A SIMPLE HYPERBOLOIDAL FOLIATION OF KERR-NEWMAN SPACETIME

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ADVANTAGES OF FOLIATIONS EXTENDING TO FUTURE NULL INFINITY $\mathcal{I}^+$

No intermediate field effects in studying radiation

Event horizon formation is revealed by monitoring increasing redshift at $\mathcal{I}^+$

For numerical treatments, $\mathcal{I}^+$ is conformally an ingoing null hypersurface so no artificial outer boundary condition is needed

Null foliations, such as Bondi-Sachs, have been a major tool for computing waveforms from black hole collisions

DISADVANTAGES OF NULL FOLIATIONS

Well suited for the exterior region of an asymptotically flat spacetime but break down due to focussing of null null rays in the strong field interior

For Schwarzschild black holes, this is no problem since the singular exterior must be excised and there are natural spherical null cones to base a perturbation treatment

For Kerr or Kerr-Newman black holes there are no natural or analytic null hypersurfaces to base a perturbation treatment

AN ALTERNATIVE IS A HYPERBOLOIDAL FOLIATION

HYPERBOLOIDAL FOLIATIONS

Spacelike hypersurfaces asymptotic to $\mathcal{I}^+$

Minkowski space example - hyperboloids $x^2+y^2+z^2-t^2=D^2$

Hyperboloidal foliations were motivated by computational implementation of Helmut Friedrich's conformal hyperbolic system for the Cauchy problem with compactified $\mathcal{I}^+$

Early constructions: Peter Hübner, Anil Zenginoglu, Oliver Rinne

Prominent hyperboloidal approaches:
Constant mean curvature foliations
Height function formulation $t+h(x^i)$

I suggested to Helmut a simple hyperboloidal foliation of Kerr spacetime based upon Kerr-Schild coordinates

KERR BLACK HOLES AND NONLINEAR RADIATION MEMORY

T. Mädler and J. Winicour Class. Quant. Grav. (2019)

KERR-NEWMAN METRIC IN OUTGOING KERR-SCHILD COORDINATES

$$x^\mu = (t, x, y, z)$$

Future directed outgoing null vector

$$k_\mu = (-1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}), \qquad k^\mu = \eta^{\mu\nu} k_\nu = (1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r})$$

Shear-free, null geodesic congruence in Minkowski space

Doubly degenerate principle null vector in Kerr-Newman

$$g_{\mu\nu} = \eta_{\mu\nu} + H k_\mu k_\nu \qquad k^\mu k_\mu = 0$$

$$g^{\mu\nu} = \eta^{\mu\nu} - H k^\mu k^\nu$$

$$A_\mu = \frac{Q r^3}{r^4 + a^2 z^2} k_\mu$$

$$H = \frac{r^2}{r^4 + a^2 z^2}(2Mr - Q^2)$$

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$

$r = const \rightarrow$ **Ellipsoids of revolution**

EXTREMAL BLACK HOLE $\quad a^2 + Q^2 = M^2$

KERR-NEWMAN HYPERBOLOIDAL FOLIATION
BACKGROUND MINKOWSKI NULL CONE

$$\tau = t - R\,, \qquad R^2 = x^2 + y^2 + z^2$$

$$\eta^{\mu\nu}\tau_{,\mu}\tau_{,\nu} = 0$$

$$g^{\mu\nu}\tau_{,\mu}\tau_{,\nu} = (\eta^{\mu\nu} - Hk^{\mu}k^{\nu})\tau_{,\mu}\tau_{,\nu} = -H(k^{\mu}\tau_{,\mu})^2$$

$$H = \frac{r^2}{r^4 + a^2 z^2}(2Mr - Q^2)$$

Region $H > 0 \Longleftrightarrow 2Mr - Q^2 > 0 \Longrightarrow g^{\mu\nu}\tau_{,\mu}\tau_{,\nu} \leq 0$

τ **is spacelike if** $k^{\mu}\tau_{,\mu} \neq 0$

τ **is null if** $k^{\mu}\tau_{,\mu} = 0$

Both $-\tau_{,\mu}$ **and** k_{μ} **on future Minkowski lightcone** $\Rightarrow -\tau_{,\mu}k^{\mu} \leq 0$

Short calculation: $\quad -\tau_{,\mu}k^{\mu} = -1 + \frac{r}{R}$

$$R^2 = r^2 + \frac{a^2(x^2 + y^2)}{r^2 + a^2} \geq r^2$$

$x^2 + y^2 > 0 \Longrightarrow \quad R > r \Longrightarrow \tau$ **is spacelike**

z**-axis** $\quad x^2 + y^2 = 0 \Longrightarrow \quad R = r \Longrightarrow \tau$ **is null**

$R \to r$ **as** $r \to \infty$ **so** τ **is everywhere asymptotically null**

Schwarzschild or Reissner-Nordstrom

$$r = R \Longrightarrow \quad k_{\mu} = -\tau_{,\mu}$$

Foliation is everywhere null and characteristic formulation is natural foliation to treat radiation and properties at $\mathcal{I}^+$

SUMMARY IN THE REGION $2Mr \geq Q^2$

OUTSIDE THE z-AXIS:

THE τ FOLIATION IS HYPERBOLOIDAL

i..e. SPACELIKE AND ASYMPTOTICALLY NULL AT $\mathcal{I}^+$

ON THE z-AXIS:

THE τ FOLIATION IS NULL

DOES THE z-AXIS CAUSE A PROBLEM?

