The moduli space of dynamical spherically symmetric black hole spacetimes and the extremal threshold

Christoph Kehle

MIT

ICERM

January 7, 2025

joint work with Y. Angelopoulos and R. Unger

MODULI SPACE

Spherically symmetric spacetimes $(\mathcal{M}^{3+1}, g)$ solving

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu} = 2T_{\mu\nu}. \tag{1}$$

In this talk: *Classical* relativity with matter fields $T = T_{\text{Maxwell}}, T_{\text{scalar field}}, T_{\text{Vlasov}}$.

Moduli space

Spherically symmetric spacetimes $(\mathcal{M}^{3+1}, g)$ solving

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu} = 2T_{\mu\nu}. \tag{1}$$

In this talk: *Classical* relativity with matter fields $T = T_{\text{Maxwell}}, T_{\text{scalar field}}, T_{\text{Vlasov}}$.

What is the moduli space?

$$\{\text{spherically symmetric solutions of Einstein–matter systems}\}/\{\text{Diff}\}$$

MODULI SPACE

Spherically symmetric spacetimes $(\mathcal{M}^{3+1}, g)$ solving

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu} = 2T_{\mu\nu}. \tag{1}$$

In this talk: *Classical* relativity with matter fields $T = T_{\mathrm{Maxwell}}, T_{\mathrm{scalar\ field}}, T_{\mathrm{Vlasov}}$.

What is the moduli space?

$$\{\text{spherically symmetric solutions of Einstein–matter systems}\}/\{\text{Diff}\}$$

Motivation: Study solutions which dynamically evolve in gravitational collapse (i.e. from asymptotically flat, regular, 1-ended data).

Moduli space

Spherically symmetric spacetimes $(\mathcal{M}^{3+1}, g)$ solving

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu} = 2T_{\mu\nu}. \tag{1}$$

In this talk: *Classical* relativity with matter fields $T = T_{\text{Maxwell}}, T_{\text{scalar field}}, T_{\text{Vlasov}}$.

What is the moduli space?

$$\{\text{spherically symmetric solutions of Einstein–matter systems}\}/\{\text{Diff}\}$$

Motivation: Study solutions which dynamically evolve in gravitational collapse (i.e. from asymptotically flat, regular, 1-ended data).

$$\mathfrak{M} = \{\text{MGHD of spherically symmetric, asym. flat data on } \mathbb{R}^3 \text{ for (1)}\}/\{\text{Diff}\}$$

MODULI SPACE

Spherically symmetric spacetimes $(\mathcal{M}^{3+1}, g)$ solving

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu} = 2T_{\mu\nu}. \tag{1}$$

In this talk: *Classical* relativity with matter fields $T = T_{\mathrm{Maxwell}}, T_{\mathrm{scalar\ field}}, T_{\mathrm{Vlasov}}$.

What is the moduli space?

$$\{\text{spherically symmetric solutions of Einstein–matter systems}\}/\{\text{Diff}\}$$

Motivation: Study solutions which dynamically evolve in gravitational collapse (i.e. from asymptotically flat, regular, 1-ended data).

$$\mathfrak{M} = \{\text{MGHD of spherically symmetric, asym. flat data on } \mathbb{R}^3 \text{ for (1)}\}/\{\text{Diff}\}$$

- Natural to parametrize the moduli space by initial data.

Moduli space

Spherically symmetric spacetimes $(\mathcal{M}^{3+1}, g)$ solving

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu} = 2T_{\mu\nu}. \tag{1}$$

In this talk: *Classical* relativity with matter fields $T = T_{\mathrm{Maxwell}}, T_{\mathrm{scalar\ field}}, T_{\mathrm{Vlasov}}$.

What is the moduli space?

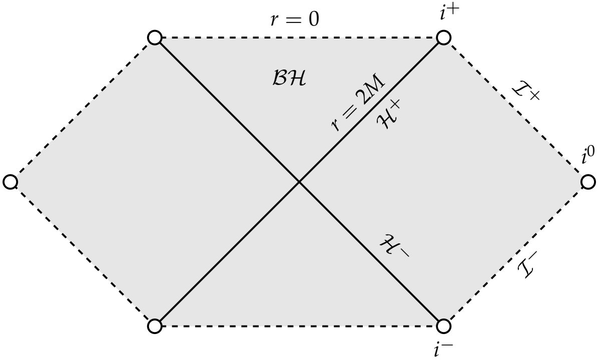
$$\{\text{spherically symmetric solutions of Einstein–matter systems}\}/\{\mathrm{Diff}\}$$

Motivation: Study solutions which dynamically evolve in gravitational collapse (i.e. from asymptotically flat, regular, 1-ended data).

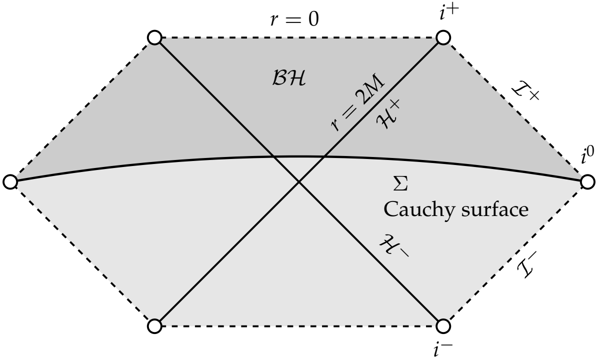
$$\mathfrak{M} = \{\text{MGHD of spherically symmetric, asym. flat data on } \mathbb{R}^3 \text{ for (1)}\}/\{\mathrm{Diff}\}$$

- Natural to parametrize the moduli space by initial data.
- Regular center excludes the Schwarzschild and Reissner–Nordström family (except Minkowski space).

Refresher on Schwarzschild

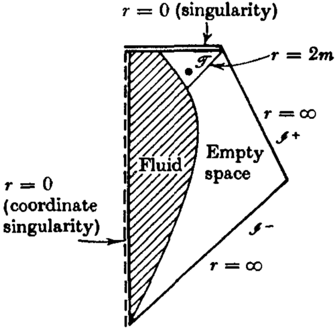


REFRESHER ON SCHWARZSCHILD



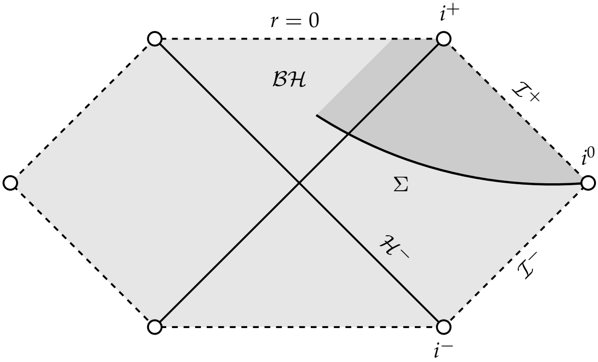
Maximally extended Schwarzschild is the unique maximal Cauchy development of the data induced on a spacelike hypersurface $\Sigma \cong \mathbb{R} \times S^2$ not on $\mathbb{R}^3$.

REFRESHER ON GRAVITATIONAL COLLAPSE



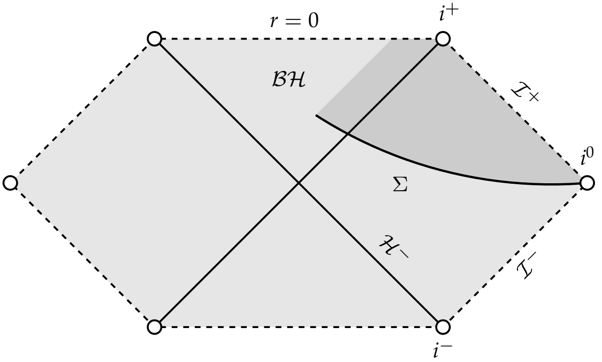
- Penrose diagram of gravitational collapse (Oppenheimer–Snyder, ’39)
- Black hole formation from regular initial data.
- Oppenheimer–Snyder collapse $\in \mathfrak{M}$.

Refresher on gravitational collapse



- Darker shaded region of Schwarzschild is isometric to the vacuum region of the Oppenheimer–Snyder collapse.
- In this sense, Schwarzschild relevant for moduli space of grav. collapse $\mathfrak{M}$.

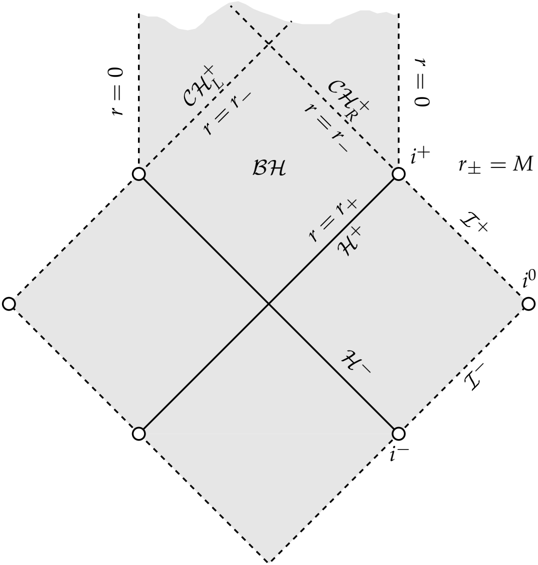
Refresher on gravitational collapse



- Darker shaded region of Schwarzschild is isometric to the vacuum region of the Oppenheimer–Snyder collapse.
- In this sense, Schwarzschild relevant for moduli space of grav. collapse $\mathfrak{M}$.
- Remark: **No** region of negative-mass Schwarzschild is relevant for the study of $\mathfrak{M}$ because elements in $\mathfrak{M}$ cannot have negative Hawking mass m

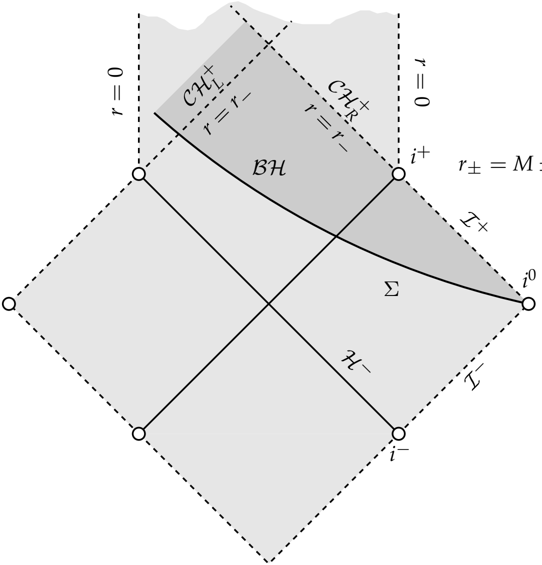
$$1 - \frac{2m}{r} = g(\nabla r, \nabla r).$$

REFRESHER ON SUBEXTREMAL REISSNER–NORDSTRÖM: $0 < |e| < M$



$r_\pm = M \pm \sqrt{M^2 - e^2}$

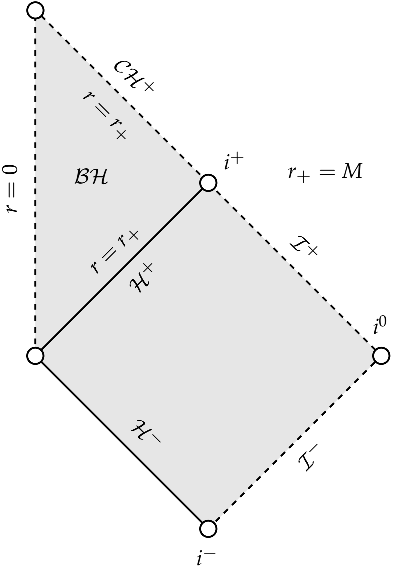
REFRESHER ON SUBEXTREMAL REISSNER–NORDSTRÖM: $0 < |e| < M$



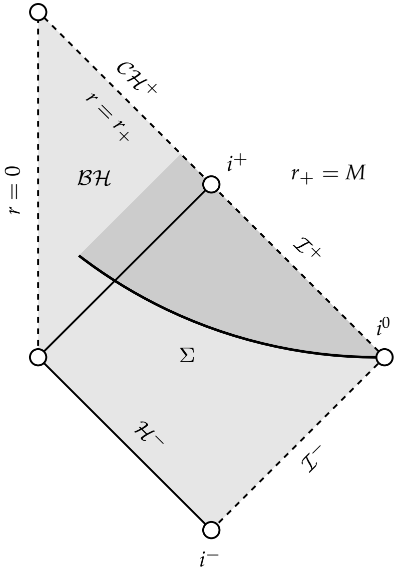
$r_\pm = M \pm \sqrt{M^2 - e^2}$

Non-negative Hawking mass $m \doteq \frac{r}{2}(1 - g(\nabla r, \nabla r))$ requires $r \geq \frac{e^2}{2M}$.

