Black hole solutions in affine null coordinates

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Extremal Black Holes and the Third Law of Black Hole Thermodynamics
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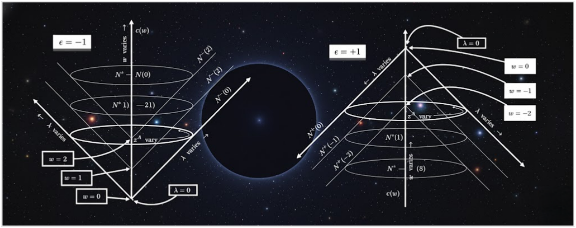


◊ **Motivation**

◊ **Affine null formulation of Einstein equations**

- Vacuum case
- Non vacuum case with spherical symmetry
- Exact solutions

◊ **Conformal affine null Einstein equations**

◊ **Rotating, Axisymmetric Black Holes**

- Null coordinates systems are based in a family of null hypersurfaces where the null coordinate x^0 labels these hypersurfaces and the angular coordinates x^A $(A = 2, 3)$ label the set of null geodesic rays. The metric reads:

$$g_{ab}dx^a dx^b = [g_{00}dx^0 + 2g_{01}dx^1 + 2g_{0A}dx^A]dx^0 + g_{AB}dx^A dx^B \quad . \tag{1}$$

- In the Bondi–Sachs metric, the areal radial coordinate r parametrizes the points along outgoing null rays:

$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB} U^A U^B\right) du^2 - 2e^{2\beta}\, du\, dr - 2r^2 h_{AB} U^B\, du\, dx^A + r^2 h_{AB}\, dx^A dx^B, \tag{2}$$

  where $\det(h_{AB}) = \det(q_{AB})$, with $q_{AB}(x^C)$ a fixed reference metric on the unit two-sphere.
- The function $\beta(u, r, x^A)$ measures the deviation from an affine parametrization of the outgoing null geodesics.
- In particular if λ is an affine parameter then

$$\partial_r \lambda(u, r, x^A) = e^{2\beta}.$$

- The areal coordinate r becomes singular where the expansion of null rays vanishes, whereas the affine coordinate λ is singular only at caustics.

- In an affine null formulation, the metric takes the form

$$ds^2 = -W\,dw^2 + 2\epsilon\,dw\,d\lambda + r^2 h_{AB}\left(dx^A - W^A dw\right)\left(dx^B - W^B dw\right). \tag{3}$$

- The covector $\ell_a = \epsilon\,dw$ generates the null hypersurfaces $w = \mathrm{const}$. The parameter $\epsilon = -1$ corresponds to a retarded null coordinate, while $\epsilon = +1$ corresponds to an advanced null coordinate.
- All metric functions r, W, W^A, and h_{AB} depend on (w, λ, x^A).

Using metrics of this type, there are four standard ways to set up a characteristic initial–boundary value problem:

- timelike–null formulation,
- vertex–null formulation,
- double–null (or $2+2$) formulation,
- affine–null formulation.

In general, the Einstein equations $E_{ab} := G_{ab} - 8\pi T_{ab} = 0$ can be decomposed into three classes:

- **Main equations:**
  - hypersurface equations,
  - evolution equations;
- **Supplementary equations** (imposed only on a boundary);
- **Trivial equation**.

Notably, for a Bondi-Sachs metrics, these equations adquire a hierarchical structure: Some main equations take the schematic form of hypersurface equations

$$\beta_{,r} = \mathcal{N}_\beta[h_{CD}] \tag{4}$$
$$(r^4 e^{-2\beta} h_{AB} U^B_{,r})_{,r} = \mathcal{N}_U[h_{CD}, \beta] \tag{5}$$
$$V_{,r} = \mathcal{N}_V[h_{CD}, \beta, U^C], \tag{6}$$

while other of the main equations take the form of evolution equations

$$M^A M^B (r h_{AB,u})_{,r} = \mathcal{N}_h[h_{CD}, \beta, U^C, V]. \tag{7}$$

Given h_{AB} on an initial null hypersurface $u = 0$, the main equations can be integrated radially in sequential order to determine the initial values of

$$h_{AB} \to \beta \to U^A \to V \to h_{AB,u} \qquad at \qquad u = 0$$

in terms of their integration constants on the boundary,. After determining $h_{AB,u}$ at $u = 0$, the hypersurface data h_{AB} can be advanced to $u = \Delta u$ by a finite difference procedure. It is the algorithm underlying the PITT null code.

In contrast, for a affine null metric the vacuum field equations take the form:

$$r^{-1} r_{,\lambda\lambda} = \mathcal{H}_r[h_{CD}] \tag{8}$$

$$(r^4 h_{AB} W^B_{,\lambda})_{,\lambda} = \mathcal{H}_W[h_{CD}, r] \tag{9}$$

$$\left(2(r^2)_{,u} - \mathcal{V}(r^2)_{,\lambda}\right)_{,\lambda} = \mathcal{H}_\mathcal{V}[h_{CD}, r, W^C] \tag{10}$$

$$M^A M^B (r h_{AB})_{,u\lambda} = \mathcal{H}_h[h_{CD}, r, W^C, \mathcal{V}], \tag{11}$$

Eq.(10) breaks the hierarchical structure due to the presence of $r_{,u}$.

However, as first shown by Jeff Winicour [Phys. Rev. D 87, 124027 (2013)], the hierarchical structure can be restored through the introduction of new variables:

$$\mathcal{Y} = \mathcal{V} - \frac{2r_{,u}}{r_{,\lambda}}, \qquad \rho = r_{,u} \tag{12}$$

and $\sigma = \frac{1}{4}m^A m^B h_{AB,\lambda}$, $\kappa = \frac{1}{4}m^A m^B h_{AB,u}$, $J = 4(r_{,\lambda}\kappa - \rho\sigma)$ With these definitions, the field equations take the form

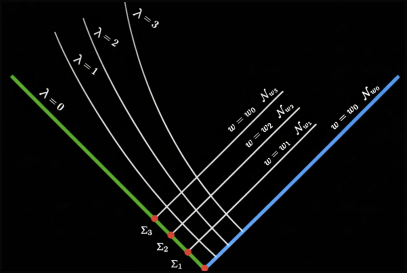
$$r^{-1}r_{,\lambda\lambda} = H_r[h_{CD}] \tag{13}$$

$$(r^4 h_{AB} W^B_{,\lambda})_{,\lambda} = H_W[h_{CD}, r] \tag{14}$$

$$\left(\mathcal{Y}(r^2)_{,\lambda}\right)_{,\lambda} = H_{\mathcal{Y}}[h_{CD}, r, W^C] \tag{15}$$

$$\left(\frac{rJ}{r_{,\lambda}}\right)_{,\lambda} = H_J[h_{CD}, r, W^C, \mathcal{Y}] \tag{16}$$

$$\left(\frac{\rho}{r_{,\lambda}}\right)_{,\lambda\lambda} = H_\rho[h_{CD}, r, J], \tag{17}$$



Affine–null formulation of GR on intersecting null hypersurfaces

Mädler Phys.Rev.D 99 (2019) 10, 104048 studied the affine null formulation when the boundary is null. In this case, the metric reads

$$ds^2 = -W\,dw^2 + 2\epsilon\,dw\,d\lambda + r^2 h_{AB}(dx^A - W^A dw)(dx^B - W^B dw), \qquad (18)$$

with boundary conditions at $\lambda = 0$: $W = W_{,\lambda} = W^A = 0, \qquad \epsilon^2 = 1.$

