Simulations of the Einstein-Maxwell-charged scalar system in null coordinates

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- Collapse simulations in (single) null coordinates (5 slides)
- Type-II charged critical collapse (3 slides)
- Extremal charged critical collapse (4 slides)

Single null coordinates

Any spacetime dimension, any symmetry, coordinates (u, x, θ^i):

$$ds^2 = -2G(du\,dx - B\,du^2) + R^2\gamma_{ij}(d\theta^i + \beta^i\,du)(d\theta^j + \beta^j\,du)$$

- surfaces $u = \text{const}$ are null cones
- ... their generators are curves of constant (u, θ^i)
- ... with tangent vector $G^{-1}\partial_x$ **(outgoing null)**
- R is fixed by $\det\gamma_{ij} \equiv \det\gamma_{ij}(\text{flat})$
- B fixes the coordinate x incrementally in u (radial shift)

Hierarchical structure of the Einstein eqns

Key geometric object: the **ingoing null vector**

$$\Xi := \partial_u - B\partial_x - \beta^i\partial_i$$

normal to the cross section $x = \text{const}$ of the null cone $u = \text{const}$

Consider γ_{ij} and R or G given

$$\left(\ln \frac{G}{R_{,x}}\right)_{,x} = \frac{R}{R_{,x}} S_G(\partial_x, \partial_i, \gamma_{ij}) \quad (1a)$$

$$R_{,xx} - (\ln G)_{,x} R_{,x} = -R S_G(\ldots) \quad (1b)$$

$$(R^4 \gamma_{ij} \beta^j{}_{,x})_{,x} = S_i(G, \ldots) \quad (2)$$

$$(R\, \Xi R)_{,x} = S_R(\beta^i, \ldots) \quad (3)$$

$$(R\, \Xi \gamma_{ij})_{,x} = S_{ij}(\Xi R, \ldots) \quad (4)$$

$$(B_{,x} - \Xi \ln G)_{,x} = S_{\mathcal{H}}(\ldots) \quad (5)$$

Two particular and three generic formulations

In **all** formulations, evolve γ_{ij} and matter, plus

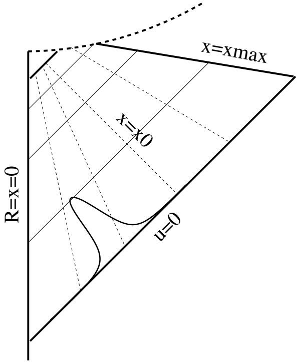
- **Bondi**: $R = x$ determines B, constrain G (needs $R_{,x} > 0$)
  - wave extraction, no origin (Babiuc+11, Moxon+23)
  - supernovae (Siebel+02)
- **affine**: $G = 1$ determines B, constrain R
  - adS problems, no origin (Chesler+11-22)
- **evolve-**R: constrain G (needs $R_{,x} > 0$), evolve R, any B
  - spherical critical collapse, $B = 0$ (Garfinkle95)
  - mildly non-spherical critical collapse, B adapted to self-similarity (G-Baumgarte-Hilditch24)
- **evolve-**G: constrain R, evolve G, any B (G-Martel25, this talk)
- **free evolution**: evolve R and G, any B
  - black hole interiors, spherical symmetry, $B = 0$, no origin (Burko-Ori97, Murata+13)

Key ingredients of the code (G-Baumgarte-Hilditch24)

- Solve timestep problem $\Delta u \lesssim (\Delta x)^2 \, \Delta\theta$ (Winicour+94)
  - pseudospectral in $Y_{\ell m}(\theta, \varphi)$
  - at grid point i, truncate to $\ell \leq \min(i, \ell_{\max})$
  - CFL limit is now $\Delta u \lesssim \Delta x$
- Make finite differencing strictly causal
  - in hierarchy equations, grid point i depends only on i, $i-1$, ...
  - in evolution equations $\partial_u = \Xi + B\partial_x + \ldots$ upwind $B\partial_x$
  - (null or future spacelike) outer boundary is trivial, so is excision
- Incremental gauge choice B
  - always fix $R = 0$ at $x = 0$
  - make $x = x_0$ ingoing null "on average": type-II critical collapse without mesh refinement (Garfinkle95)
  - use non-spherical part of B to keep coordinates regular

Outlook: axisymmetric type-II critical collapse

- New evolve-G formulation allows evolving through horizon with arbitrary B
- Use this to push axisymmetric type-II critical collapse to larger non-sphericity and higher fine-tuning
- Long-term goal: vacuum critical collapse
- Worry: do **outgoing** coordinate null cones form caustics?