Refresher on extremal Reissner–Nordström: $0 < |e| = M$

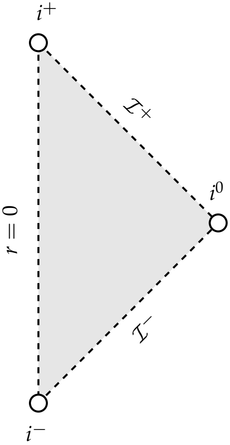


REFRESHER ON EXTREMAL REISSNER–NORDSTRÖM: $0 < |e| = M$

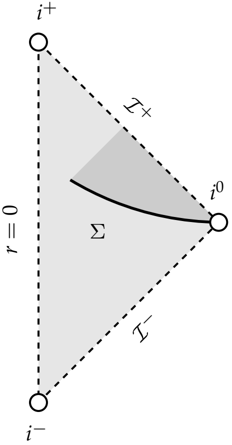


Non-negative Hawking mass requires $r \geq \frac{e^2}{2M}$.

Refresher on superextremal Reissner–Nordström: $0 < M < |e|$

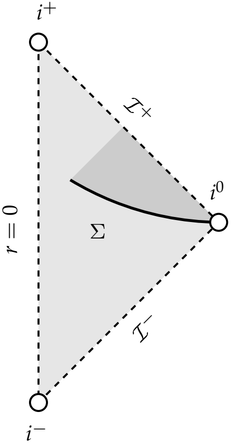


REFRESHER ON SUPEREXTREMAL REISSNER–NORDSTRÖM: $0 < M < |e|$



Non-negative Hawking mass requires $r \geq \frac{e^2}{2M}$.

Refresher on superextremal Reissner–Nordström: $0 < M < |e|$



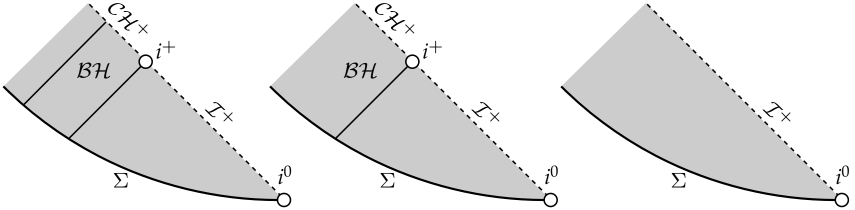
Non-negative Hawking mass requires $r \geq \frac{e^2}{2M}$.

The naked singularity of extremal Reissner–Nordström is dynamically ***inaccessible!***

Reissner–Nordström family arises dynamically

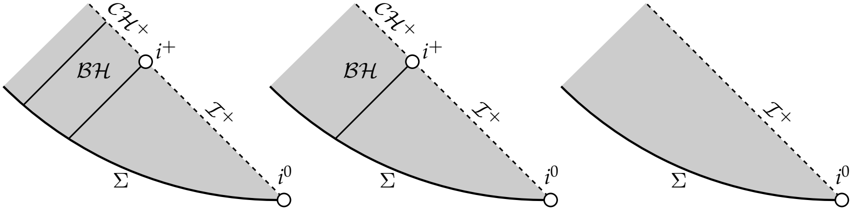
Theorem (K.–Unger '22).

There exist regular, spherically symmetric data on $\mathbb{R}^3$ for the Einstein–Maxwell–charged scalar field model (i.e. elements in $\mathfrak{M}$) whose MGHD contains the darker shaded regions of Reissner–Nordström for $|e| \leq M$, $|e| = M$ and $|e| > M$.



(a) $|e| < M$: black hole (b) $|e| = M$: black hole (c) $|e| > M$: dispersive

Theorem (K.–Unger '22).
There exist regular, spherically symmetric data on $\mathbb{R}^3$ for the Einstein–Maxwell–charged scalar field model (i.e. elements in $\mathfrak{M}$) whose MGHD contains the darker shaded regions of Reissner–Nordström for $|e| \leq M$, $|e| = M$ and $|e| > M$.

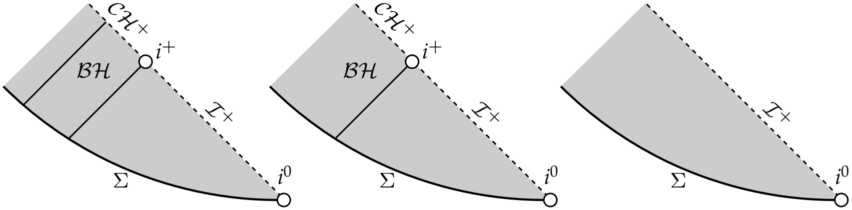


(a) $|e| < M$: black hole (b) $|e| = M$: black hole (c) $|e| > M$: dispersive

- More generally, all regions of the Reissner–Nordström family with Hawking mass $m > 0$ arise in gravitational collapse.

Reissner–Nordström family arises dynamically

Theorem (K.–Unger ’22).
There exist regular, spherically symmetric data on $\mathbb{R}^3$ for the Einstein–Maxwell–charged scalar field model (i.e. elements in $\mathfrak{M}$) whose MGHD contains the darker shaded regions of Reissner–Nordström for $|e| \le M$, $|e| = M$ and $|e| > M$.



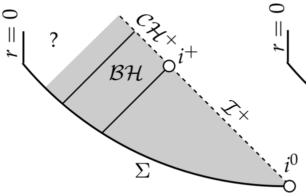
(a) $|e| < M$: black hole (b) $|e| = M$: black hole (c) $|e| > M$: dispersive

- More generally, all regions of the Reissner–Nordström family with Hawking mass $m > 0$ arise in gravitational collapse.
- If local charge-mass inequality (e.g. $\mathfrak{m} \ge |\mathfrak{e}|$) holds, then no sphere with

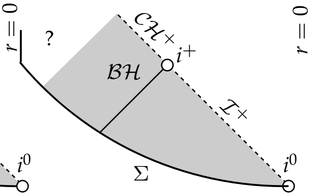
$$r \le \frac{e^2}{M}$$

can arise in gravitational collapse [Reall’24, McSharry–Reall–’25].

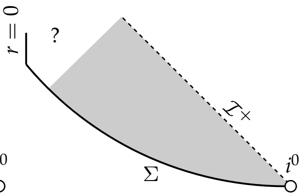
CRITICAL COLLAPSE?



(a) $|e| < M$: black hole

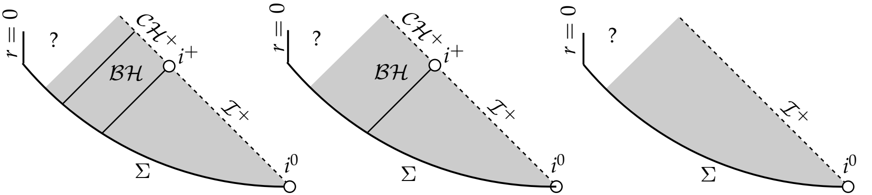


(b) $|e| = M$: black hole



(c) $|e| > M$: **??**

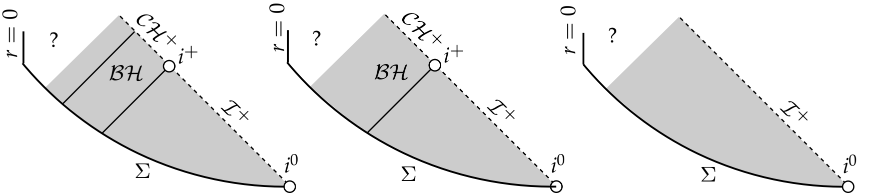
Critical Collapse?



(a) $|e| < M$: black hole (b) $|e| = M$: black hole (c) $|e| > M$: **??**

- This suggests that extremal black holes could arise on the black hole formation threshold: **Extremal critical collapse.**

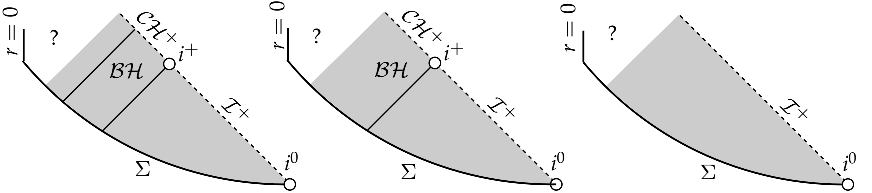
Critical Collapse?



(a) $|e| < M$: black hole (b) $|e| = M$: black hole (c) $|e| > M$: **??**

- This suggests that extremal black holes could arise on the black hole formation threshold: **Extremal critical collapse.**
- Need to understand global properties of solutions to the EMCSF model around extremal Reissner–Nordström. This is difficult! (Dejan's talk!)

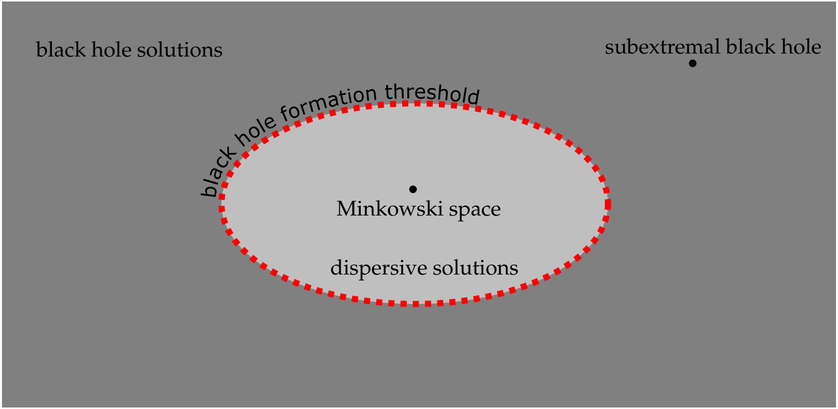
CRITICAL COLLAPSE?