The null vectors ℓ^a and n^a have the coordinate expressions

$$\ell^a \partial_a = \partial_\lambda \ , \tag{19}$$

$$n^a \partial_a = -\epsilon \partial_w - \frac{1}{2} W \partial_\lambda - \epsilon W^A \partial_A \ . \tag{20}$$

where in particular $n^a \partial_a|_{\mathcal{B}} = -\epsilon \partial_w$. The expansion rates, $\theta_{(\ell)} := \nabla_a \ell^a$ and $\theta_{(n)} := \nabla_a n^a$, for both null vectors are

$$\theta_{(\ell)} = \frac{2 r_{,\lambda}}{r} \tag{21}$$

$$\theta_{(n)} = -\epsilon \partial_w \ln r^2 - \frac{(r^2 W)_{,\lambda}}{2 r^2} - \frac{\epsilon \mathcal{D}_A (r^2 W^A)}{r^2} \ . \tag{22}$$

At the boundary $\mathcal{B}$, $\theta_{(n)} = \frac{2 r_{,w}}{r}$.

The initial value problem at two intersecting null hypersurfaces for the vaccum Einstein equations split into three different groups:

(i) a hierarchical set of differential equations along the null rays on the boundary $\mathcal{B}$ where $\lambda = 0$.

(ii) a hierarchical set of hypersurface equations on the null hypersurfaces. $\mathcal{N}_{w_0}$ where $w = w_0$

(iii) two evolution equations to propagate the initial data from a given null hypersurface $\mathcal{N}_{w_0}$ to a null hypersurface $\mathcal{N}_{w_0+\Delta w}$.

This set of equations requires the following initial-boundary data

- Initial data on Σ, that are functions of x^A, only

$$h_{AB} \ , \ \sigma_{AB} \ , \ r, \ W^A_{,\lambda} \ , \ \rho_\ell = r_{,y} \ , \ \rho_n = r_{,w} \tag{23}$$

- free data on $\mathcal{B}$ which are functions depending on w and x^A

$$n_{AB} = h_{AB,w} \tag{24}$$

- free data on $\mathcal{N}_{w_0}$ which are functions depending on λ and x^A

$$\sigma_{AB} = h_{AB,y} \tag{25}$$

Can the formalism be generalized to include matter fields?

In Phys. Rev. D 104 (2021), 084048, we showed that, under spherical symmetry, the affine-null hierarchy can be restored in the presence of a Maxwell field without sources.
We introduce an electromagnetic potential $A_a = (A_w, A_\lambda, 0, 0)$ with Faraday tensor $F_{ab} = 2A_{[b,a]}$. In the adapted null gauge,

$$A_a = \alpha(w, \lambda)\, dw. \tag{26}$$

The field equations are

$$R_{ab} = 8\pi \left(T_{ab} - \tfrac{1}{2} g_{ab} T^c{}_c \right).$$

with

$$T_{ab} = -\frac{1}{4\pi} \left(F_{ac} F_b{}^c - \tfrac{1}{4} g_{ab} F^{cd} F_{cd} \right),,$$

determined by F_{ab}.

- The fundamental metric and matter variables are prescribed as independent constants on Σ_{w_0}:

$$A := \alpha_{,\lambda}\Big|_{\Sigma_{w_0}}\ ,\quad r_0 := r\Big|_{\Sigma_{w_0}}\ ,\quad N_0 = N\Big|_{\Sigma_{w_0}}\ ,\quad \Theta_0 = \Theta\Big|_{\Sigma_{w_0}}\ , \tag{27}$$

- The non-extremal Reissner-Nordström metric: Let us choose now the data on Σ_0 such that $N_0 = 0$, and $\Theta_0 \neq 0$. The Bondi mass is given by:

$$\begin{aligned} m_B(w) :=& \lim_{\substack{\lambda\to\infty \\ w=\text{const}}} \frac{r}{2}(1 + g^{ab} r_{,a} r_{,b}) \\ =& \frac{r_0}{2}(1 + 2N_0\Theta_0) + \frac{Q^2}{2r_0} = m|_{\Sigma_0} + \frac{Q^2}{2r_0} \equiv m = \text{const.} \end{aligned} \tag{28}$$

The final solution is:

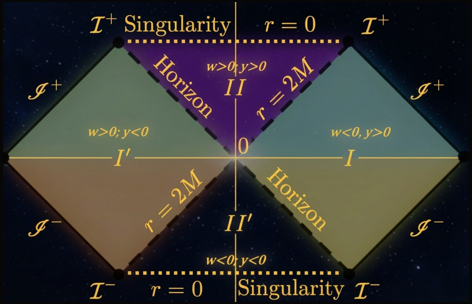
$$ds^2 = -\frac{1 + 2\kappa_H(r + r_0)}{r^2}\lambda^2 dw^2 - 2dwd\lambda + r^2 d\Omega^2$$

with $r = r_H - \kappa_H w\lambda$.

If $Q = 0$ this solution reduces to

$$ds^2 = \frac{2\lambda^2}{8m^2 - w\lambda} dw^2 - 2dwd\lambda + \left(2m - \frac{w\lambda}{4m}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{29}$$

Israel obtained this metric by analyzing the null geodesics in the standard Schwarzschild metric representation adopting the w coordinate to the null structure. The past and future horizons in the above metric are given by $\lambda = 0$ and $w = 0$ respectively.



The case of a scalar field was studied by Winicour, Mädler, and collaborators in Phys. Rev. D 100 (2019), 104017; Phys. Rev. D 110 (2024), 044061. This framework was also employed to analyze critical collapse, where affine null coordinates allow the evolution to be followed beyond the formation of the event horizon and up to the physical singularity.

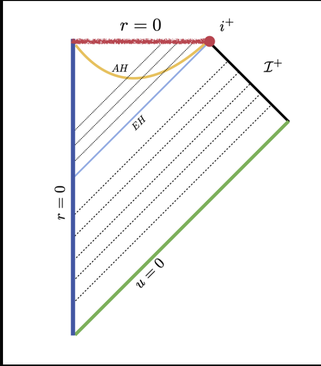


Figure extracted from PRD 110 (2024) 4, 044061 (Mädler, Baake, Hosseini, Winicour)

In (Phys. Rev. D **111** (2025) 124001) we extended the affine-null formulation to the Einstein–Maxwell system coupled to a massless complex scalar field.
The field equations are

$$E_{ab} := 0 = R_{ab} - \frac{\kappa}{4\pi}\left(g^{cd}F_{ac}F_{bd} - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right) - \kappa\left[\frac{1}{2}(\overline{\mathcal{D}_a\Phi})(\mathcal{D}_b\Phi) + \frac{1}{2}(\overline{\mathcal{D}_b\Phi})(\mathcal{D}_a\Phi)\right],$$
$$E_b := 0 = \nabla_a F^a{}_b + 4\pi j_b,$$
$$E := 0 = \mathcal{D}^a\mathcal{D}_a\Phi, \qquad \bar{E} := 0 = \overline{\mathcal{D}^a\mathcal{D}_a\Phi},$$

where $\mathcal{D}_c = \nabla_c + iqA_c$ is the gauge-covariant derivative, and q is the scalar charge and with current $j_b = \frac{iq}{2}\left(\bar{\Phi}\,\mathcal{D}_b\Phi - \Phi\,\overline{\mathcal{D}_b\Phi}\right)$. Its integral over a three-volume $\mathbb{V}$ defines the charge function

$$Q = \int_{\mathbb{V}} j^a d\Sigma_a = \frac{1}{4\pi}\oint_{\Sigma} F^{ab}\,\ell_{[a}n_{b]}\,\sqrt{\det(g_{AB})}\,dx^2\,dx^3. \tag{30}$$