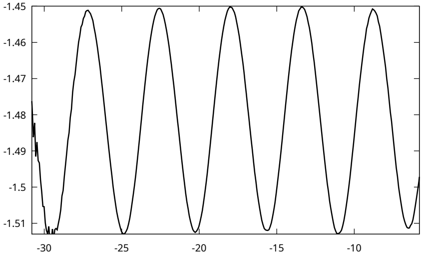
Spherical Einstein-Maxwell-scalar

- $D_a = \nabla_a + iqA_a$
- Maxwell gauge $A_x = 0$
- initial data $\phi(0, x) = p\, e^{i\omega x}$ Gaussian(x)
- collapse diagnostic Hawking compactness $C = 2M/R = 1 - |\nabla R|^2 > 1$
- conserved local mass M
- conserved local charge Q
- $\mathcal{M} := M + Q^2/(2R)$ constant in RN
- collapse threshold p_* to 15 digits
- tune x_0 to 2-4 digits
- expect $M(p) \sim (p - p_*)^{0.374}$ and $Q \sim (p - p_*)^{0.883}$ (G-Martín-García96)

Real versus complex initial data

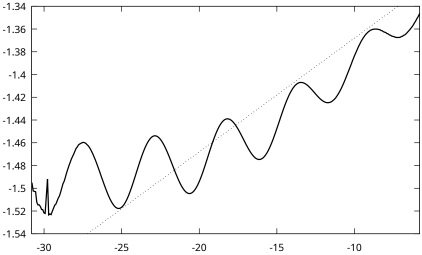
Mass fine structure

$\ln M_{\mathrm{FMOTS}}(p)$ versus $\ln(p - p_*)$, expected $M(p) \sim (p - p_*)^{0.374}$ taken out



real scalar field

$\ln M$ versus $\ln(p - p_*)$



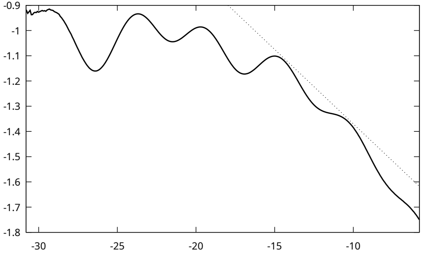
charged complex scalar field

dashed: fit to $M \sim (p - p_*)^{0.384}$ at low fine-tuning (Baumgarte's talk)

Universality but critical exponent modified a bit

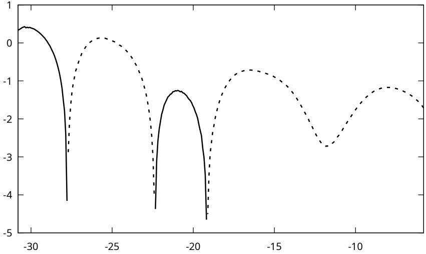
Charge fine structure

In Q_{FMOTS} versus $\ln(p - p_*)$, expected $Q \sim (p - p_*)^{0.883}$ (no change of sign) (G-Martín-García96) taken out



charged scalar field (as before)

dashed: fit to $Q \sim (p - p_*)^{0.824}$ at low fine-tuning

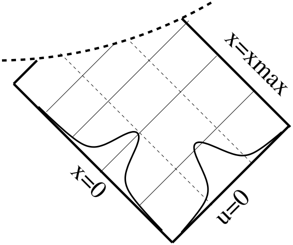


charged scalar field (q times 10)

dashed: Q opposite sign

Critical exponent universal but not fine-structure

General null rectangle setup (G-M25)



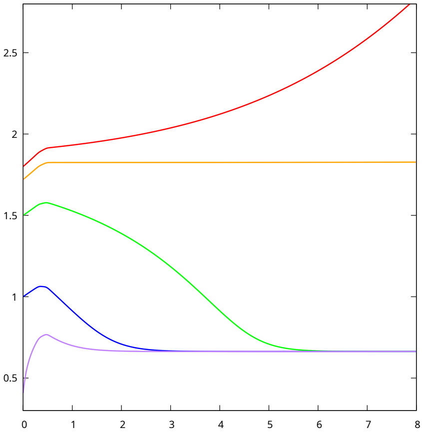
- spherical Einstein-Maxwell-scalar
- corner data R_*, Q_*, $\mathcal{M}_*$
- boundary data $\phi(x)$ and/or $\phi(u)$
- double-null gauge $B = 0$