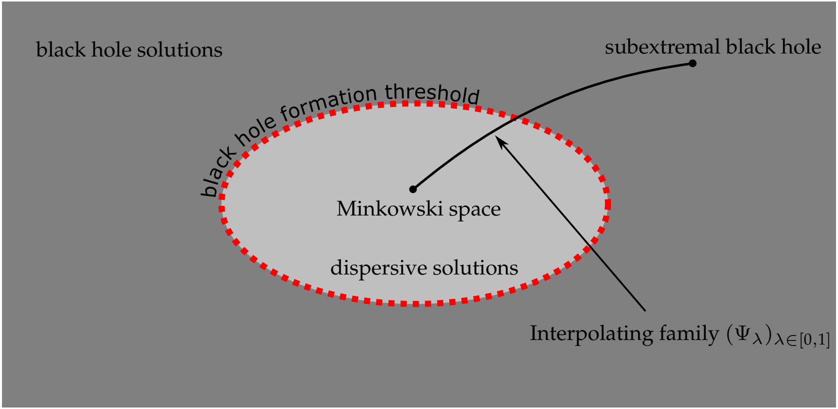
(a) $|e| < M$: black hole (b) $|e| = M$: black hole (c) $|e| > M$: **??**

- This suggests that extremal black holes could arise on the black hole formation threshold: **Extremal critical collapse.**
- Need to understand global properties of solutions to the EMCSF model around extremal Reissner–Nordström. This is difficult! (Dejan's talk!)
- More accessible in the Einstein–Maxwell–Vlasov system: Exploit localization in physical space.

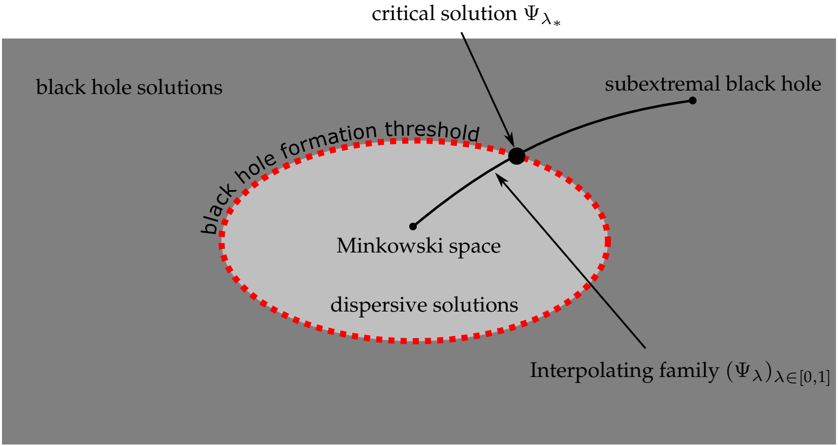
CARTOON PICTURE OF MODULI SPACE OF GRAV. COLLAPSE



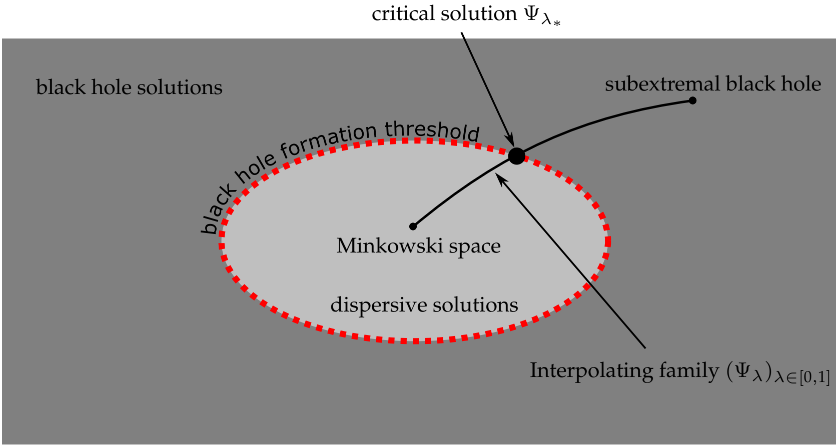
CARTOON PICTURE OF MODULI SPACE OF GRAV. COLLAPSE



Cartoon picture of moduli space of grav. collapse

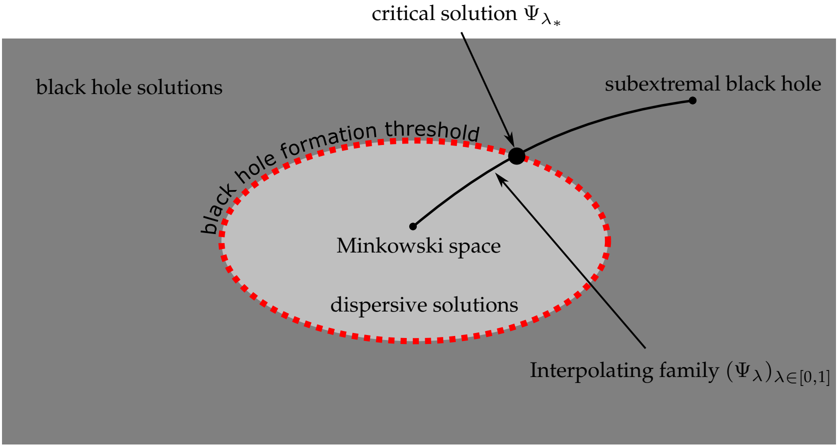


CARTOON PICTURE OF MODULI SPACE OF GRAV. COLLAPSE



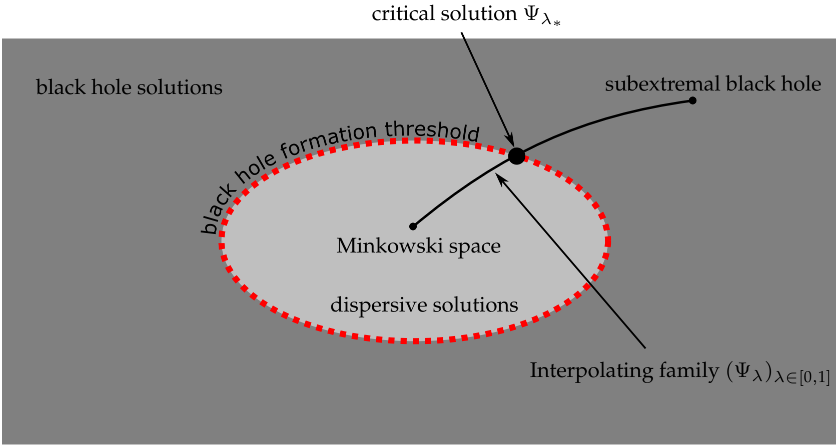
Numerics for sph. symm. Einstein-scalar field: Ψ_{λ_*} leads to a **naked singularity**
[CHOPTUIK '93, ...]

CARTOON PICTURE OF MODULI SPACE OF GRAV. COLLAPSE



Numerics for sph. symm. Einstein-scalar field: Ψ_{λ_*} leads to a **naked singularity**
[CHOPTUIK '93, ...]
Also numerics suggesting **star-like objects** as Ψ_{λ_*} for Einstein–Klein–Gordon/Vlasov
[BRADY, CHAMBERS, GONCALVES, REIN, RENDALL, SCHAEFFER, EAST...]

Cartoon picture of moduli space of grav. collapse

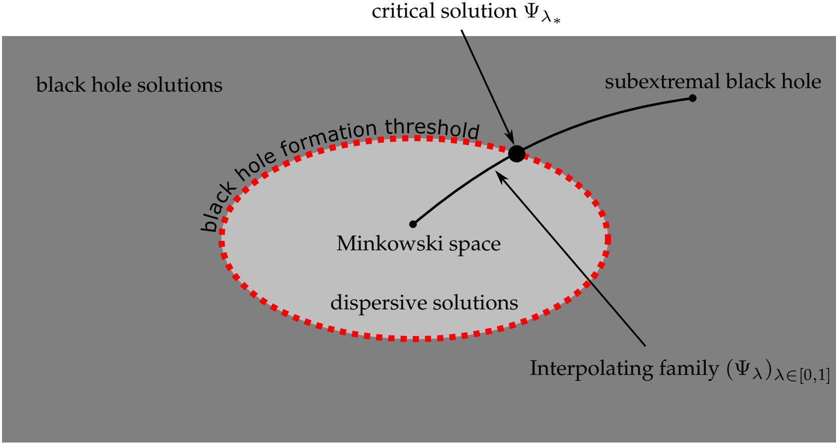


Numerics for sph. symm. Einstein-scalar field: Ψ_{λ_*} leads to a **naked singularity**
[Choptuik '93, . . .]
Also numerics suggesting **star-like objects** as Ψ_{λ_*} for Einstein–Klein–Gordon/Vlasov
[Brady, Chambers, Goncalves, Rein, Rendall, Schaeffer, East. . .]

It is an open problem to make any of these numerics rigorous!

(Upcoming work of [Cicortas–Rodnianski] in $2+1D$!)

Cartoon picture of moduli space of grav. collapse



Numerics for sph. symm. Einstein-scalar field: Ψ_{λ_*} leads to a **naked singularity** [Choptuik '93, …]
Also numerics suggesting **star-like objects** as Ψ_{λ_*} for Einstein–Klein–Gordon/Vlasov [Brady, Chambers, Goncalves, Rein, Rendall, Schaeffer, East…]

It is an open problem to make any of these numerics rigorous!

(Upcoming work of [Cicortas–Rodnianski] in $2+1D$!)

Although "much newer", extremal critical collapse is *more accessible*.

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} + \int_{P_x^{\mathfrak{m}}} p_\mu p_\nu f \, d\mu_x^{\mathfrak{m}}\right), \\ \nabla^\alpha F_{\mu\alpha} = \mathfrak{e}\int_{P_x^{\mathfrak{m}}} p_\mu f \, d\mu_x^{\mathfrak{m}}, \\ p^\mu \frac{\partial}{\partial x^\mu} f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f = -\mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} f. \end{cases}$$

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} + \int_{P_x^{\mathfrak{m}}} p_\mu p_\nu f \, d\mu_x^{\mathfrak{m}}\right), \\ \nabla^\alpha F_{\mu\alpha} = \mathfrak{e} \int_{P_x^{\mathfrak{m}}} p_\mu f \, d\mu_x^{\mathfrak{m}}, \\ p^\mu \frac{\partial}{\partial x^\mu} f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f = -\mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} f. \end{cases}$$

Theorem (K.–Unger '24).

