel: "non-extremal" means "there exists a trapped surface"

Theorem (McSharry-HSR 2025) Let Σ be a compact spacelike surface with $\partial\Sigma = S \cup T$ where T is an outer trapped surface. Assume that matter satisfied the local mass-charge inequality on Σ. Then S cannot be a surface with the same metric, Maxwell field and extrinsic curvature as a horizon cross-section of extremal RN . In other words, this is impossible:



Anti-de Sitter black holes

$d = 4$ Einstein-Maxwell theory with negative cosmological constant and charged matter satisfying local mass-charge inequality.

Supercovariant derivative modified: now acts on charged spinor field.

Extremal RN-AdS is not supersymmetric but there exists a 1-parameter family of supersymmetric ($\Rightarrow$ extremal) Kerr-Newman-AdS black holes (Kostalecky-Perry 1995).

McSharry-HSR 2025: we proved that the third law holds for these black holes.

Extremal BTZ is also supersymmetric so our methods can be used to prove a third law, for matter satisfying DEC (to appear).

Vacuum black holes

Our results suggest that whether or not the third law is violated might depend on the matter model.

However, Kehle and Unger conjecture that it should be possible to form a black hole that is exactly *extremal Kerr* after a finite advanced time, starting from regular *vacuum* initial data (gravitational collapse of gravitational waves).

If correct, this implies that the third law is false for any matter model.

They proved (2023) that regular vacuum initial data can give a spacetime that is exactly a slowly-rotating ($|a| \ll M$) non-extremal Kerr black hole after a finite advanced time.

Constructing third law violating solutions in 4d vacuum looks difficult.

5d vacuum gravity (Crump, Gadioux, Santos & HSR)

In 5d vacuum gravity the analogue of Kerr is the Myers-Perry solutions, parameterized by M, J_1, J_2 where J_i are angular momenta in two orthogonal 2-planes.

Generically the isometry group is $\mathbb{R} \times U(1) \times U(1)$ and the solution depends on two coordinates, just like Kerr.

However if $J_1 = \pm J_2$ then the isometry group is $\mathbb{R} \times SU(2) \times U(1)$ and the metric depends on only one coordinate:

$$-f(r)dt^2 + g(r)dr^2 + \frac{r^2}{4}\left[\sigma_1^2 + \sigma_2^2 + h(r)(\sigma_3 + \Omega(r)dt)^2\right]$$

where σ_i are left-invariant 1-forms on $SU(2) \sim S^3$.

This belongs to much larger class of *dynamical* spacetimes with $SU(2)$ isometry group (with S^3 orbits), depending on only 2 coordinates (e.g. Bizon, Chmaj & Schmidt 2005).

We'll seek third-law violating solutions within the class of $SU(2)$-symmetric spacetimes.

These describe a Schwarzschild black hole evolving to an extremal MP black hole. Can also construct solutions describing gravitational collapse to form extremal MP.

We start from an Ansatz for a subset of $SU(2)$-symmetric spacetimes for which the equations of motion are qualitatively similar to those of Kehle & Unger for the Einstein-Maxwell-charged scalar system:

Charged scalar $\Phi \leftrightarrow$ some deformation of S^3

Maxwell 1-form $\leftrightarrow$ twist 1-form describing rotation of S^3

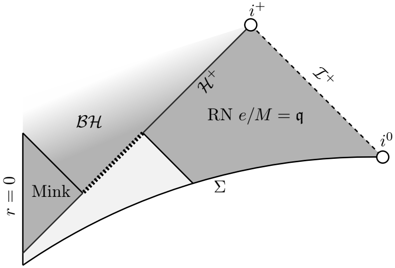
Quasilocal charge $Q \leftrightarrow$ quasilocal angular momentum J

We try to construct solutions using characteristic gluing (Aretakis, Czimek & Rodnianski 2023-25)

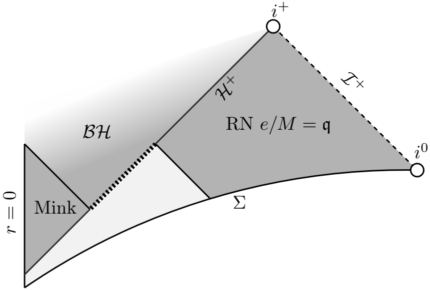
Characteristic gluing

Einstein-Maxwell-charged scalar in spherical symmetry.

Kehle & Unger construct solutions by specifying data along pairs of characteristic (i.e. null) surfaces and solving forwards/backwards in time. Simplest example describes formation of RN, either subextremal or extremal:



(image credit: Kehle and Unger)



Double null coordinates (U, V).

The free data is $\Phi(V)$ on the dotted line. Must be chosen to ensure matching of Q and sufficient differentiability across the horizon.

Q and $\partial_U^j \Phi$ satisfy transport eqs along dotted line so uniquely determined by either their Minkowski (above) or RN (below) values. Hence these will agree everywhere along the dotted line if they agree at the corners. For a C^k solution this must be true for $j \leq k$ so have $2k + 1$ real equations.

Summary: to achieve C^k gluing we need to choose $\Phi(V)$ such that $2k+1$ equations hold.