INGOING KERR-SCHILD FOLIATION

Horizon penetrating coordinates $\hat{x}^\mu = (\hat{t}, x, y, z)$

$$\hat{g}_{\mu\nu} = \hat{\eta}_{\mu\nu} + H\hat{k}_\mu \hat{k}_\nu$$

Future directed ingoing null vector

$$\hat{k}_\mu = (-1, -\frac{rx + ay}{r^2 + a^2}, -\frac{ry - ax}{r^2 + a^2}, -\frac{z}{r})$$

Associated Minkowski outgoing null hypersurface

$$\hat{\tau} = \hat{t} - R$$

Relation to outgoing hyperboloidal foliation

$$\hat{\tau} = \tau + 2(r^* - r)$$

Tortoise coordinate

$$r^* \to r + 2m\ln(\frac{r}{2m}), \qquad r \to \infty$$

$$\hat{\tau} = const \Longrightarrow \tau = const - 2(r^* - r) \to const - 4m\ln(\frac{r}{2m}) \to -\infty$$

$\hat{\tau}$ **foliation is spacelike and asymptotes to spatial infinity**

Holds even in ingoing Kerr-Schild representation of Schwarzschild

Hyperboloidal foliations have been constructed in ingoing Kerr-Schild coordinates using height function technique

Rodrigo Macedo, Hyperboloidal Framework for the Kerr Spacetime, Class Quant Grav (2020)

TREATMENT OF z-AXIS

Treatment of the z axis in the τ hyperboloidal foliation has an analogy to the treatments of the polar singularity in spherical coordinates

One standard approach is to introduce two stereographic patches covering the north and south hemispheres

Requires communication between spin weighted fields in the overlap between the north and south patches

Successfully implemented in the finite difference Pitt null code (Yosef Zlochower's PhD thesis)

Another approach has been implemented by the SXS collaboration in their quasi-spectral null code SpECTRE (Geoffrey Lovelace's talk)

The axis is placed off grid and regularity conditions at the poles are enforced on the spin-weighted fields

Another possible approach

Use of a regular reference metric (Thomas Baumgarte)

NORTH-SOUTH HYPERBOLOIDAL PATCHES

Offset Minkowski coordinates - North patch $D > 0$

$$\tau = t - R\,, \quad R^2 = x^2 + y^2 + (z + D)^2$$

$$\eta^{\mu\nu}\tau_{,\mu}\tau_{,\nu} = 0$$

$$k^\mu \tau_{,\mu} = 1 - \frac{r}{R}$$

$$R^2 = x^2 + y^2 + z^2 + D(2z + D) = r^2 + \frac{a^2(x^2 + y^2)}{r^2 + a^2} + D(2z + D)$$

z-axis with $2z > -D$

$$R^2 = r^2 + D(2z + D) > r^2 \Longrightarrow k^\mu \tau_{,\mu} > 0$$

NORTH PATCH IS EVERYWHERE HYPERBOLOIDAL

Offset Minkowski coordinates - South patch $D < 0$

$$\tau = t - R\,, \quad R^2 = x^2 + y^2 + (z + D)^2$$

$$\eta^{\mu\nu}\tau_{,\mu}\tau_{,\nu} = 0 \qquad k^\mu \tau_{,\mu} = 1 - \frac{r}{R}$$

$$R^2 = r^2 + \frac{a^2(x^2 + y^2)}{r^2 + a^2} + D(2z + D)$$

z-axis with $2z < -D = |D|$

$$R^2 = r^2 + |D(2z + D)| > r^2 \Longrightarrow k^\mu \tau_{,\mu} > 0$$

SOUTH PATCH IS EVERYWHERE HYPERBOLOIDAL

THE τ FOLIATION IS GLOBALLY HYPERBOLOIDAL IN THE KERR-NEWMAN EXTERIOR

MODEL PROBLEM FOR SPHERICAL HARMONICS ON τ FOLIATION

Minkowski space $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad r^2 = x^2 + y^2 + z^2$

Almost hyperboloidal foliation $\tau = t - \sqrt{r^2 - \frac{k^2(x^2+y^2)}{r^2}}$