There exists a smooth 1*-parameter family of solutions* $\{D_\lambda\}_{\lambda\in[0,1]}$ *and a critical value* $\lambda_* \in (0,1)$ *such that:*

THEOREM: EXTREMAL CRITICAL COLLAPSE

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2 \left(g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} + \int_{P_x^{\mathfrak{m}}} p_\mu p_\nu f \, d\mu_x^{\mathfrak{m}} \right), \\ \nabla^\alpha F_{\mu\alpha} = \mathfrak{e} \int_{P_x^{\mathfrak{m}}} p_\mu f \, d\mu_x^{\mathfrak{m}}, \\ p^\mu \frac{\partial}{\partial x^\mu} f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f = -\mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} f. \end{cases}$$

Theorem (K.–Unger '24).
There exists a smooth 1*-parameter family of solutions* $\{D_\lambda\}_{\lambda \in [0,1]}$ *and a critical value* $\lambda_* \in (0,1)$ *such that:*

- *If* $0 \le \lambda < \lambda_*$*, the solution **disperses** to Minkowski space and **no** black hole forms.*

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} + \int_{P_x^{\mathfrak{m}}} p_\mu p_\nu f \, d\mu_x^{\mathfrak{m}}\right), \\ \nabla^\alpha F_{\mu\alpha} = \mathfrak{e}\int_{P_x^{\mathfrak{m}}} p_\mu f \, d\mu_x^{\mathfrak{m}}, \\ p^\mu \frac{\partial}{\partial x^\mu} f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f = -\mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} f. \end{cases}$$

Theorem (K.–Unger ’24).
There exists a smooth 1*-parameter family of solutions* $\{D_\lambda\}_{\lambda\in[0,1]}$ *and a critical value* $\lambda_* \in (0,1)$ *such that:*

- *If* $0 \le \lambda < \lambda_*$*, the solution* ***disperses*** *to Minkowski space and* ***no*** *black hole forms.*
- *If* $\lambda = \lambda_*$*, an* ***extremal*** *black hole forms.*

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} + \int_{P_x^{\mathfrak{m}}} p_\mu p_\nu f \, d\mu_x^{\mathfrak{m}}\right), \\ \nabla^\alpha F_{\mu\alpha} = \mathfrak{e}\int_{P_x^{\mathfrak{m}}} p_\mu f \, d\mu_x^{\mathfrak{m}}, \\ p^\mu \frac{\partial}{\partial x^\mu} f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f = -\mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} f. \end{cases}$$

Theorem (K.–Unger '24).

There exists a smooth 1*-parameter family of solutions* $\{D_\lambda\}_{\lambda\in[0,1]}$ *and a critical value* $\lambda_* \in (0,1)$ *such that:*

- *If* $0 \le \lambda < \lambda_*$*, the solution* ***disperses*** *to Minkowski space and* ***no*** *black hole forms.*
- *If* $\lambda = \lambda_*$*, an* ***extremal*** *black hole forms.*
- *If* $\lambda_* < \lambda \le 1$*, a* ***subextremal*** *black hole forms.*

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} + \int_{P_x^{\mathfrak{m}}} p_\mu p_\nu f \, d\mu_x^{\mathfrak{m}}\right), \\ \nabla^\alpha F_{\mu\alpha} = \mathfrak{e}\int_{P_x^{\mathfrak{m}}} p_\mu f \, d\mu_x^{\mathfrak{m}}, \\ p^\mu \frac{\partial}{\partial x^\mu} f - \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f = -\mathfrak{e}F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} f. \end{cases}$$

Theorem (K.–Unger '24).

There exists a smooth 1*-parameter family of solutions* $\{D_\lambda\}_{\lambda\in[0,1]}$ *and a critical value* $\lambda_* \in (0,1)$ *such that:*

- *If* $0 \leq \lambda < \lambda_*$*, the solution* ***disperses*** *to Minkowski space and* ***no*** *black hole forms.*
- *If* $\lambda = \lambda_*$*, an* ***extremal*** *black hole forms.*
- *If* $\lambda_* < \lambda \leq 1$*, a* ***subextremal*** *black hole forms.*

There exist ***extremal*** *black holes on the black hole formation threshold!*

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} + \int_{P_x^{\mathfrak{m}}} p_\mu p_\nu f\, d\mu_x^{\mathfrak{m}}\right), \\ \nabla^\alpha F_{\mu\alpha} = \mathfrak{e}\int_{P_x^{\mathfrak{m}}} p_\mu f\, d\mu_x^{\mathfrak{m}}, \\ p^\mu \frac{\partial}{\partial x^\mu} f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f = -\mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} f. \end{cases}$$

Theorem (K.–Unger '24).

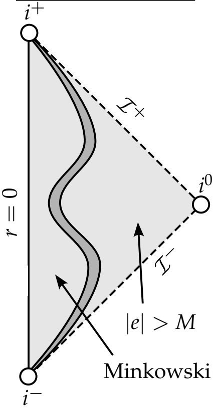
There exists a smooth 1*-parameter family of solutions* $\{D_\lambda\}_{\lambda\in[0,1]}$ *and a critical value* $\lambda_* \in (0,1)$ *such that:*

- *If* $0 \le \lambda < \lambda_*$*, the solution* ***disperses*** *to Minkowski space and* ***no*** *black hole forms.*
- *If* $\lambda = \lambda_*$*, an* ***extremal*** *black hole forms.*
- *If* $\lambda_* < \lambda \le 1$*, a* ***subextremal*** *black hole forms.*

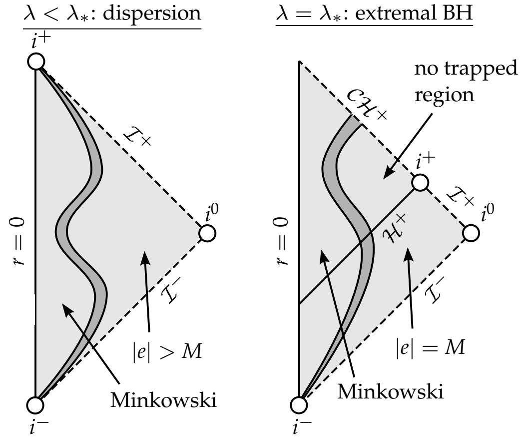
There exist ***extremal*** *black holes on the black hole formation threshold!*

Penrose diagram: Extremal critical collapse

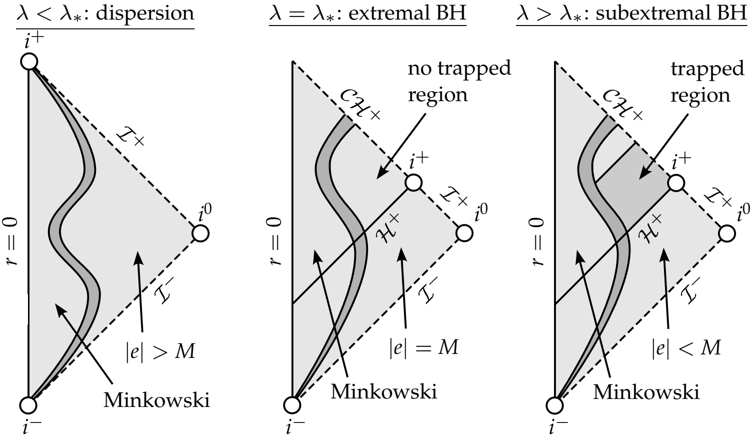
$\lambda < \lambda_*$: dispersion



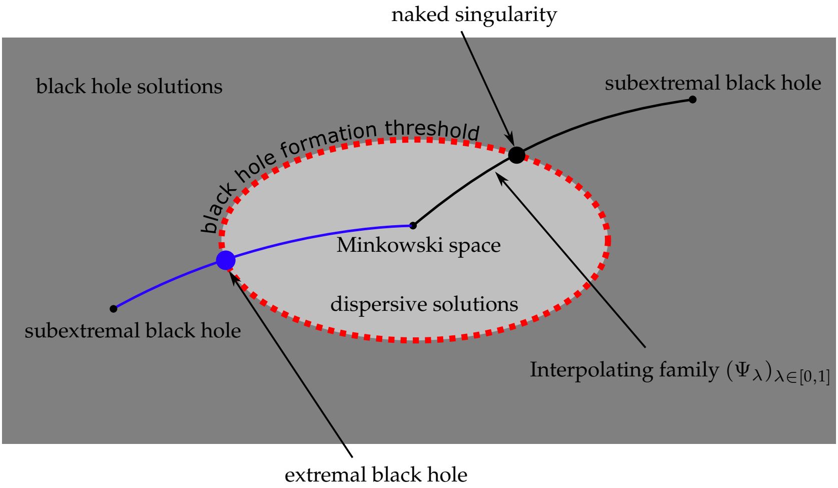
PENROSE DIAGRAM: EXTREMAL CRITICAL COLLAPSE



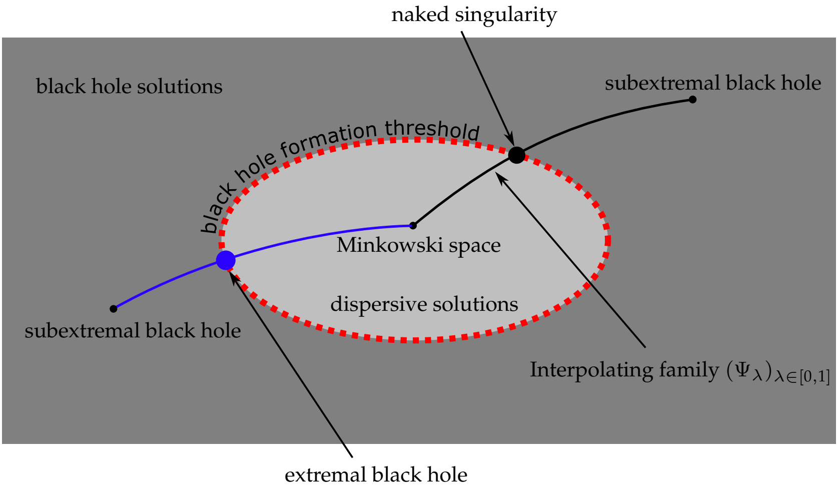
Penrose diagram: Extremal critical collapse



CARTOON PICTURE OF MODULI SPACE



Cartoon picture of moduli space

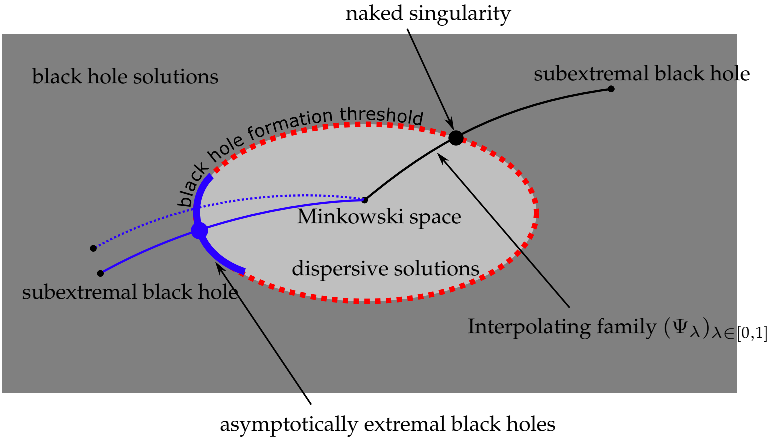


Recently, East numerically observed both charged Vlasov stars and extremal black holes on the threshold for the Einstein–Maxwell–Vlasov system [East'25].