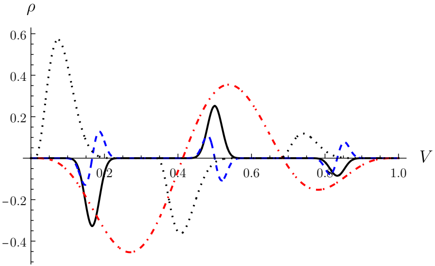
Kehle & Unger make an Ansatz for $\Phi(V)$ containing $2k+1$ real parameters α_i

For large eM (e = scalar charge), matching of Q defines a sphere S^{2k} in $\mathbb{R}^{2k+1}$. The remaining equations are odd under $\alpha_i \to -\alpha_i$ (arises from $\Phi \to -\Phi$ symmetry). The Borsuk-Ulam theorem then guarantees existence of a solution for any k.

The proof doesn't tell us anything about the form of the solution!

Numerics for RN (Gadioux, Santos & HSR 2025)

Profile solving C^1 gluing constraints for 4 different Ansätze:



Can also include a mass parameter m for the scalar field: the largest m/e for which we achieve C^k gluing is about 0.2, 0.04, 0.01 for $k = 0, 1, 2$, in agreement with my result (that gluing is impossible for $m/e \geq 1$).

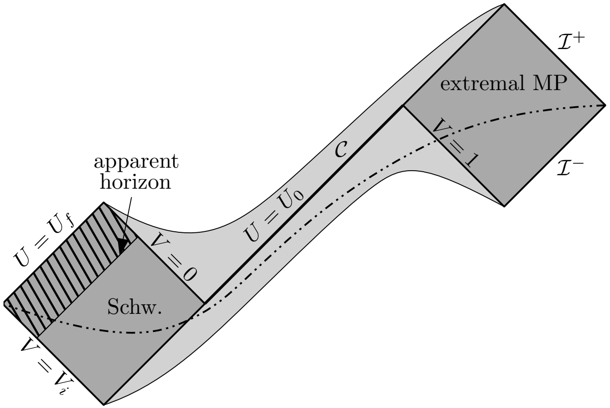
Back to 5d gravity

The Kehle-Unger proof exploits the presence of a free parameter e (eM is taken large), and also the $\Phi \to -\Phi$ symmetry. We don't have these in our 5d gravity system. So we use numerics to look for a solution of gluing.

In the RN case we could increase e until we found a numerical solution - not possible here!

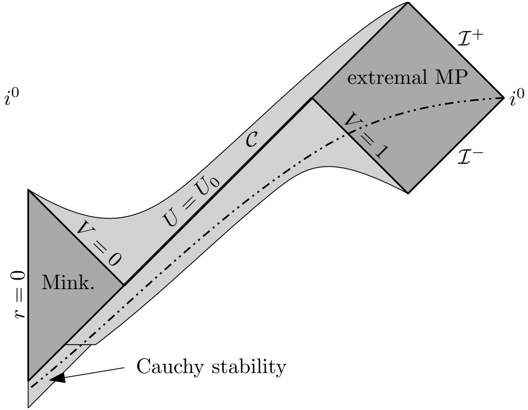
Numerically, the difficulty of C^k gluing increases with k.

Third law violation in 5d vacuum gravity



C^2 gluing of Schwarzschild to extremal Myers-Perry. Violation of third law!

Gravitational collapse to extremal MP



Gluing of Minkowski to extremal Myers-Perry. Formation of extremal MP in gravitational collapse of gravitational waves.

Solution loses 2 derivatives at origin: need C^4 gluing for C^2 solution. We're nearly there...

Discussion

The third law can be violated for Einstein-Maxwell theory coupled to matter with a large charge to mass ratio.

I've proved that the third law holds for *supersymmetric* black holes if matter satisfies the local mass-charge inequality. This covers extremal Reissner-Norstrom (if $\Lambda = 0$) or supersymmetric Kerr-Newman-AdS (if $\Lambda < 0$).

I think this should generalize to supersymmetric black holes for various other theories in various dimensions (e.g. extremal BTZ).

Could a black hole formed in collapse approach a supersymmetric black hole *asymptotically*, i.e., at infinite advanced time?

Is the result sharp? If the local mass-charge inequality is violated (in a sensible matter model) then do there exist third law violating solutions? (Kehle student: yes, for massive charged dust.)

We've used numerical methods to determine the gluing solution on the horizon, for third law violating solutions of Einstein-Maxwell-scalar or 5d vacuum gravity. It should be possible to construct the solutions away from the horizon using numerics.

We've shown that third law is violated in 5d vacuum gravity. Seems likely that same is true in 4d, as conjectured by Kehle and Unger. Constructing these solutions using gluing will be much harder than in 5d but is worth investigating numerically.

If third law is violated in 4d vacuum gravity then extremal Kerr will violate third law even in theories where extremal RN does not, i.e., third law violated by non-susy black holes but not by susy holes.

A different version of the third law asserts that entropy should vanish at zero temperature. Violated classically but recent work (Iliesu, Turiaci, Murthy, . . .) suggests that quantum effects might enforce this version of the third law for non-susy holes but not for susy ones, i.e., the opposite of the above situation!