Consider the metric

$$ds^2 = -V(w,\lambda)\,dw^2 + 2\epsilon\,dw\,d\lambda + r^2(w,\lambda)\,q_{AB}\,dx^A dx^B.$$

For the charge function (30), we find $Q = -\epsilon r^2 \alpha_{,\lambda}$. The nonzero components of E_{ab} are

$$(E_{ww})^{[S]} \quad 0 = r_{,ww} - \frac{V(r^2 V_{,\lambda})_{,\lambda}}{4r} - \frac{\epsilon}{2}(r_{,w}V_{,\lambda} - r_{,\lambda}V_{,w}) + \frac{1}{2}\kappa r\left[|\Phi_{,w}|^2 + \frac{VQ^2}{8\pi r^4} + q^2\alpha^2|\Phi|^2 + iq\alpha\left(\Phi\bar{\Phi}_{,w} - \bar{\Phi}\Phi_{,w}\right)\right],$$

$$(E_{w\lambda})^{[T]} \quad 0 = r_{,w\lambda} + \frac{\epsilon}{4r}(r^2 V_{,\lambda})_{,\lambda} - \frac{1}{2}\kappa r\left[\frac{\epsilon Q^2}{8\pi r^4} - \frac{1}{2}\left(\bar{\Phi}_{,w}\Phi_{,\lambda} + \bar{\Phi}_{,\lambda}\Phi_{,w}\right) - \frac{iq\alpha}{2}(\Phi\bar{\Phi}_{,\lambda} - \bar{\Phi}\Phi_{,\lambda})\right]$$

$$(E_{\lambda\lambda})^{[M]} \quad 0 = r_{,\lambda\lambda} + \frac{1}{2}\kappa r|\Phi_{,\lambda}|^2,$$

$$(q^{AB}E_{AB})^{[M]} \quad 0 = -(Vrr_{,\lambda} + 2\epsilon r r_{,w} - \lambda)_{,\lambda} - \frac{\kappa}{8\pi}\frac{Q^2}{r^2}.$$

and those of E_a are

$$(E_w)^{[S]} \quad : \quad 0 = Q_{,w} + \epsilon V Q_{,\lambda} + 4\pi q^2 r^2 \alpha |\Phi|^2 + 2i\pi q r^2 \left(\Phi \bar{\Phi}_{,w} - \bar{\Phi}\Phi_{,w}\right) , \qquad (31)$$

$$(E_\lambda)^{[M]} \quad : \quad 0 = Q_{,\lambda} + 2\pi r^2 q i (\overline{\Phi}\Phi_{,\lambda} - \Phi\overline{\Phi}_{,\lambda}). \qquad (32)$$

while E gives

$$E^{[M]} : 0 = 0 = (r^2\Phi_{,w})_{,\lambda} + (r^2\Phi_{,\lambda})_{,w} + \epsilon(r^2 V \Phi_{,\lambda})_{,\lambda} + 2iqr\alpha(r\Phi)_{,\lambda} - i\epsilon q \Phi Q \, . \qquad (33)$$

- **main equations**

$$E_{\lambda\lambda} \, , \quad q^{AB} E_{AB} \, , \quad E_\lambda \, , \quad E \, . \qquad (34)$$

- **supplementary equations :**

$$E_{ww} \, , \quad E_w . \qquad (35)$$

- **trivial equation :**

$$E_{w\lambda} . \qquad (36)$$

CAN THE HIERARCHY BE RESTORED?

In recent work with T. Mädler and R. Gannouji (Phys. Rev. D **111** (2025) 124001), the hierarchical structure is restored by introducing the following auxiliary fields:

$$\rho = r_{,w} \;, \tag{37a}$$

$$Z = Vr r_{,\lambda} + 2\epsilon r \rho - \lambda \;, \tag{37b}$$

$$\mathcal{L} = \frac{2r\left(r_{,\lambda}\Phi_{,w} - \rho\Phi_{,\lambda}\right) + \epsilon(Z+\lambda)\Phi_{,\lambda}}{r_{,\lambda}}. \tag{37c}$$

$$\mathcal{J} = i(\overline{\Phi}\Phi_{,\lambda} - \overline{\Phi}_{,\lambda}\Phi). \tag{37d}$$

Allowing to cast the main equations into

$$r_{,\lambda\lambda} = -\frac{1}{2}\kappa r|\Phi_{,\lambda}|^2 \,, \tag{38a}$$

$$Q_{,\lambda} = -2\pi r^2 q \mathcal{J} \,, \tag{38b}$$

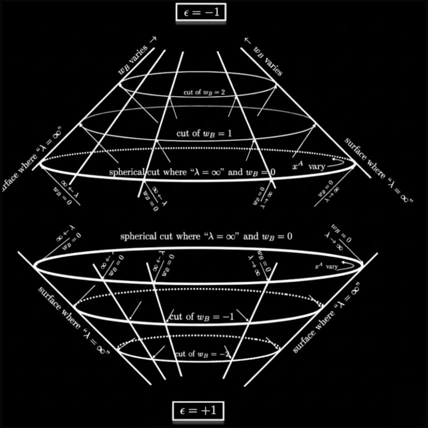
$$\alpha_{,\lambda} = -\epsilon\frac{Q}{r^2} \,, \tag{38c}$$

$$Z_{,\lambda} = -\frac{\kappa}{8\pi}\frac{Q^2}{r^2} \,, \tag{38d}$$

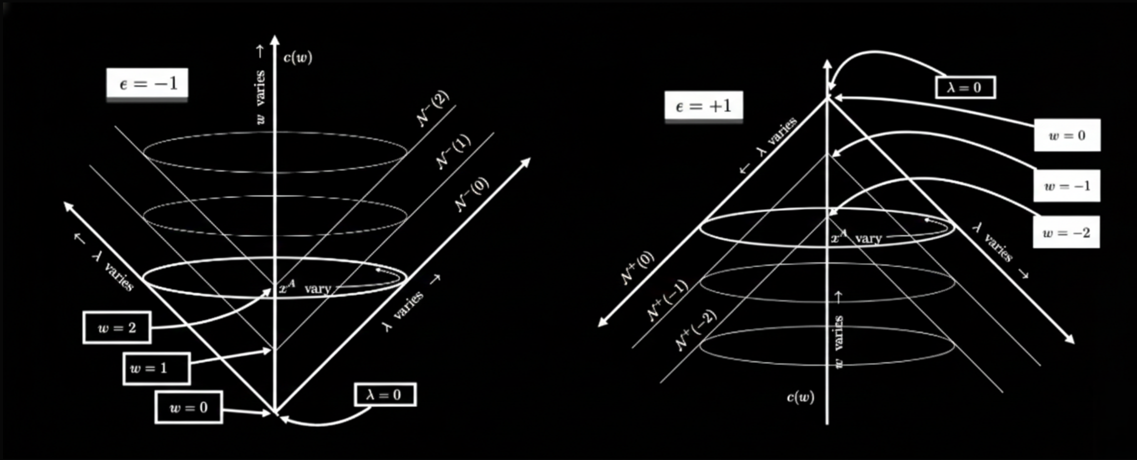
$$\mathcal{L}_{,\lambda} = -\frac{\epsilon(\lambda + Z)\Phi_{,\lambda}}{r} - 2iq\alpha(r\Phi)_{,\lambda} + i\frac{\epsilon q \Phi Q}{r} \,, \tag{38e}$$