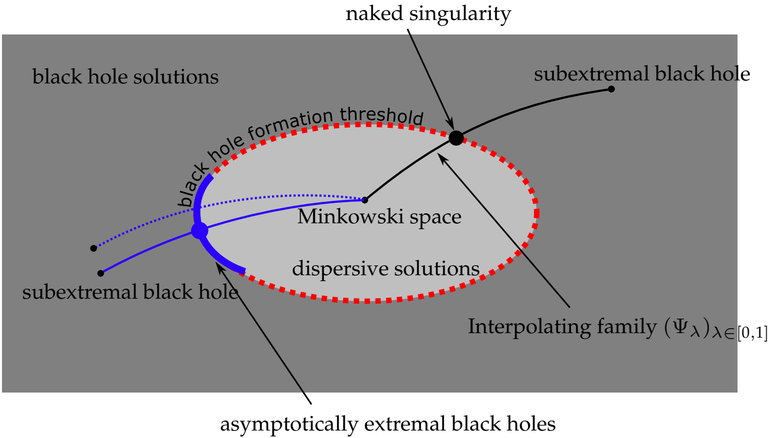
Conjecture.
*Extremal critical collapse is a **stable** phenomenon.*

Conjecture.

*Extremal critical collapse is a **stable** phenomenon.*



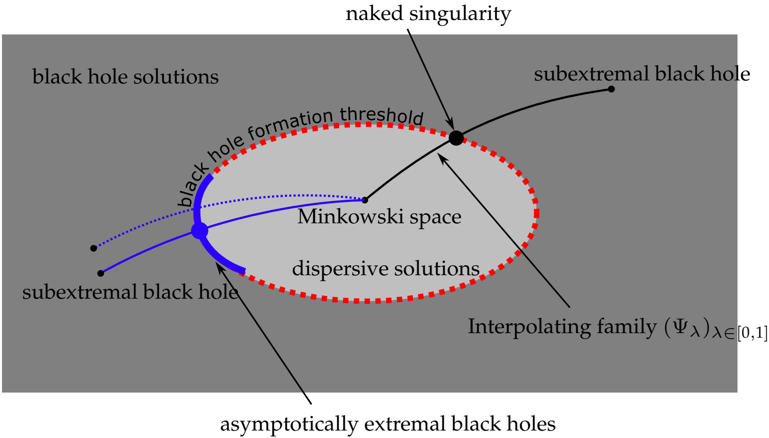
Conjecture.
*Extremal critical collapse is a **stable** phenomenon.*



- This is also a non-trivial statement about the **interiors** of black holes.

Conjecture.

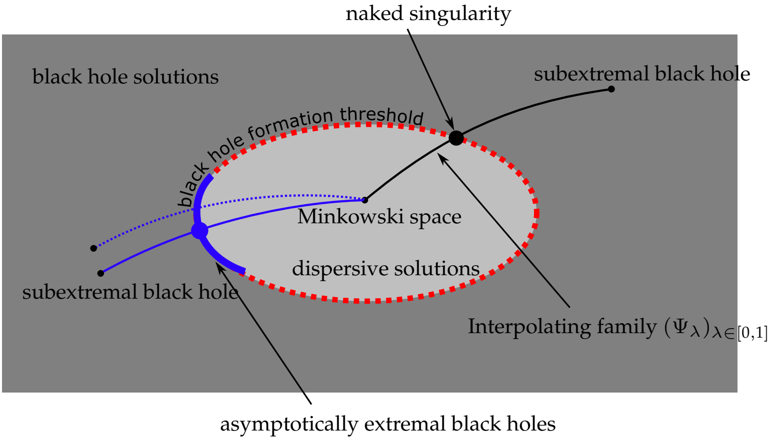
Extremal critical collapse is a ***stable*** *phenomenon.*



- This is also a non-trivial statement about the **interiors** of black holes.
- Further difficulty: **Aretakis instability** associated to extremal horizons

Conjecture.

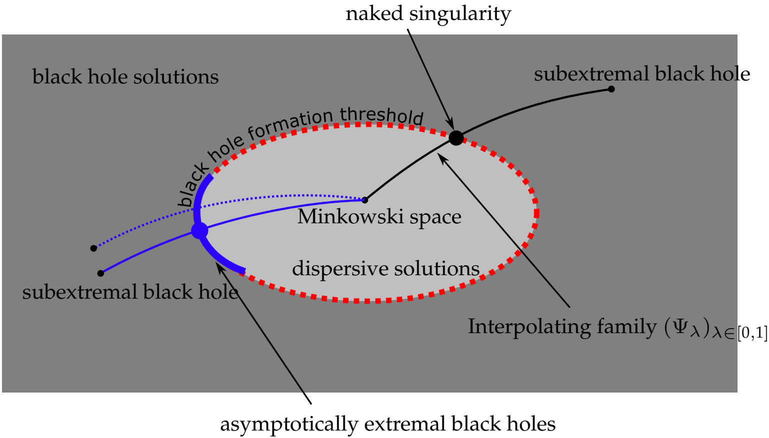
Extremal critical collapse is a ***stable*** *phenomenon.*



- This is also a non-trivial statement about the **interiors** of black holes.
- Further difficulty: **Aretakis instability** associated to extremal horizons
- The black hole formation threshold is not expected to be smooth.

Conjecture.

*Extremal critical collapse is a **stable** phenomenon.*



- This is also a non-trivial statement about the **interiors** of black holes.
- Further difficulty: **Aretakis instability** associated to extremal horizons
- The black hole formation threshold is not expected to be smooth.
- Back to the Einstein-(neutral) scalar field model as in [MURATA–REALL–TANAHASHI'13]

Einstein–Maxwell–(neutral) scalar field

$$\mathrm{Ric}(g) - \tfrac{1}{2}R(g)g = 2(T^{\mathrm{EM}} + T^{\mathrm{SF}}),$$
$$dF = 0, \quad d \star F = 0, \quad \Box_g \phi = 0,$$
$$T^{\mathrm{EM}}_{\mu\nu} \doteq F_{\mu\alpha}F^{\alpha}{}_{\nu} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \quad T^{\mathrm{SF}}_{\mu\nu} \doteq \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi.$$

$$\partial_u\partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r\partial_v r}{r} + \frac{\Omega^2 Q^2}{4r^3},$$
$$\partial_u\partial_v \log\Omega^2 = \frac{\Omega^2}{2r^2} + \frac{2\partial_u r\partial_v r}{r^2} - \frac{\Omega^2 Q^2}{r^4} - 2\partial_u\phi\partial_v\phi,$$

and Raychaudhuri's equations

$$\partial_u\left(\frac{\partial_u r}{\Omega^2}\right) = -\frac{r}{\Omega^2}(\partial_u\phi)^2, \quad \partial_v\left(\frac{\partial_v r}{\Omega^2}\right) = -\frac{r}{\Omega^2}(\partial_v\phi)^2.$$

$$\partial_u\partial_v\phi = -\frac{\partial_v r\partial_u\phi}{r} - \frac{\partial_u r\partial_v\phi}{r}.$$

It is useful to eliminate Ω for ϖ and have

$$\boxed{(\phi, r, \varpi, Q)}$$

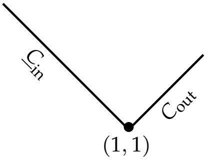
as unknowns. Here

$$\varpi \doteq m + \frac{Q^2}{2r}, \quad Q = \mathrm{const}.$$

This is the renormalized Hawking/Dougan–Mason mass in spherical symmetry.

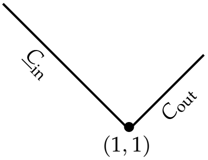
Definition of the moduli space $\mathfrak{M}$

Fix $M_0 > 0$ once and for all. Characteristic data posed on $C = \underline{C}_{\text{in}} \cup C_{\text{out}}$:
$\underline{C}_{\text{in}} = [1, 99M_0] \times \{1\}$, $C_{\text{out}} = \{1\} \times [1, \infty)$.



Definition of the moduli space $\mathfrak{M}$

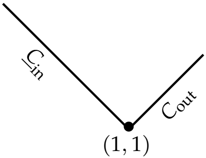
Fix $M_0 > 0$ once and for all. Characteristic data posed on $C = \underline{C}_{\text{in}} \cup C_{\text{out}}$:
$\underline{C}_{\text{in}} = [1, 99M_0] \times \{1\}$, $C_{\text{out}} = \{1\} \times [1, \infty)$.



Initial data gauge: $\partial_v r = 1$ on C_{out} and $\partial_u r = -1$ on C_{in}.

Definition of the moduli space $\mathfrak{M}$

Fix $M_0 > 0$ once and for all. Characteristic data posed on $C = \underline{C}_{\text{in}} \cup C_{\text{out}}$:
$\underline{C}_{\text{in}} = [1, 99M_0] \times \{1\}$, $C_{\text{out}} = \{1\} \times [1, \infty)$.



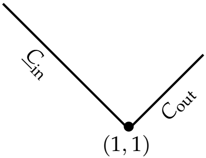
Initial data gauge: $\partial_v r = 1$ on C_{out} and $\partial_u r = -1$ on C_{in}.
Free data:

- $(r_\circ, \varpi_\circ, \rho_\circ)$, $\rho_\circ = Q_\circ/\varpi_\circ$ on the bifurcation sphere $(1,1)$.
- $\phi_\circ$ on $C = C_{\text{in}} \cup C_{\text{out}}$
- Initial data

$$\Psi \doteq (\phi_\circ, r_\circ, \varpi_\circ, \rho_\circ) \in C^2_w(C) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathfrak{X} \times \mathbb{R} = \mathfrak{Z}$$

Definition of the moduli space $\mathfrak{M}$

Fix $M_0 > 0$ once and for all. Characteristic data posed on $C = \underline{C}_{\text{in}} \cup C_{\text{out}}$:
$\underline{C}_{\text{in}} = [1, 99M_0] \times \{1\}$, $C_{\text{out}} = \{1\} \times [1, \infty)$.



Initial data gauge: $\partial_v r = 1$ on C_{out} and $\partial_u r = -1$ on C_{in}.
Free data:

- $(r_\circ, \varpi_\circ, \rho_\circ)$, $\rho_\circ = Q_\circ / \varpi_\circ$ on the bifurcation sphere $(1, 1)$.
- $\phi_\circ$ on $C = C_{\text{in}} \cup C_{\text{out}}$
- Initial data

$$\Psi \doteq (\phi_\circ, r_\circ, \varpi_\circ, \rho_\circ) \in C^2_w(C) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathfrak{X} \times \mathbb{R} = \mathfrak{Z}$$

We restrict to

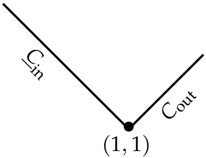
$$\Psi \in \mathfrak{M} \doteq B^{\mathfrak{X}}_{\varepsilon}(x_0) \times [-10, 10]_{\rho_\circ} \subset \mathfrak{Z}$$

where

$$x_0 = (0, 100M_0, M_0).$$

DEFINITION OF THE MODULI SPACE $\mathfrak{M}$

Fix $M_0 > 0$ once and for all. Characteristic data posed on $C = \underline{C}_{\text{in}} \cup C_{\text{out}}$:
$\underline{C}_{\text{in}} = [1, 99M_0] \times \{1\}$, $C_{\text{out}} = \{1\} \times [1, \infty)$.