Asymptotically null $\quad \tau \to t - r$ **as** $r \to \infty$

Hyperboloidal $\quad \tau > t - r$ **for** $x^2 + y^2 > 0$

Null on z-axis $\quad \tau = t - r$ **for** $x^2 + y^2 - 0$

SPHERICAL COORDINATES ON τ FOLIATION

$$y^\mu = (\tau = t - R, R, \theta, \varphi) \quad R = \sqrt{r^2 - \frac{k^2(x^2+y^2)}{r^2}} = \sqrt{r^2 - k^2 \sin^2\theta},$$

$$x + iy = r \sin\theta e^{i\varphi}, \quad z = r\cos\theta$$

$$ds^2 = -d\tau^2 - 2d\tau dR + (\frac{R^2}{r^2} - 1)dR^2 + \frac{2k^2 R \sin\theta\cos\theta}{r^2} dR d\theta$$
$$+(r^2 + \frac{k^4 \sin^2\theta\cos^2\theta}{r^2})d\theta^2 + r^2\sin^2\theta d\varphi^2$$

$r^2 = R^2 + k^2\sin^2\theta$ **is well behaved function of** $\ell = 0$ **and** $\ell = 2$ **spherical harmonics**

$\mathcal{I}^+$ **conformally compactified with coordinate** $L = 1/R$ **and** $g_{\mu\nu} \to L^2 g_{\mu\nu}$ **where** $L = 0$ **at** $\mathcal{I}^+$

Multipole solutions are well behaved spherical functions

$$\frac{1}{r} = \frac{1}{\sqrt{R^2 + k^2\sin^2\theta}}, \quad \frac{x}{r^3} = \frac{\sin\theta\cos\varphi}{R^2 + k^2\sin^2\theta}, \quad \frac{z}{r^3} = \frac{\cos\theta}{R^2 + k^2\sin^2\theta}$$

Spherical waves well behaved outside origin

$$\frac{F(t \pm r)}{r} = \frac{F(\tau + R \pm r)}{r}$$

Physical fields have regular spherical harmonic expansions on τ-foliation

KERR-NEWMAN HYPERBOLOIDAL SPHERICAL COORDINATES

Background Minkowski spherical coordinates adapted to τ foliation

$$y^\mu = (\tau, R, \theta, \varphi) \qquad \tau = t - R$$

$$R = \sqrt{x^2 + y^2 + z^2} \quad x + iy = R \sin\theta e^{i\varphi}, \quad z = R\cos\theta$$

$$\eta_{\mu\nu} dy^\mu dy^\nu = -d\tau^2 - 2d\tau dR + R^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + H k_\mu k_\nu$$

$$H = \frac{r^2}{r^4 + a^2 R^2 \cos^2\theta}(2Mr - Q^2)$$

$$R^2 = r^2 + \frac{a^2 R^2 \sin^2\theta}{r^2 + a^2}$$

$$k_\mu dy^\mu = -d\tau - (1 - \frac{r}{R})dR + \frac{r^2 - R^2}{r}\cot\theta d\theta + \frac{r^2 - R^2}{a} d\varphi$$

$$k^\mu \partial_\mu = (1 - \frac{r}{R})\partial_\tau + \frac{r}{R}\partial_R - \frac{1}{r}(1 - \frac{r^2}{R^2})\cot\theta \partial_\theta - \frac{1}{a\sin^2\theta}(1 - \frac{r^2}{R^2})\partial_\phi$$

∂_τ and ∂_φ are Killing vectors

Kerr congruence reduces to Minkowski null cone congruence for $a = 0 \Rightarrow r = R$

Kerr-Newman is asymptotically Minkowski
Compactified using coordinate $L = 1/R$
with conformal mapping $g_{\mu\nu} \to L^2 g_{\mu\nu}$ $\qquad L = 0$ **at $\mathcal{I}^+$**

STABILITY OF KERR-NEWMAN BLACK HOLES ANALYTIC RESULTS

(Articles by Elena Giorgi)

Kerr-Newman perturbations pose difficulties not found in Kerr or Reissner-Nordstrom black holes

Electromagnetic and gravitational perturbations cannot be decoupled

The linear perturbation system reduces to two coupled PDE's which cannot be solved by separation of variables (Chandrasekhar)

The stability of linearized perturbations of the subextremal Kerr-Newman case is not fully established analytically

M. Reiris (2013) INSTABILITY OF THE EXTREME KERR-NEWMAN BLACK-HOLES
Indirect proof showing extreme Reissner-Nordstrom black hole are unstable under electro-vacuum perturbations

Talk by Zachary Gelles

Extensive results for scalar wave perturbations which are uncoupled to the Einstein-Maxwell system and allow mode analysis through separation of variables

D. Civin PhD thesis (2014) Stability of Kerr-Newman black holes under linear scalar perturbations

NUMERICAL SIMULATIONS INDICATE STABILITY OF SUBEXTREMAL KERR-NEWMAN BLACK HOLES

COMPUTATION OF QUASINORMAL MODES FROM COUPLED PDEs

P. Pani, E. Berti, L. Gualtieri (2013) SCALAR, ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATIONS OF KERR-NEWMAN BLACK HOLES IN THE SLOW-ROTATION LIMIT
To linear order in the spin, the quasinormal modes are computed to be stable.