$$\begin{aligned} V_{,\lambda\lambda} = & -\frac{1}{\lambda}\left(\frac{\lambda^2}{r^2}\right)_{,\lambda} + \frac{2Zr_{,\lambda}}{r^3} + \frac{\kappa}{2\pi}\frac{Q^2}{r^4} \\ & -\frac{\kappa\epsilon}{2r}\left(\overline{\Phi_{,\lambda}}\mathcal{L} + \overline{\mathcal{L}}\Phi_{,\lambda}\right) + \kappa q \epsilon \alpha \mathcal{J} \,, \end{aligned} \tag{38f}$$

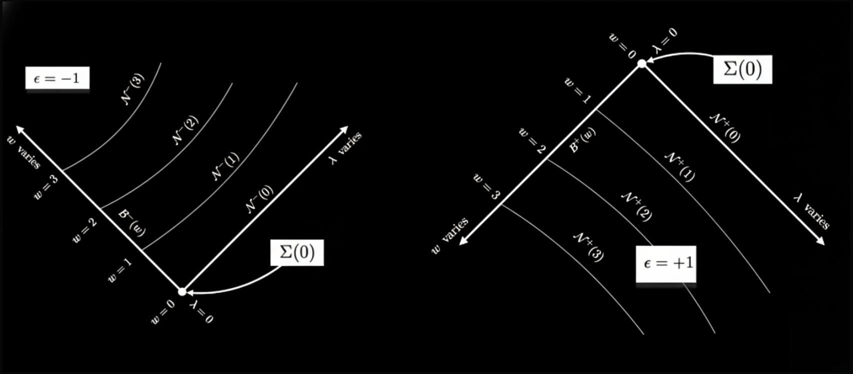
$$\Phi_{,w} = \frac{\mathcal{L}}{2r} - \frac{\epsilon}{2}V\Phi_{,\lambda} \,. \tag{38g}$$



(Phys. Rev. D **111** (2025) 124001)



(Phys. Rev. D **111** (2025) 124001)


ϵ = −1
Σ(0)
Σ(0)
ϵ = +1

A solution of the Einstein-Maxwell-scalar field equations on the domain $\mathcal{B}^\epsilon(w) \cup \mathscr{N}^\epsilon(w)$ can be found for the double-null CIBVP with the following specifications:

- On $\mathscr{N}^\epsilon(0)$ specify a free (sufficiently differentiable) complex function $F_\Phi(\lambda)$;
- On $\mathcal{B}^\epsilon(0)$ specify a free (sufficiently differentiable) complex function $\mathcal{N}_\mathcal{B}(w)$;
- The following fields are trivial everywhere on $\mathcal{B}^\epsilon(w)$

$$\alpha(w,0) = V(w,0) = V_{,\lambda}(w,0) = 0. \tag{39}$$

- On the common intersection $\Sigma_0 = \mathcal{B}^\epsilon(0) \cap \mathscr{N}^\epsilon(0)$ specify the values for the following fields

$$\Phi(0,0)\ , \quad r(0,0)\ , \quad r_{,w}(0,0)\ , \quad r_{,\lambda}(0,0)\ , \quad Q(0,0). \tag{40}$$

where in particular $r(0,0) > 0$.

This double-null CIBVP can be solved as follows:

1. Given $\mathcal{N}_{\mathcal{B}}(w)$ and $\Phi(0,0)$, determine the scalar field on the boundary $\mathcal{B}^{\epsilon}(w)$ from
$$\Phi_{,w}(w,0) = \mathcal{N}_{\mathcal{B}}(w).$$
2. Using $\mathcal{N}_{\mathcal{B}}(w)$, $\Phi(w,0)$, and the initial values $r(0,0)$, $r_{,w}(0,0)$, $Q(0,0)$, and $r_{,\lambda}(0,0)$, solve the boundary hierarchy to obtain r, Q, and $r_{,\lambda}$ on $\mathcal{B}^{\epsilon}(w)$:
$$\begin{aligned} E_{ww}|_{\mathcal{B}} \rightarrow \qquad & 0 = r_{,ww} + \tfrac{1}{2}\kappa r|\mathcal{N}_{\mathcal{B}}|^2, \\ E_{w}|_{\mathcal{B}} \rightarrow \qquad & 0 = Q_{,w} + 2i\pi q r^2 \left(\bar{\Phi}\mathcal{N}_{\mathcal{B}} - \Phi\bar{\mathcal{N}}_{\mathcal{B}}\right), \qquad \text{on } \mathcal{B}^{\epsilon}. \\ q^{AB}E_{AB}|_{\mathcal{B}} \rightarrow \qquad & 0 = 2\epsilon(r r_{,\lambda})_{,w} - 1 + \tfrac{\kappa}{8\pi}\tfrac{Q^2}{r^2}, \end{aligned}$$
3. From r, $r_{,w}$, and $\mathcal{N}_{\mathcal{B}}$ determine Z and $\mathcal{L}$, thus fixing all boundary data for the hypersurface equations (41a)–(41f).
4. On the initial null hypersurface $w=0$, use $\Phi(0,0)$ and $\mathcal{F}_{\Phi}$ to integrate
$$\Phi_{,\lambda}(0,\lambda) = \mathcal{F}_{\Phi}(\lambda).$$

5. Given initial data $\Phi(0, \lambda)$ on $\mathscr{N}^{\epsilon}(0)$ $(w = 0)$, integrate hierarchically the hypersurface equations (41a)–(41f) to obtain

$$r \to Q \to \alpha \to Z \to \mathcal{L} \to V \quad \text{at } w = 0,$$

using $\mathcal{F}_{\Phi}(\lambda)$ and the boundary data at $w = 0$.

Namely,

$$
\begin{aligned}
r_{,\lambda\lambda} &= -\frac{1}{2}\kappa r|\Phi_{,\lambda}|^2 \,, &\text{(41a)}\\
Q_{,\lambda} &= -2\pi r^2 q\mathcal{J} \,, &\text{(41b)}\\
\alpha_{,\lambda} &= -\epsilon\frac{Q}{r^2} \,, &\text{(41c)}\\
Z_{,\lambda} &= -\frac{\kappa}{8\pi}\frac{Q^2}{r^2} \,, &\text{(41d)}\\
\mathcal{L}_{,\lambda} &= -\frac{\epsilon(\lambda+Z)\Phi_{,\lambda}}{r} - 2iq\alpha(r\Phi)_{,\lambda} + i\frac{\epsilon q\Phi Q}{r} \,, &\text{(41e)}\\
V_{,\lambda\lambda} &= -\frac{1}{\lambda}\left(\frac{\lambda^2}{r^2}\right)_{,\lambda} + \frac{2Zr_{,\lambda}}{r^3} + \frac{\kappa}{2\pi}\frac{Q^2}{r^4} \\
&\quad -\frac{\kappa\epsilon}{2r}\left(\overline{\Phi_{,\lambda}}\mathcal{L} + \overline{\mathcal{L}}\Phi_{,\lambda}\right) + \kappa q\epsilon\alpha\mathcal{J} \,, &\text{(41f)}
\end{aligned}
$$

6. Compute $\Phi_{,w}(0,\lambda)$ algebraically from r, $\mathcal{L}$, V, and $\Phi_{,\lambda}$ via

$$\Phi_{,w} = \frac{\mathcal{L}}{2r} - \frac{\epsilon}{2} V \Phi_{,\lambda},$$

   and integrate in w to obtain $\Phi(\Delta w, \lambda)$ on $\mathscr{N}^{\epsilon}(\Delta w)$.
7. Use $\Phi(\Delta w, \lambda)$ as new characteristic data on $\mathscr{N}^{\epsilon}(\Delta w)$ and repeat step 4 at $w = \Delta w$.