Initial data gauge: $\partial_v r = 1$ on C_{out} and $\partial_u r = -1$ on C_{in}.
Free data:

- $(r_\circ, \varpi_\circ, \rho_\circ)$, $\rho_\circ = Q_\circ/\varpi_\circ$ on the bifurcation sphere $(1, 1)$.
- $\phi_\circ$ on $C = C_{\text{in}} \cup C_{\text{out}}$
- Initial data

$$\Psi \doteq (\phi_\circ, r_\circ, \varpi_\circ, \rho_\circ) \in C^2_w(C) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathfrak{X} \times \mathbb{R} = \mathfrak{Z}$$

We restrict to

$$\Psi \in \mathfrak{M} \doteq B^{\mathfrak{X}}_{\varepsilon}(x_0) \times [-10, 10]_{\rho_\circ} \subset \mathfrak{Z}$$

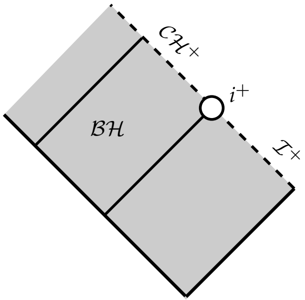
where

$$x_0 = (0, 100M_0, M_0).$$

Any $\Psi \in \mathfrak{M}$ gives rise to a unique MGHD in the future of $\underline{C}_{\text{in}} \cup C_{\text{out}}$.

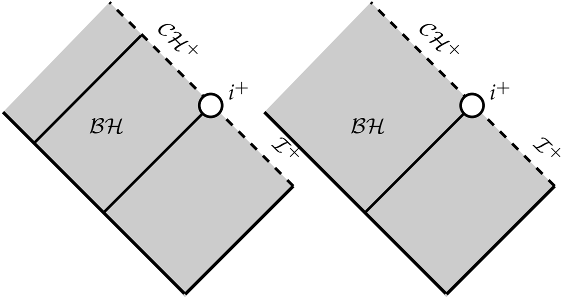
Setup inspired by [MURATA–REALL–TANAHASHI '13].

The Reissner–Nordström family $(x_0, \rho)_{\rho \in [-10,10]}$



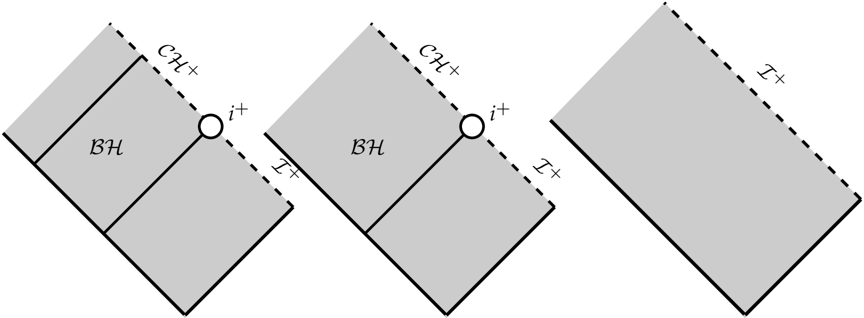
(a) MGHD of $\Psi = (x_0, 9/10)$

THE REISSNER–NORDSTRÖM FAMILY $(x_0, \rho)_{\rho \in [-10,10]}$



(a) MGHD of $\Psi = (x_0, 9/10)$ (b) MGHD of $\Psi = (x_0, 1)$

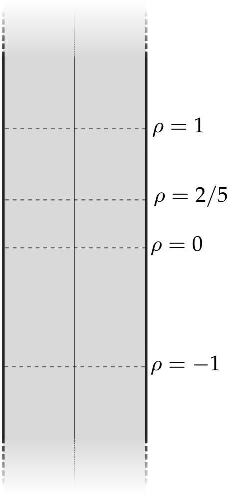
THE REISSNER–NORDSTRÖM FAMILY $(x_0, \rho)_{\rho \in [-10,10]}$



(a) MGHD of $\Psi = (x_0, 9/10)$ (b) MGHD of $\Psi = (x_0, 1)$ (c) MGHD of $\Psi = (x_0, 11/10)$

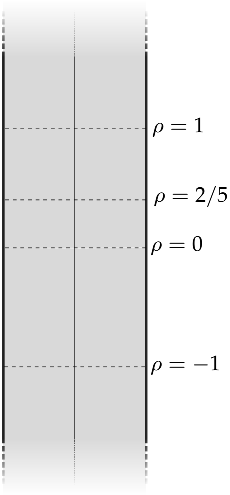
Critical collapse in the Reissner–Nordström family.

Illustration of the moduli space $\mathfrak{M}$



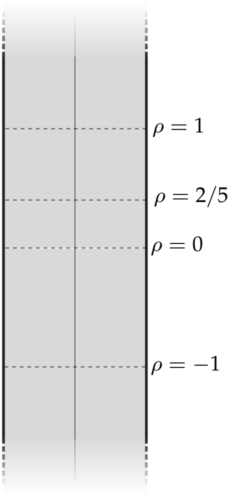
- The vertical axis is $\Psi = (x_0, \rho_\circ)$, where $\rho_\circ \in [-10, 10]$ and $x_0 = (0, 100M_0, M_0)$.

Illustration of the moduli space $\mathfrak{M}$



- The vertical axis is $\Psi = (x_0, \rho_\circ)$, where $\rho_\circ \in [-10, 10]$ and $x_0 = (0, 100M_0, M_0)$.
- Perturbing in the horizontal axis is also making $\phi_\circ \neq 0$.

ILLUSTRATION OF THE MODULI SPACE $\mathfrak{M}$



- The vertical axis is $\Psi = (x_0, \rho_\circ)$, where $\rho_\circ \in [-10, 10]$ and $x_0 = (0, 100M_0, M_0)$.
- Perturbing in the horizontal axis is also making $\phi_\circ \neq 0$.
- This is only a (small) open subset of the full moduli space which itself is a subset of the Banach space $\mathfrak{Z}$.

A PRIORI DICHOTOMY

$$\mathfrak{M}_{\text{black}} \doteq \{\Psi \in \mathfrak{M} : \text{MGHD of } \Psi \text{ contains a black hole region.}\}$$
$$\mathfrak{M}_{\text{dispersive}} \doteq \{\Psi \in \mathfrak{M} : \text{MGHD of } \Psi \text{ is asymptotically flat.}\}$$

$$\mathfrak{M}_{\text{black}} \doteq \{\Psi \in \mathfrak{M} : \text{MGHD of } \Psi \text{ contains a black hole region.}\}$$
$$\mathfrak{M}_{\text{dispersive}} \doteq \{\Psi \in \mathfrak{M} : \text{MGHD of } \Psi \text{ is asymptotically flat.}\}$$

Theorem (Dafermos ’05).

$$\mathfrak{M} = \mathfrak{M}_{\text{black}} \sqcup \mathfrak{M}_{\text{disp}}$$

This is a general result exploiting the

- *monotonicities* of Raychaudhuri’s equations,
- the *semilinearity* and *subcriticality* of Einstein equations in spherical symmetry in an initial data gauge (no teleological gauge) and away from the center.

Main theorem

Theorem (Angelopoulos–K.–Unger, upcoming).

1. ***Asymptotic stability and foliation by stable manifolds of*** $\mathfrak{M}_{\mathrm{black}}$

Main theorem

Theorem (Angelopoulos–K.–Unger, upcoming).

1. ***Asymptotic stability and foliation by stable manifolds of*** $\mathfrak{M}_{\text{black}}$
   $\mathfrak{M}_{\text{black}}$ *is foliated by* C^1 *hypersurfaces* $\mathfrak{M}^{\sigma}_{\text{stab}}$, *indexed by* $\sigma \in [-1, 1]$.

   $$\mathfrak{M}^{\sigma}_{\text{stab}} = \{\Psi \in \mathfrak{M}_{\text{black}} : \mathscr{P}(\Psi) = \sigma\}$$

   where $\mathscr{P} : \mathfrak{M}_{\text{black}} \to [-1, 1]$ *is the final* ***signed charge-to-mass ratio*** *of the black hole formed from* Ψ.

2. ***Threshold property of*** $\mathfrak{M}^{1}_{\text{stab}} \sqcup \mathfrak{M}^{-1}_{\text{stab}}$

Theorem (Angelopoulos–K.–Unger, upcoming).

1. ***Asymptotic stability and foliation by stable manifolds of*** $\mathfrak{M}_{\text{black}}$
   $\mathfrak{M}_{\text{black}}$ *is foliated by* C^1 *hypersurfaces* $\mathfrak{M}^{\sigma}_{\text{stab}}$, *indexed by* $\sigma \in [-1, 1]$.

$$\mathfrak{M}^{\sigma}_{\text{stab}} = \{\Psi \in \mathfrak{M}_{\text{black}} : \mathscr{P}(\Psi) = \sigma\}$$

   where $\mathscr{P} : \mathfrak{M}_{\text{black}} \to [-1, 1]$ *is the final* ***signed charge-to-mass ratio*** *of the black hole formed from* Ψ.

2. ***Threshold property of*** $\mathfrak{M}^{1}_{\text{stab}} \sqcup \mathfrak{M}^{-1}_{\text{stab}}$
   The black hole threshold $\partial\mathfrak{M}_{\text{black}}$ *is characterized by:*

$$\partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{1}_{\text{stab}} \sqcup \mathfrak{M}^{-1}_{\text{stab}},$$

   the set of asymptotically extremal black holes.

Theorem (Angelopoulos–K.–Unger, upcoming).

1. ***Asymptotic stability and foliation by stable manifolds of*** $\mathfrak{M}_{\mathrm{black}}$
   $\mathfrak{M}_{\mathrm{black}}$ *is foliated by* C^1 *hypersurfaces* $\mathfrak{M}^{\sigma}_{\mathrm{stab}}$*, indexed by* $\sigma \in [-1, 1]$.

$$\mathfrak{M}^{\sigma}_{\mathrm{stab}} = \{\Psi \in \mathfrak{M}_{\mathrm{black}} : \mathscr{P}(\Psi) = \sigma\}$$

   where $\mathscr{P} : \mathfrak{M}_{\mathrm{black}} \to [-1, 1]$ *is the final* ***signed charge-to-mass ratio*** *of the black hole formed from* Ψ.

2. ***Threshold property of*** $\mathfrak{M}^{1}_{\mathrm{stab}} \sqcup \mathfrak{M}^{-1}_{\mathrm{stab}}$
   The black hole threshold $\partial\mathfrak{M}_{\mathrm{black}}$ *is characterized by:*

$$\partial\mathfrak{M}_{\mathrm{black}} = \mathfrak{M}^{1}_{\mathrm{stab}} \sqcup \mathfrak{M}^{-1}_{\mathrm{stab}},$$

   the set of asymptotically extremal black holes.

3. ***Universality and scaling laws with scaling index*** $1/2$
   The final area, temperature, and event horizon location satisfy "universal scaling laws" with index $1/2$.

Main theorem

Theorem (Angelopoulos–K.–Unger, upcoming).

1. ***Asymptotic stability and foliation by stable manifolds of*** $\mathfrak{M}_{\text{black}}$
   $\mathfrak{M}_{\text{black}}$ *is foliated by* C^1 *hypersurfaces* $\mathfrak{M}^{\sigma}_{\text{stab}}$*, indexed by* $\sigma \in [-1, 1]$*.*

   $$\mathfrak{M}^{\sigma}_{\text{stab}} = \{\Psi \in \mathfrak{M}_{\text{black}} : \mathscr{P}(\Psi) = \sigma\}$$

   where $\mathscr{P} : \mathfrak{M}_{\text{black}} \to [-1, 1]$ *is the final* ***signed charge-to-mass ratio*** *of the black hole formed from* Ψ*.*

2. ***Threshold property of*** $\mathfrak{M}^{1}_{\text{stab}} \sqcup \mathfrak{M}^{-1}_{\text{stab}}$
   The black hole threshold $\partial\mathfrak{M}_{\text{black}}$ *is characterized by:*

   $$\partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{1}_{\text{stab}} \sqcup \mathfrak{M}^{-1}_{\text{stab}},$$

   the set of asymptotically extremal black holes.