Z. Mark, H. Yang, A. Zimmerman, y. Chen Y (2015) THE QUASINORMAL MODES OF WEAKLY CHARGED KERR-NEWMAN SPACETIMES
Compliments the slow rotation result that the quasinormal modes are stable for small charge

O. Dias, M. Godazgar. J. Santos (2015) LINEAR MODE STABILITY OF THE KERR-NEWMAN BLACK HOLE AND ITS QUASI- NORMAL MODES
Up to 99.999 of extremality, Kerr-Newman black holes are found to be linear mode stable

FULL NUMERICAL EVOLUTION

M. Zilhao, V. Cardoso, C. Herdeiro, Luis Lehner, U. Sperhake (2014) TESTING THE NONLINEAR STABILITY OF KERR-NEWMAN BLACK HOLES
Stability of fully nonlinear Einstein-Maxwell perturbation with full parameter space up to $.995$ of extremality for $\ell = m = 2$ perturbation

KERR-SCHILD HYPERBOLOIDAL FOLIATION IS ALTERNATIVE FRAMEWORK FOR STUDY OF KERR-NEWMAN BLACK HOLES

ALL analytic and numerical stability studies have been based on Boyer-Lindquist coordinates

Except for scalar perturbations, PDE simplifications of Boyer-Lindquist no longer holds for Kerr-Newman

Hyperboloidal foliation offers a different approach in which compactified $\mathcal{I}^+$ yields unambiguous waveforms

No outer boundary condition needed at $\mathcal{I}^+$ as opposed to Boyer-Lindquist boundary at large spatial distance

Hyperboloidal approach opens the door for new perturbation studies

Simplest, straightforward but least interesting is simulation of linearized scalar waves

More interesting: Full Einstein-Maxwell linear or nonlinear perturbations of Kerr-Newman background

A well-posed hyperbolic evolution system can be set up in generalized harmonic and generalized Lorentz gauges determined by Kerr-Newman Einstein and Maxwell fields

The principle parts of the Einstein and Maxwell fields are uncoupled wave operators

Would the foliation evolve hyperboloidal in the nonlinear regime?

HAVING A WELL-POSED EVOLUTION SYSTEM IS JUST THE BEGINNING

REQUIREMENTS OF A CAUCHY EVOLUTION CODE

Initial perturbation satisfying the constraints

An inner boundary excising the singular region

ONE SIMPLE APPROACH

Unperturbed Kerr-Newman initial data

A time like inner boundary on which the perturbation can be prescribed for a well-posed generalized harmonic system

$R = const$ **worldtube remains timelike if**

$$g^{\mu\nu} R_{,\mu} R_{,\nu} = 1 - H(k^\mu R_{,\mu})^2 = 1 - H(r/R)^2 > 0$$

On the z**-axis, where** $R = r$**, this reduces to** $R > r_+$**, the past branch of the horizon**

There are perturbations on timelike worldtubes, $R = const > r_+$**, outside of which there exists a well-posed IBVP for all subextremal initial Kerr-Newman data**

The free boundary data is the conformal 2-metric

$$h_{AB}(\tau, y^C)/\sqrt{h} \qquad y^A = (\theta, \varphi)$$

on the foliation of the boundary

CONCLUSION
GENERALIZED HARMONIC PERTURBATION OF KERR-NEWMAN CAN IN PRINCIPLE BE CARRIED OUT IN BOTH THE LINEAR AND NONLINEAR REGIME

IN THE LINEAR REGIME, IN WHICH THE BONDI MASS AND HORIZON LOCATION ARE UNAFFECTED, THE EVOLUTION SHOULD COVER THE ENTIRE EXTERIOR OF THE $R = const$ WORLDTUBE

IN THE NONLINEAR REGIME?

THAT'S WHAT THE E IN ICERM STANDS FOR EXPERIMENTAL

Roy Kerr's original approach to discover the spinning black hole metric was based upon the degenerate principal null vectors

This led in a natural way to its formulation in Kerr-Schild coordinates

THE GEOMETRIC SIMPLICITY OF KERR-SCHILD COORDINATES MIGHT STILL BE USEFUL

I LOOK FORWARD TO YOUR COMMENTS