Let us consider possible timelike Killing vectors ξ.
An ansatz for the Killing vector ξ left invariant under spatial rotations is $\xi = \xi^w \partial_w + \xi^\lambda \partial_\lambda$ with the components ξ^w and ξ^λ only dependent on w and λ. The general solution is

$$\xi = A(w)\partial_w + \left[-A_{,w}(w)\lambda + B(w)\right]\partial_\lambda \,, \tag{42}$$

where A and B are free functions. It is convenient to introduce the scalar function

$$z = A(w)\lambda - \int^w B(\hat{w})d\hat{w} \,, \tag{43}$$

which is invariant under the action of the vector field ξ that is the Lie derivative $\mathscr{L}_\xi z = \xi^a \partial_a z = 0$.

Finally, inserting into the remaining Killing equations gives us

$$V = \frac{\hat{V}[z(w,\lambda)]}{A^2(w)} - \frac{2\epsilon(A_{,w}\lambda - B)}{A}\,, \tag{44}$$

in which $\hat{V}(\cdot)$ is an arbitrary differentiable function.
Some choices for A and B are

A	B	ξ	Invariant z	r	V
$A(w)$	$B(w)$	$A(w)\partial_w - [A_{,w}(w)\lambda - B(w)]\,\partial_\lambda$	$A(w)\lambda - \int^w B(\hat{w})d\hat{w}$	$\hat{r}(z)$	$\frac{\hat{V}(z)-2\epsilon A(w)[A_{,w}(w)\lambda - B(w)]}{A^2}$
1	0	∂_w	λ	$\hat{r}(z)$	$\hat{V}(z)$
kw	0	$k(w\partial_w - \lambda\partial_\lambda)$	$kw\lambda$	$\hat{r}(z)$	$\lambda^2\frac{\hat{V}(z)-2\epsilon kz}{z^2} = \lambda^2\tilde{V}(z)$
$\frac{1}{2}\left(\frac{w}{w+k}\right)^2$	1	$\frac{1}{2}\left(\frac{w}{w+k}\right)^2\partial_w + \left[-\frac{kw\lambda}{(w+k)^3}+1\right]\partial_\lambda$	$\frac{1}{2}\left(\frac{w}{w+k}\right)^2\lambda - w$	$\hat{r}(z)$	$\frac{4(w+k)^4\hat{V}(z)}{w^4} + \frac{4\epsilon}{w^2}\left[(w+k)^2 - \frac{kw\lambda}{w+k}\right]$

Selective choices for A and B to build a Killing vector

$$r = m_e - w + \frac{w^2}{2(m_e - w)^2}\,\lambda, \tag{45a}$$

$$V = \frac{4m_e\lambda^2}{(m_e - w)^5}\,\frac{-\lambda w^3 + (m_e + 2w)(m_e - w)^3}{r^2}. \tag{45b}$$

- The future horizon is at $w = 0$ while the past horizon is located at

$$\lambda(w) = \frac{2(w - m_e)^2}{w}.$$

- Conformal compactification provides a natural framework for studying global aspects of Einstein's equations, as it gives a precise definition of infinity.
- On the other hand, in numerical work, infinities are often treated using compactified coordinates in physical spacetime and regularized fields.

QUESTIONS:

- Can the conformal equations be written in a simple hierarchical form?
- Are these conformal equations equivalent to the regularized equations obtained in compactified physical space?

Yes: Phys.Rev.D 112 (2025) 2, 024056

Let $\mathcal{M} = \tilde{\mathcal{M}} \cup \mathcal{I}$ be the conformal completion of the physical spacetime $\tilde{\mathcal{M}}$, where $\mathcal{I} = \partial\tilde{\mathcal{M}}$ is null infinity.

There exists a smooth conformal factor Ω on $\mathcal{M}$ such that $\Omega > 0$ in $\tilde{\mathcal{M}}$, $\Omega = 0$ and $\nabla_a\Omega \neq 0$ on $\mathcal{I}$.

The unphysical fields are defined by

$$g_{ab} = \Omega^2 \tilde{g}_{ab}, \qquad \Phi = \Omega^{-1}\tilde{\Phi},$$

with Ω otherwise arbitrary.

Unphysical coordinates are denoted by x^a, and ∇_a is the Levi–Civita connection of g_{ab}. The physical and unphysical Ricci tensors satisfy

$$\tilde{R}_{ab} = R_{ab} + \frac{2\nabla_a \nabla_b \Omega}{\Omega} + g_{ab}\left(\frac{\nabla^c \nabla_c \Omega}{\Omega} - \frac{3\nabla^c \Omega \nabla_c \Omega}{\Omega^2}\right). \qquad (46)$$

Using the physical field equations, the conformal Einstein–scalar system on $(\mathcal{M}, g_{ab}, \Omega)$ becomes

$$0 = -\kappa(\nabla_a \Omega\, \Phi)(\nabla_b \Omega\, \Phi) + R_{ab} + \frac{2\nabla_a \nabla_b \Omega}{\Omega} + g_{ab}\left(\frac{\nabla^c \nabla_c \Omega}{\Omega} - \frac{3\nabla^c \Omega \nabla_c \Omega}{\Omega^2}\right),$$

$$0 = \Omega^3 \nabla_a \nabla^a \Phi + \Omega^2 \Phi\, \nabla_a \nabla^a \Omega - 2\Omega\Phi\, \nabla^a \Omega\, \nabla_a \Omega.$$

The spherically symmetric affine–null physical metric reads

$$d\tilde{s}^2_{AN} = -W\, d\tilde{u}^2 - 2\, d\tilde{u}\, d\tilde{\lambda} + r^2 q_{AB} d\tilde{x}^A d\tilde{x}^B. \qquad (47)$$

With $\tilde{\lambda} \in [0, \infty)$, choose the conformal factor

$$\Omega(\tilde{\lambda}) = \frac{1}{1+\tilde{\lambda}}.$$

The unphysical metric $g_{ab} = \Omega^2 \tilde{g}_{ab}$ becomes

$$ds^2_{AN} = -\frac{W}{(1+\tilde{\lambda})^2} d\tilde{u}^2 - 2\, d\tilde{u}\, d\left(\frac{\tilde{\lambda}}{1+\tilde{\lambda}}\right) + \left(\frac{r}{1+\tilde{\lambda}}\right)^2 q_{AB} d\tilde{x}^A d\tilde{x}^B.$$

This naturally defines conformal coordinates

$$x^a = (u, x, x^A) = \left(\tilde{u}, \frac{\tilde{\lambda}}{1+\tilde{\lambda}}, \tilde{x}^A\right), \qquad x \in [0, 1],$$

so that $\tilde{\lambda} = \frac{x}{1-x}$ and $\Omega = 1 - x$.