3. ***Universality and scaling laws with scaling index*** $1/2$
   The final area, temperature, and event horizon location satisfy "universal scaling laws" with index $1/2$*.*

4. ***Aretakis instability at the threshold***
   The Aretakis instability is present for generic data in $\mathfrak{M}^{\pm 1}_{\text{stab}}$ *and a transient horizon instability is exhibited near* $\mathfrak{M}^{\pm 1}_{\text{stab}}$*.*

Main theorem

Theorem (Angelopoulos–K.–Unger, upcoming).

1. ***Asymptotic stability and foliation by stable manifolds of*** $\mathfrak{M}_{\mathrm{black}}$
   $\mathfrak{M}_{\mathrm{black}}$ is foliated by C^1 hypersurfaces $\mathfrak{M}^{\sigma}_{\mathrm{stab}}$, indexed by $\sigma \in [-1, 1]$.

   $$\mathfrak{M}^{\sigma}_{\mathrm{stab}} = \{\Psi \in \mathfrak{M}_{\mathrm{black}} : \mathscr{P}(\Psi) = \sigma\}$$

   *where $\mathscr{P} : \mathfrak{M}_{\mathrm{black}} \to [-1, 1]$ is the final **signed charge-to-mass ratio** of the black hole formed from Ψ.*

2. ***Threshold property of*** $\mathfrak{M}^{1}_{\mathrm{stab}} \sqcup \mathfrak{M}^{-1}_{\mathrm{stab}}$
   The black hole threshold $\partial\mathfrak{M}_{\mathrm{black}}$ is characterized by:

   $$\partial\mathfrak{M}_{\mathrm{black}} = \mathfrak{M}^{1}_{\mathrm{stab}} \sqcup \mathfrak{M}^{-1}_{\mathrm{stab}},$$

   the set of asymptotically extremal black holes.

3. ***Universality and scaling laws with scaling index*** $1/2$
   The final area, temperature, and event horizon location satisfy "universal scaling laws" with index $1/2$.

4. ***Aretakis instability at the threshold***
   The Aretakis instability is present for generic data in $\mathfrak{M}^{\pm 1}_{\mathrm{stab}}$ and a transient horizon instability is exhibited near $\mathfrak{M}^{\pm 1}_{\mathrm{stab}}$.

- The theorem can be viewed as the spherically symmetric analog of a conjecture [Dafermos–Holzegel–Rodnianski–Taylor '21]. See Mihalis' talk.

Theorem (Angelopoulos–K.–Unger, upcoming).

1. ***Asymptotic stability and foliation by stable manifolds of*** $\mathfrak{M}_{\text{black}}$
   $\mathfrak{M}_{\text{black}}$ is foliated by C^1 hypersurfaces $\mathfrak{M}^{\sigma}_{\text{stab}}$, indexed by $\sigma \in [-1, 1]$.

$$\mathfrak{M}^{\sigma}_{\text{stab}} = \{\Psi \in \mathfrak{M}_{\text{black}} : \mathscr{P}(\Psi) = \sigma\}$$

   where $\mathscr{P} : \mathfrak{M}_{\text{black}} \to [-1, 1]$ is the final ***signed charge-to-mass ratio*** *of the black hole formed from Ψ.*

2. ***Threshold property of*** $\mathfrak{M}^{1}_{\text{stab}} \sqcup \mathfrak{M}^{-1}_{\text{stab}}$
   The black hole threshold $\partial\mathfrak{M}_{\text{black}}$ is characterized by:

$$\partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{1}_{\text{stab}} \sqcup \mathfrak{M}^{-1}_{\text{stab}},$$

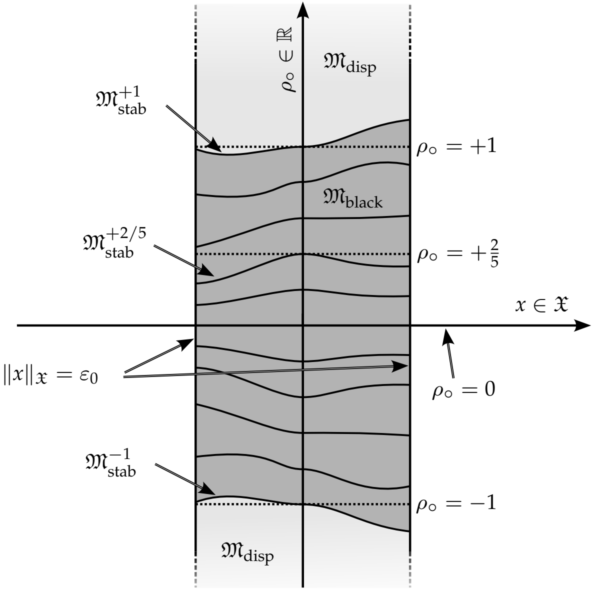
   the set of asymptotically extremal black holes.

3. ***Universality and scaling laws with scaling index*** $1/2$
   The final area, temperature, and event horizon location satisfy "universal scaling laws" with index $1/2$.

4. ***Aretakis instability at the threshold***
   The Aretakis instability is present for generic data in $\mathfrak{M}^{\pm 1}_{\text{stab}}$ and a transient horizon instability is exhibited near $\mathfrak{M}^{\pm 1}_{\text{stab}}$.

- The theorem can be viewed as the spherically symmetric analog of a conjecture [DAFERMOS–HOLZEGEL–RODNIANSKI–TAYLOR '21]. See Mihalis' talk.
- Asymptotic stability in the subextremal case proved before by [DR05],[LUK–OH'19]. Builds on large body of works of [ANGELOPOULOS–ARETAKIS–GAJIC].

Illustration of main theorem

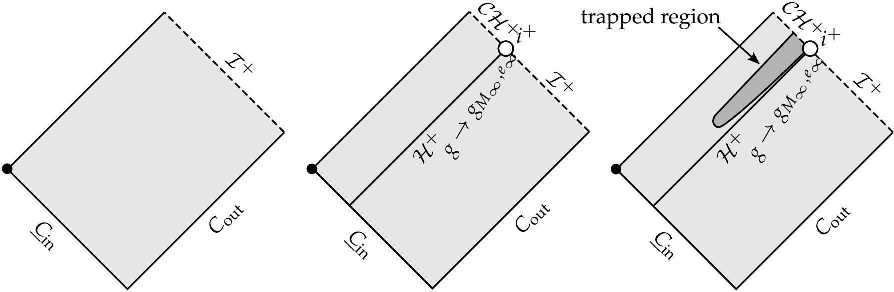


TRICHOTOMY

$\Psi \in \mathfrak{M}_{\text{disp}}$: no black hole

$\Psi \in \mathfrak{M}^{\sigma}_{\text{stab}}$: $|\sigma| = 1$

$\Psi \in \mathfrak{M}^{\sigma}_{\text{stab}}$: $\sigma \in (-1, 1)$



UNIVERSAL SCALING LAWS

If $\Psi \in \mathfrak{M}_{\text{black}}$, define:

- $\mathscr{M}(\Psi)$ final mass
- $\mathscr{A}(\Psi)$ final area
- $\mathscr{K}(\Psi)$ final surface gravity ($= 2\pi\mathscr{T}(\Psi)$ temperature)
- $\mathscr{U}(\Psi)$ u-coordinate of $\mathcal{H}^+$

Universal scaling laws

If $\Psi \in \mathfrak{M}_{\text{black}}$, define:

- $\mathscr{M}(\Psi)$ final mass
- $\mathscr{A}(\Psi)$ final area
- $\mathscr{K}(\Psi)$ final surface gravity ($= 2\pi\mathscr{T}(\Psi)$ temperature)
- $\mathscr{U}(\Psi)$ u-coordinate of $\mathcal{H}^+$

Let $\{\Psi_p\}_{p\in[0,1]}$ be a smooth one-parameter family of data sets in $\mathfrak{M}$ which crosses $\partial\mathfrak{M}_{\text{black}}$ such that $\Psi_p \in \mathfrak{M}_{\text{black}}$ for $p \in [0, p_*]$ and $\Psi_{p_*} \in \partial\mathfrak{M}_{\text{black}}$.

Universal scaling laws

If $\Psi \in \mathfrak{M}_{\text{black}}$, define:

- $\mathscr{M}(\Psi)$ final mass
- $\mathscr{A}(\Psi)$ final area
- $\mathscr{K}(\Psi)$ final surface gravity ($= 2\pi\mathscr{T}(\Psi)$ temperature)
- $\mathscr{U}(\Psi)$ u-coordinate of $\mathcal{H}^+$

Let $\{\Psi_p\}_{p\in[0,1]}$ be a smooth one-parameter family of data sets in $\mathfrak{M}$ which crosses $\partial\mathfrak{M}_{\text{black}}$ such that $\Psi_p \in \mathfrak{M}_{\text{black}}$ for $p \in [0, p_*]$ and $\Psi_{p_*} \in \partial\mathfrak{M}_{\text{black}}$. Then

$$\mathscr{A}(\Psi_p) - \mathscr{A}(\Psi_{p_*}) = \sqrt{2c}8\pi\mathscr{M}(\Psi_{p_*})^2|p - p_*|^{1/2} + o(|p - p_*|^{1/2}),$$

If $\Psi \in \mathfrak{M}_{\text{black}}$, define:

- $\mathscr{M}(\Psi)$ final mass
- $\mathscr{A}(\Psi)$ final area
- $\mathscr{K}(\Psi)$ final surface gravity ($= 2\pi\mathscr{T}(\Psi)$ temperature)
- $\mathscr{U}(\Psi)$ u-coordinate of $\mathcal{H}^+$

Let $\{\Psi_p\}_{p\in[0,1]}$ be a smooth one-parameter family of data sets in $\mathfrak{M}$ which crosses $\partial\mathfrak{M}_{\text{black}}$ such that $\Psi_p \in \mathfrak{M}_{\text{black}}$ for $p \in [0, p_*]$ and $\Psi_{p_*} \in \partial\mathfrak{M}_{\text{black}}$. Then

$$\mathscr{A}(\Psi_p) - \mathscr{A}(\Psi_{p_*}) = \sqrt{2c}8\pi\mathscr{M}(\Psi_{p_*})^2|p - p_*|^{1/2} + o(|p - p_*|^{1/2}),$$

$$\mathscr{K}(\Psi_p) = \frac{\sqrt{2c}}{\mathscr{M}(\Psi_{p_*})}|p - p_*|^{1/2} + o(|p - p_*|^{1/2}),$$

Universal scaling laws

If $\Psi \in \mathfrak{M}_{\text{black}}$, define:

- $\mathscr{M}(\Psi)$ final mass
- $\mathscr{A}(\Psi)$ final area
- $\mathscr{K}(\Psi)$ final surface gravity ($= 2\pi\mathscr{T}(\Psi)$ temperature)
- $\mathscr{U}(\Psi)$ u-coordinate of $\mathcal{H}^+$