Example of toy extremal critical collapse (Kehle-Unger-Angelopoulos)

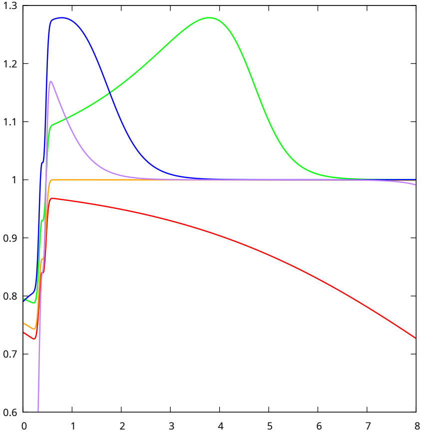
- $\mathcal{M}_* = 1,\ Q_* = 1.1,\ R_* = 100$
- $\phi(u, 0) = 0,\ \phi(0, x) = p\,\mathrm{Gaussian}(x),\ q = 0$
- ... so ϕ real and $Q = 1.1$ everywhere
- $R \in [0.4, 100]$ on $x = 0$, up to $R \in [100, 108]$ on $u = 0$

Well supercritical case $p = 1.6$

- $\mathcal{M}(u,0) = 1$ increases to $\mathcal{M}(u, x_{\max}) \simeq 1.245$ for all u
- $Q(u,x) = 1.1$ everywhere
- $r_\pm = \mathcal{M} \pm \sqrt{\mathcal{M}^2 - Q^2} \simeq 1.83, 0.66$
- snapshots at $u = 98.2$, 98.28 (event horizon), 98.5, 99.0, 99.6



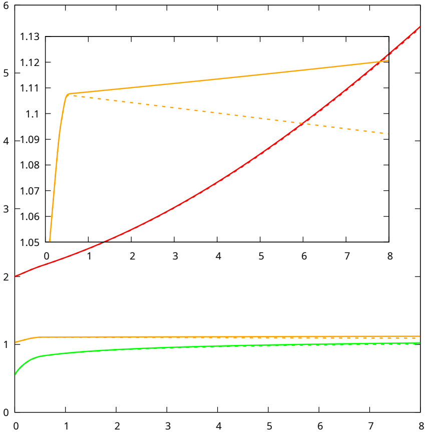
R versus x



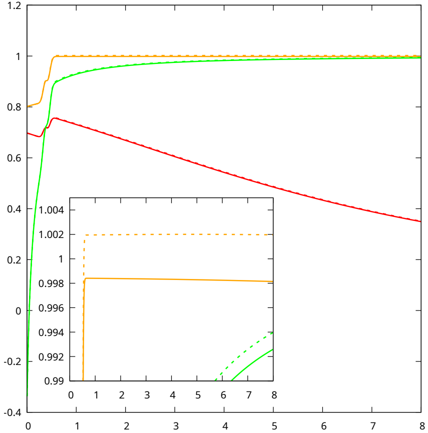
$C := 2M/R$ versus x

Near sub-critical $p = 1.02$ and near-supercritical $p = 1.03$

- $\mathcal{M}(u, 0) = 1$, $\mathcal{M}(u, x_{\text{max}}) \simeq 1.099$ (1.101) for $p = 1.02$ (1.03)
- $r_\pm \simeq 1.15, 1.05$ (for $p = 1.03$)
- snapshots at $u = 98.0$, 98.97, 99.45



R versus x



$C := 2M/R$ versus x

Outlook: spherical extremal critical collapse

- Extremal BHs at the threshold of collapse: true for Einstein-Maxwell-Vlasov (Kehle-Unger22, East25)
- Einstein-Maxwell-charged scalar: true for toy extremal critical collapse on null rectangle, unclear for collapse from data on $\mathbb{R}^3$
- Backscatter from the scalar field that made $\mathcal{M}_*, Q_*$ could yet make trapped surfaces inside the extremal BH, so that it is **not** a threshold solution
- Play with data on a regular null cone (unsuccessful so far)
- Evolve characteristic gluing data for extremal horizon (Kehle-Unger24, Gadioux+25) to see if trapped surfaces form
- Do the same in 4+1 twisting vacuum (Reall-Santos+)