- In conformal coordinates x^a, the affine parameter satisfies $\tilde{\lambda} = 0$ at the vertex, and future null infinity is located at $x = 1$.
- The unphysical metric in coordinates x^a takes the form

$$ds^2 = -(1-x)\mathcal{W}\, du^2 - 2\, du\, dx + \mathcal{R}^2 q_{AB}\, dx^A dx^B, \tag{48}$$

where the conformally regular variables are defined by

$$\mathcal{R} := (1-x)\, r, \qquad \mathcal{W} := (1-x)\, W.$$

- The physical and conformal scalar fields are related by

$$\tilde{\Phi} = (1-x)\, \Phi.$$

- As in the physical affine-null formulation, the Einstein equations do not admit a natural hierarchical structure.

$$
\begin{aligned}
0 &= \mathcal{R}_{,uu} + \frac{1}{2}(1-x)^2\left[\left(\frac{\mathcal{W}}{1-x}\right)_{,x}\mathcal{R}_{,u} - \left(\frac{\mathcal{R}}{1-x}\right)_{,y}\mathcal{W}_{,u}\right] \\
&\quad -\frac{1}{4}(1-x)^3\mathcal{W}\left[\mathcal{R}\left(\frac{\mathcal{W}}{1-x}\right)_{,x}\right]_{,x} + \frac{\kappa}{2}\mathcal{R}[(1-x)\Phi_{,u}]^2,
\end{aligned}
\tag{49a}
$$

$$
0 = \mathcal{R}_{,xx} + \frac{\kappa}{2}\mathcal{R}\Big[(1-x)\Phi_{,x} - \Phi\Big]^2, \tag{49b}
$$

$$
0 = q_{AB}\left\{-\frac{1}{2}\left[(1-x)(\mathcal{R}^2)_{,x}\mathcal{W} - 2x - 2(\mathcal{R}^2)_{,u}\right]_{,x} - \frac{(1-x)^2}{\mathcal{R}^2}\left[\frac{\mathcal{R}^4\mathcal{W}}{(1-x)^2}\right]_{,x} + \frac{2(\mathcal{R}^2)_{,u}}{1-x}\right\},
$$

$$
0 = -(\mathcal{R}^2\Phi_{,u})_{,x} - (\mathcal{R}^2\Phi_{,x})_{,u} + \left[\mathcal{R}^2(1-x)\mathcal{W}\Phi_{,x}\right]_{,x} + \left\{\frac{(\mathcal{R}^2)_{,u}}{(1-x)} - (1-x)\left[\frac{\mathcal{R}^2\mathcal{W}}{1-x}\right]_{,x}\right\}\Phi.
$$

In order to restore a hierarchy, we start by introducing the field

$$
\mathcal{Y} = (1-x)\left(\frac{\mathcal{R}}{1-x}\right)_{,x}\mathcal{W} - \frac{2\mathcal{R}_{,u}}{1-x}, \tag{50}
$$

This casts $q^{AB}E_{AB} = 0$ into

$$0 = \left(\frac{\mathcal{R}\mathcal{Y}_{,x}}{1-x} - \frac{x}{1-x} \right)_{,x} . \qquad (51)$$

implying

$$\mathcal{Y} = \frac{x + (1-x)\mathcal{Z}_0(u)}{\mathcal{R}}, \qquad (52)$$

with $\mathcal{Z}_0(u)$ being a function of integration. Next, definition of

$$\mathcal{L} = 2\mathcal{R}\Phi_{,u} - \frac{\mathcal{W}[(1-x)\Phi]_{,x}}{(1-x)\mathcal{R}_{,x} + \mathcal{R}} \qquad (53)$$

gives a first order evolution equation for the unphysical scalar field Φ, i.e.

$$\Phi_{,u} = \frac{1}{2}\left\{ \frac{\mathcal{L}}{\mathcal{R}} + \mathcal{W}[(1-x)\Phi]_{,x} \right\}, \qquad (54)$$

and casts the main equation for Φ into a hypersurface equation.

After a lengthy and tedious calculation we arrive at

$$0 = \mathcal{R}_{,xx} + \frac{\kappa\mathcal{R}}{2}\{[(1-x)\Phi]_{,x}\}^2, \tag{55a}$$

$$\mathcal{L}_{,x} = \frac{[x+(1-x)\mathcal{Z}_0][(1-x)\Phi]_{,x}}{\mathcal{R}}, \tag{55b}$$

$$\mathcal{W}_{,xx} = \kappa\frac{[(1-x)\Phi]_{,x}\mathcal{L}}{\mathcal{R}} + \frac{2[\mathcal{R}+(1-x)\mathcal{R}_{,x}]\mathcal{Z}_0}{\mathcal{R}^3} - \frac{1}{x}\left(\frac{x^2}{\mathcal{R}^2}\right)_{,x}, \tag{55c}$$

$$\Phi_{,u} = \frac{1}{2}\left\{\frac{\mathcal{L}}{\mathcal{R}} + \mathcal{W}[(1-x)\Phi]_{,x}\right\}, \tag{55d}$$

which is regular at the boundary $\mathcal{I}$, i.e. it has no singularities at $x=1$. The system (55) is completed with $E_{uu}|_{\mathcal{B}} = 0$ and $q^{AB}E_{AB}|_{\mathcal{B}} = 0$ evaluated a some value $x \in [0,1)$, which are needed to fix the boundary values for the hypersurface fields $\mathcal{R}$, $\mathcal{Z}$, $\mathcal{L}$ and $\mathcal{W}$.

What about rotating spacetimes in this framework?

A generic line element for an affine-null metric defined with respect to a family of outgoing null hypersurfaces $u = const$ is

$$g_{ab}dx^a dx^b = -W du^2 - 2du d\lambda + R^2 h_{AB}(dx^A - W^A du)(dx^B - W^B du). \quad (56)$$

h_{AB} is transverse-traceless and has only two degrees of freedom:

$$h_{AB}dx^A dx^B = \left(e^{2\gamma} d\theta^2 + \frac{\sin^2\theta}{e^{2\gamma}} d\phi^2\right)\cosh(2\delta) + 2\sin\theta\sinh(2\delta)d\theta d\phi \ . \quad (57)$$

In this situation, the vacuum Einstein equations $R_{ab} = 0$ split into:

- supplementary equations $S_i = 0$, with

$$S_i = (R_{uu}, R_{u\theta}, R_{u\phi}),$$

- one trivial equation $R_{u\lambda} = 0$,
- six main equations.

We assume axisymmetry and stationarity, with Killing vectors ∂_u and ∂_ϕ, implying u- and ϕ-independent metric functions. The associated Komar charges are the mass

$$K_m = \frac{1}{8\pi} \lim_{\lambda \to \infty} \oint \left(W_{,\lambda} - R^2 h_{AB} W^A W^B_{,\lambda} \right) R^2 \, d^2 q,$$

and the angular momentum

$$K_L = -\frac{1}{16\pi} \lim_{\lambda \to \infty} \oint \left(R^4 h_{\phi B} W^B_{,\lambda} \right) d^2 q,$$

with $d^2 q = \sin\theta \, d\theta \, d\phi$.