Let $\{\Psi_p\}_{p\in[0,1]}$ be a smooth one-parameter family of data sets in $\mathfrak{M}$ which crosses $\partial\mathfrak{M}_{\text{black}}$ such that $\Psi_p \in \mathfrak{M}_{\text{black}}$ for $p \in [0, p_*]$ and $\Psi_{p_*} \in \partial\mathfrak{M}_{\text{black}}$. Then

$$\mathscr{A}(\Psi_p) - \mathscr{A}(\Psi_{p_*}) = \sqrt{2c}8\pi\mathscr{M}(\Psi_{p_*})^2|p - p_*|^{1/2} + o(|p - p_*|^{1/2}),$$

$$\mathscr{K}(\Psi_p) = \frac{\sqrt{2c}}{\mathscr{M}(\Psi_{p_*})}|p - p_*|^{1/2} + o(|p - p_*|^{1/2}),$$

$$\mathscr{U}(\Psi_p) - \mathscr{U}(\Psi_{p_*}) = -\sqrt{2c}(1 + O(\varepsilon_0))\mathscr{M}(\Psi_{p_*})|p - p_*|^{1/2} + o(|p - p_*|^{1/2})$$

If $\Psi \in \mathfrak{M}_{\text{black}}$, define:

- $\mathscr{M}(\Psi)$ final mass
- $\mathscr{A}(\Psi)$ final area
- $\mathscr{K}(\Psi)$ final surface gravity ($= 2\pi\mathscr{T}(\Psi)$ temperature)
- $\mathscr{U}(\Psi)$ u-coordinate of $\mathcal{H}^+$

Let $\{\Psi_p\}_{p\in[0,1]}$ be a smooth one-parameter family of data sets in $\mathfrak{M}$ which crosses $\partial\mathfrak{M}_{\text{black}}$ such that $\Psi_p \in \mathfrak{M}_{\text{black}}$ for $p \in [0, p_*]$ and $\Psi_{p_*} \in \partial\mathfrak{M}_{\text{black}}$. Then

$$\mathscr{A}(\Psi_p) - \mathscr{A}(\Psi_{p_*}) = \sqrt{2c}8\pi\mathscr{M}(\Psi_{p_*})^2|p-p_*|^{1/2} + o(|p-p_*|^{1/2}),$$

$$\mathscr{K}(\Psi_p) = \frac{\sqrt{2c}}{\mathscr{M}(\Psi_{p_*})}|p-p_*|^{1/2} + o(|p-p_*|^{1/2}),$$

$$\mathscr{U}(\Psi_p) - \mathscr{U}(\Psi_{p_*}) = -\sqrt{2c}(1+O(\varepsilon_0))\mathscr{M}(\Psi_{p_*})|p-p_*|^{1/2} + o(|p-p_*|^{1/2})$$

as $p \nearrow p_*$, where $c \doteq \left| \frac{d^-}{dp^-}\Big|_{p=p_*} \mathscr{P}(\Psi_p)\right|$.

Universal scaling laws

If $\Psi \in \mathfrak{M}_{\text{black}}$, define:

- $\mathscr{M}(\Psi)$ final mass
- $\mathscr{A}(\Psi)$ final area
- $\mathscr{K}(\Psi)$ final surface gravity ($= 2\pi\mathscr{T}(\Psi)$ temperature)
- $\mathscr{U}(\Psi)$ u-coordinate of $\mathcal{H}^+$

Let $\{\Psi_p\}_{p\in[0,1]}$ be a smooth one-parameter family of data sets in $\mathfrak{M}$ which crosses $\partial\mathfrak{M}_{\text{black}}$ such that $\Psi_p \in \mathfrak{M}_{\text{black}}$ for $p \in [0, p_*]$ and $\Psi_{p_*} \in \partial\mathfrak{M}_{\text{black}}$. Then

$$\mathscr{A}(\Psi_p) - \mathscr{A}(\Psi_{p_*}) = \sqrt{2c}8\pi\mathscr{M}(\Psi_{p_*})^2|p-p_*|^{1/2} + o(|p-p_*|^{1/2}),$$

$$\mathscr{K}(\Psi_p) = \frac{\sqrt{2c}}{\mathscr{M}(\Psi_{p_*})}|p-p_*|^{1/2} + o(|p-p_*|^{1/2}),$$

$$\mathscr{U}(\Psi_p) - \mathscr{U}(\Psi_{p_*}) = -\sqrt{2c}(1+O(\varepsilon_0))\mathscr{M}(\Psi_{p_*})|p-p_*|^{1/2} + o(|p-p_*|^{1/2})$$

as $p \nearrow p_*$, where $c \doteq \left| \frac{d^-}{dp^-}\Big|_{p=p_*} \mathscr{P}(\Psi_p)\right|$.

Scaling of $\mathscr{K}$ measured numerically in this setting by [Murata–Reall–Tanahashi'13].

1. If $\Psi \in \partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{\pm 1}_{\text{stab}}$, then

$$\lim_{v\to\infty} Y(r\phi)|_{\mathcal{H}^+} = H_0[\phi]$$

$$\lim_{v\to\infty} v^{-1} \cdot Y^2(r\phi)|_{\mathcal{H}^+} = -2M^{-2}H_0[\phi],$$

1. If $\Psi \in \partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{\pm 1}_{\text{stab}}$, then

$$\lim_{v\to\infty} Y(r\phi)|_{\mathcal{H}^+} = H_0[\phi]$$

$$\lim_{v\to\infty} v^{-1} \cdot Y^2(r\phi)|_{\mathcal{H}^+} = -2M^{-2}H_0[\phi],$$

where for an open and dense subset of $\partial\mathfrak{M}_{\text{black}}$, $H_0[\phi] \neq 0$.

1. If $\Psi \in \partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{\pm 1}_{\text{stab}}$, then

$$\lim_{v\to\infty} Y(r\phi)|_{\mathcal{H}^+} = H_0[\phi]$$

$$\lim_{v\to\infty} v^{-1} \cdot Y^2(r\phi)|_{\mathcal{H}^+} = -2M^{-2}H_0[\phi],$$

   where for an open and dense subset of $\partial\mathfrak{M}_{\text{black}}$, $H_0[\phi] \neq 0$.
2. Let $\{\Psi_p\}_{p\in[0,1]} \subset \mathfrak{M}$ be a family crossing the threshold and $\beta = \beta(\Psi_p)$ is the inverse temperature of the black hole formed from Ψ_p.

1. If $\Psi \in \partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{\pm 1}_{\text{stab}}$, then

$$\lim_{v\to\infty} Y(r\phi)|_{\mathcal{H}^+} = H_0[\phi]$$

$$\lim_{v\to\infty} v^{-1} \cdot Y^2(r\phi)|_{\mathcal{H}^+} = -2M^{-2}H_0[\phi],$$

   where for an open and dense subset of $\partial\mathfrak{M}_{\text{black}}$, $H_0[\phi] \neq 0$.

2. Let $\{\Psi_p\}_{p\in[0,1]} \subset \mathfrak{M}$ be a family crossing the threshold and $\beta = \beta(\Psi_p)$ is the inverse temperature of the black hole formed from Ψ_p. Then

$$Y(r\phi)|_{\mathcal{H}^+}(v) = e^{-4\pi v/\beta} H_0^\flat[\phi] + O(\varepsilon v^{-1+\delta}),$$

Nonlinear Aretakis instability and transient instabilities

1. If $\Psi \in \partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{\pm 1}_{\text{stab}}$, then

$$\lim_{v\to\infty} Y(r\phi)|_{\mathcal{H}^+} = H_0[\phi]$$

$$\lim_{v\to\infty} v^{-1} \cdot Y^2(r\phi)|_{\mathcal{H}^+} = -2M^{-2}H_0[\phi],$$

   where for an open and dense subset of $\partial\mathfrak{M}_{\text{black}}$, $H_0[\phi] \neq 0$.

2. Let $\{\Psi_p\}_{p\in[0,1]} \subset \mathfrak{M}$ be a family crossing the threshold and $\beta = \beta(\Psi_p)$ is the inverse temperature of the black hole formed from Ψ_p. Then

$$Y(r\phi)|_{\mathcal{H}^+}(v) = e^{-4\pi v/\beta} H_0^\flat[\phi] + O(\varepsilon v^{-1+\delta}),$$

$$|Y^2(r\phi)|_{\mathcal{H}^+}(v)| \gtrsim |H_0^\flat[\phi]| v + O(\varepsilon v^{\delta}) \text{ for } v \leq \beta,$$

   where for generic curves $\{\Psi_p\}_{p\in[0,1]}$, $H_0^\flat[\phi] \neq 0$.

Nonlinear Aretakis instability and transient instabilities

1. If $\Psi \in \partial\mathfrak{M}_{\text{black}} = \mathfrak{M}^{\pm 1}_{\text{stab}}$, then

$$\lim_{v\to\infty} Y(r\phi)|_{\mathcal{H}^+} = H_0[\phi]$$

$$\lim_{v\to\infty} v^{-1}\cdot Y^2(r\phi)|_{\mathcal{H}^+} = -2M^{-2}H_0[\phi],$$

   where for an open and dense subset of $\partial\mathfrak{M}_{\text{black}}$, $H_0[\phi] \neq 0$.

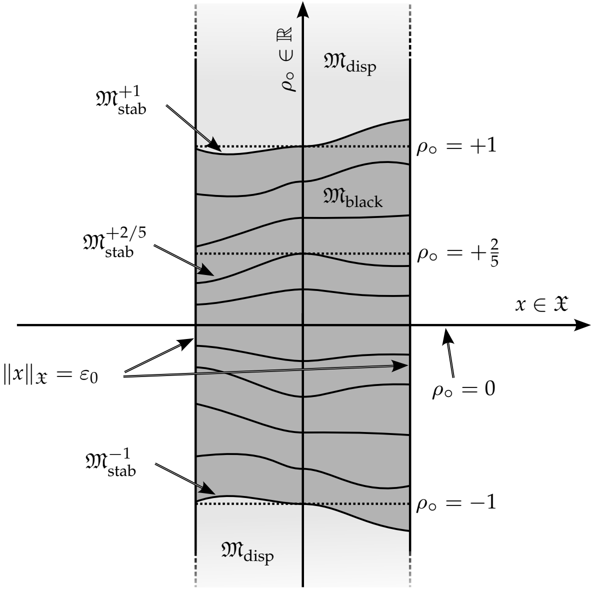
2. Let $\{\Psi_p\}_{p\in[0,1]} \subset \mathfrak{M}$ be a family crossing the threshold and $\beta = \beta(\Psi_p)$ is the inverse temperature of the black hole formed from Ψ_p. Then

$$Y(r\phi)|_{\mathcal{H}^+}(v) = e^{-4\pi v/\beta}H_0^\flat[\phi] + O(\varepsilon v^{-1+\delta}),$$

$$|Y^2(r\phi)|_{\mathcal{H}^+}(v)| \gtrsim |H_0^\flat[\phi]|v + O(\varepsilon v^{\delta}) \text{ for } v \leq \beta,$$

   where for generic curves $\{\Psi_p\}_{p\in[0,1]}$, $H_0^\flat[\phi] \neq 0$.

Numerical evidence for this transient instability timescale given in [Murata–Reall–Tanahashi'13].



Thank you for your attention!