Let us assume there is a smooth one parameter family of stationary and axially symmetric metrics $g_{ab}(\varepsilon)$, where ε is a small parameter such that $\varepsilon = 0$ is a corresponds to a (static) spherically symmetric spacetime solution of the vacuum Einstein equations. Then there is an expansion of the metric fields like

$$R(\lambda,\theta) = r(\lambda) + R_{[1]}(\lambda,\theta)\varepsilon + R_{[2]}(\lambda,\theta)\varepsilon^2 + R_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4), \quad (58a)$$

$$W(\lambda,\theta) = V(\lambda) + W_{[1]}(\lambda,\theta)\varepsilon + W_{[2]}(\lambda,\theta)\varepsilon^2 + W_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4), \quad (58b)$$

$$W^A(\lambda,\theta) = W^A_{[1]}(\lambda,\theta)\varepsilon + W^A_{[2]}(\lambda,\theta)\varepsilon^2 + W^A_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4), \quad (58c)$$

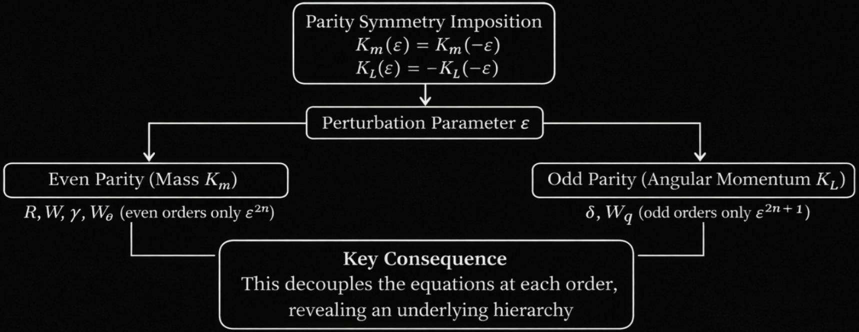
$$\gamma(\lambda,\theta) = \gamma_{[1]}(\lambda,\theta)\varepsilon + \gamma_{[2]}(\lambda,\theta)\varepsilon^2 + \gamma_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4), \quad (58d)$$

$$\delta(\lambda,\theta) = \delta_{[1]}(\lambda,\theta)\varepsilon + \delta_{[2]}(\lambda,\theta)\varepsilon^2 + \delta_{[3]}(\lambda,\theta)\varepsilon^3 + O(\varepsilon^4). \quad (58e)$$

In PRD 107 (2023) 10, 104010 we recovered the slowly rotating limit of the Kerr metric in affine null coordinates using a Master equation for δ and γ and the expansions:

$$R(\lambda,\theta) = r(\lambda)) + R[1]\varepsilon + R[2]\varepsilon^2 + \dots \qquad W_q(\lambda,\theta) = W_q[1]\varepsilon + W_q[3]\varepsilon^3 + \dots$$

$$W(\lambda,\theta) = V(\lambda) + W[1]\varepsilon + W[2]\varepsilon^2 + \dots \qquad \delta(\lambda,\theta) = \delta[1]\varepsilon + \delta[3]\varepsilon^3 + \dots$$



$$
\begin{aligned}
R(\lambda,\theta) &= \lambda - \frac{3m^2\sin^4\theta}{40\lambda^5}a^4 + O(a^6) \\
W(\lambda,\theta) &= 1 - \frac{2m}{\lambda} + \left[\frac{2m}{\lambda^3} + \left(-\frac{3m}{\lambda^3} + \frac{3m^2}{\lambda^4}\right)\sin^2\theta\right]a^2 \\
&\quad + \left[-\frac{2m}{\lambda^5} + \left(\frac{10m}{\lambda^5} - \frac{3m^2}{\lambda^6}\right)\sin^2\theta + \left(-\frac{35m}{4\lambda^5} + \frac{21m^2}{4\lambda^6} - \frac{9m^3}{10\lambda^7}\right)\sin^4\theta\right]a^4 + O(a^6) \\
W^\theta(\lambda,\theta) &= \left\{-\frac{3m}{\lambda^4}a^2 + \left[\frac{5m}{\lambda^6} - \left(\frac{35m}{4\lambda^6} + \frac{3m^2}{10\lambda^7}\right)\sin^2\theta\right]a^4\right\}\sin\theta\cos\theta + O(a^6) \\
W^\phi(\lambda,\theta) &= \frac{2m}{\lambda^3}a + \left[-\frac{4m}{\lambda^5} + \left(\frac{5m}{\lambda^5} - \frac{m^2}{\lambda^6}\right)\sin^2\theta\right]a^3 + O(a^5) \\
\gamma(\lambda,\theta) &= \left(-\frac{m\sin^2\theta}{2\lambda^3}\right)a^2 + \left[\frac{9m\sin^2\theta}{4\lambda^5} + \left(-\frac{21m}{8\lambda^5} - \frac{m^2}{4\lambda^6}\right)\sin^4\theta\right]a^4 + O(a^6) \\
\delta(\lambda,\theta) &= -\frac{5m\cos\theta\sin^2\theta}{4\lambda^4}a^3 + O(a^5)
\end{aligned}
$$

- From the explicit metric, the location of the past horizon

$$\lambda = \lambda_H(\theta)$$

can be determined, and the metric is regular there (see PRD 107 (2023) 10, 104010 for the explicit expression).

- As expected in a Bondi frame, the future horizon is not covered by this coordinate chart.

Key question

Does there exist a Israel like coordinate system which covers not only the exterior region but also a neighbourhood of *both* horizons?

We start with the stationary and axisymmetric metric expressed in a Bondi-like coordinate system (with affine parameter λ:

$$x^\mu = (u, \lambda, \theta, \phi), \qquad A, B \in \{\theta, \phi\},$$

and metric

$$ds^2 = g_{uu}\, du^2 - 2\, du\, d\lambda + 2\, g_{uA}\, du\, dx^A + g_{AB}\, dx^A dx^B, \tag{60}$$

We assume all components depend only on (λ, θ):

$$g_{\mu\nu} = g_{\mu\nu}(\lambda, \theta) \quad \text{(no } u \text{ or } \phi \text{ dependence)}. \tag{61}$$

Let a past Killing horizon $\mathcal{B}$ be the null hypersurface given by the level set

$$f(x^\mu) = \lambda - \lambda_H(\theta) = 0, \tag{62}$$

for a smooth function $\lambda_H(\theta)$, and let the Killing generator:

$$\chi^a = \partial_u + \Omega_H\, \partial_\phi, \qquad \chi_a = g_{au} + \Omega_H\, g_{a\phi}. \tag{63}$$

On a past Killing horizon one has the standard horizon identities

$$\chi^2 := g_{uu} + 2\Omega_H g_{u\phi} + \Omega_H^2 g_{\phi\phi} = 0 \quad \text{on } \mathcal{B}, \qquad \partial_\lambda \chi^2\big|_{\mathcal{B}} = -2\kappa_H, \tag{64}$$

$\partial_\lambda \chi^2\big|_{\mathcal{B}} = -2\kappa_H$ follows from the definition of surface gravity given by

$$\nabla_a \chi^2 = 2\kappa_H \chi_a\big|_{\mathcal{B}}; \tag{65}$$

therefore by contraction with $\ell^a = \partial_\lambda$

$$\ell^a \nabla_a \chi^2 = 2\kappa_H \ell^a \chi_a\big|_{\mathcal{B}}; \tag{66}$$

but $\chi_a \ell^a = \chi_\lambda = g_{\lambda u} + \Omega_H g_{\lambda\phi} = -1$.
Moreover, comparing χ_a with the horizon normal $s_a = \nabla_a f = (0, 1, -\partial_\theta \lambda_H, 0)$, i.e. $\chi_a = \alpha \nabla_a f$ we obtain:

$$g_{u\phi} + \Omega_H g_{\phi\phi} = 0, \qquad \partial_\theta \lambda_H = g_{u\theta} + \Omega_H g_{\theta\phi} \quad \text{on } \mathcal{B}. \tag{67}$$

Define new coordinates (w, y, Θ, Φ) by

$$\lambda = \lambda_H(\Theta) - \kappa_H\, w\, y, \qquad w = -e^{-\kappa_H u}, \qquad \phi = \Phi + \Omega_H u, \qquad \Theta = \theta. \tag{68}$$

In these new coordinates, the Killing vector field χ^a reads $\chi^a = \kappa_H(y\partial_y - w\partial_w)$. Define

$$F_\phi(\lambda, \theta) := g_{u\phi}(\lambda, \theta) + \Omega_H\, g_{\phi\phi}(\lambda, \theta), \tag{69}$$

$$F_\theta(\lambda, \theta) := \partial_\theta \lambda_H(\theta) - g_{u\theta}(\lambda, \theta) - \Omega_H\, g_{\theta\phi}(\lambda, \theta). \tag{70}$$

On $\mathcal{B}$, one has $F_\phi|_{\mathcal{B}} = F_\theta|_{\mathcal{B}} = 0$. With the change of variables (68), the mixed coefficient reads

$$g_{ww}(w, y, \Theta) = \frac{\chi^2(\lambda_H(\Theta) - \kappa_H wy, \Theta) - 2\kappa_H^2 wy}{\kappa_H^2 w^2}, \tag{71}$$

$$g_{w\Theta}(w, y, \Theta) = \frac{F_\theta\left(\lambda_H(\Theta) - \kappa_H wy, \Theta\right)}{\kappa_H w}, \tag{72}$$

$$g_{w\Phi}(w, y, \Theta) = -\frac{F_\phi\left(\lambda_H(\Theta) - \kappa_H wy, \Theta\right)}{\kappa_H w}, \tag{73}$$

and they extend smoothly to neighborhoods of $w = 0$ and $y = 0$.

We start from the Kerr-Newman metric in Boyer–Lindquist coordinates. A Bondi-like coordinate system $(\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi})$ is constructed by defining a null coordinate u satisfying

$$g^{ab}\nabla_a u \nabla_b u = 0, \quad (74)$$

$$g^{ab}\nabla_a u \nabla_b \lambda = -1, \quad (75)$$

$$g^{ab}\nabla_a u \nabla_b \phi = 0, \quad (76)$$

$$g^{ab}\nabla_a u \nabla_b \theta = 0. \quad (77)$$

Slow-rotation expansion

In the slow-rotation regime, the solution can be constructed iteratively to arbitrary order in the spin parameter a, assuming the expansion

$$u = u_{\mathrm{RN}} + \sum_{i=1}^{N} f_i(\hat{r}, \hat{\theta})\, a^i,$$

where u_{RN} denotes the Reissner–Nordström null coordinate.

Let $\lambda' = -\kappa_0 wy + r_0$ with κ_0 and r_0 the surface gravity and location of the event horizon of the RN black hole. Up to second order in a:

$$\begin{aligned}
W(w,y,\Theta) &= -\frac{y^2\left(2\kappa_0\lambda' + 2\kappa_0 r_0 - 1\right)}{(\lambda')^2} + a^2 y^2 F(\lambda',\Theta),\\
W^{(\Theta)}(w,y,\Theta) &= -\frac{y\,a^2}{3\,(\lambda')^5 r_0}\,\sin\Theta\,\cos\Theta\left(\kappa_0(\lambda')^2 r_0 + \kappa_0(\lambda')r_0^2 - 8\kappa_0 r_0^3 - 5(\lambda')^2 - 5(\lambda')r_0 + 4r_0^2\right),\\
W^{(\Phi)}(w,y,\Theta) &= -\frac{y\,a}{r_0^2(\lambda')^4}\left(2\kappa_0 r_0^4 + (\lambda')^3 + (\lambda')^2 r_0 + (\lambda')r_0^2 - r_0^3\right),\\
R(w,y,\Theta) &= \lambda' + \left[-\frac{\sin^2\Theta}{12\,(\lambda')^3 r_0}\left(2\kappa_0(\lambda')^3 r_0 - 2\kappa_0 r_0^4 - 10(\lambda')^3 + r_0^3\right)\right.\\
&\quad\left.-\frac{2\kappa_0^2(\lambda')r_0^2 - 2\kappa_0^2 r_0^3 - 2\kappa_0(\lambda')r_0 + 3\kappa_0 r_0^2 + \lambda' - r_0}{2\,r_0^4\kappa_0^2}\right]a^2,\\
\gamma(w,y,\Theta) &= \frac{\sin^2\Theta}{4\,(\lambda')^4}\left(2\kappa_0(\lambda')r_0 - 2\kappa_0 r_0^2 - 2\lambda' + r_0\right) r_0\,a^2,\\
\delta &= 0.
\end{aligned}$$

- Using these expressions, we can compute the double-null data that characterize the slowly rotating limit of the Kerr–Newman metric.
- As shown by Rácz, given two null horizons H^+ and H^- intersecting at a bifurcation sphere, the expansion, shear, $\tau := \ell_{a;b}\, m^a n^b$, and a set of curvature scalars uniquely determine the solution of the Einstein–Maxwell equations.
- In particular, on the bifurcation sphere Σ_0,

$$\tau\big|_{\Sigma_0} = -\sqrt{2}\,(\kappa_0 r_0 + 1)\, a \left(\frac{a \sin(2\Theta)}{r_0^3} - i\, \frac{\sin\Theta}{2 r_0^2} \right). \tag{78}$$

- Similarly, the value of the Weyl curvature scalar Ψ_4 on H^+ is

$$\Psi_4\big|_{H^+} = -\frac{3\kappa_0^3\, y^2 \sin^2\Theta\, a^2}{r_0^3}. \tag{79}$$

- These results motivate the question of whether a hierarchical formulation of the Einstein–Maxwell equations can be constructed. The answer is yes! (Bridera, Gallo, Mädler: Upcoming)

- We formulated an affine–null formulation (ANF) of the Einstein equations with matter, using a suitable set of auxiliary variables.
- We constructed Israel-type coordinates for slowly rotating spacetimes and analyzed the small-spin limit of Kerr-Newman.
- We derived a hierarchical affine–null formulation of the Einstein–scalar field equations in conformal space, naturally yielding regular fields at null infinity; extensions beyond spherical symmetry remain open.
- We have also constructed an affine–null formulation of the Einstein–Maxwell equations for general spacetimes, providing a flexible framework for future dynamical studies.