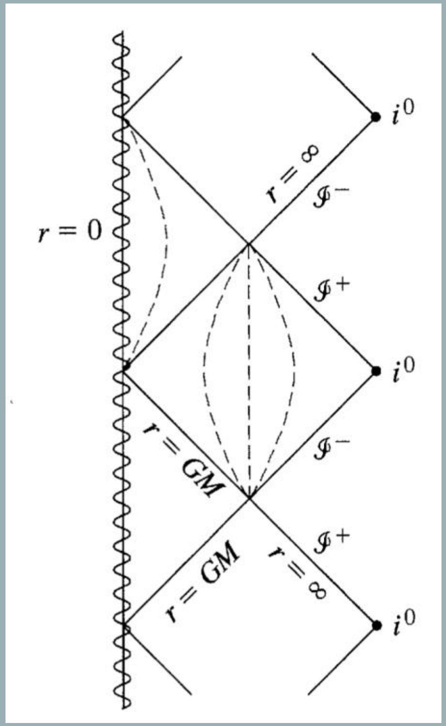
THE NONLINEAR DEVELOPMENT OF CHARGED HORIZON INSTABILITIES

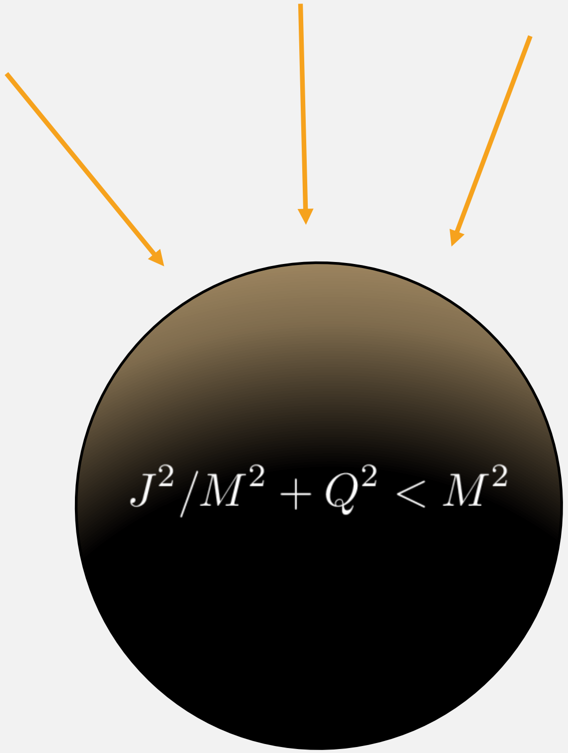


Zack Gelles & Frans Pretorius

Image Credit: Carroll

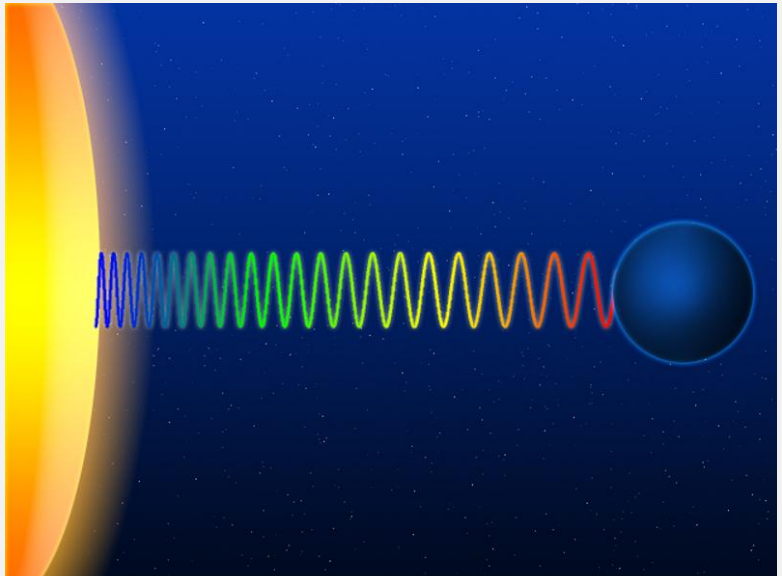
BLACK HOLE STABILITY

- Stationary black holes in nature believed to be described by *M*, *Q*, *J* alone (no-hair conjecture)
- What happens if you perturb a black hole?
  - Charged, rotating black holes are known to be linearly stable in the sub-extremal case (e.g. Dafermos, Holzegel, & Rodnianski 2016; Giorgi 2020; Andersson, Backdahl, Blue, & Ma 2019)
- Open questions: Extremal black holes?



(IN)STABILITY OF EXTREMAL BLACK HOLES

- Extremal black holes $(J^2/M^2 + Q^2 = M^2)$ are linearly *unstable* to perturbations: Aretakis Instability (2010, 2011)
  - Energy density of a scalar field does not decay on extremal horizon (fixed background metric)
- Occurs due to the lack of gravitational redshift at extremality (exponential → power-law)



$$E \sim e^{-\kappa V}$$

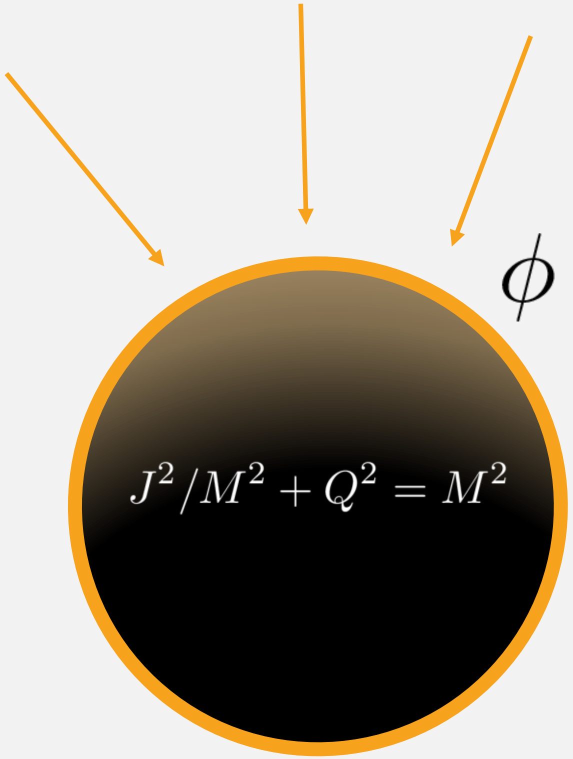
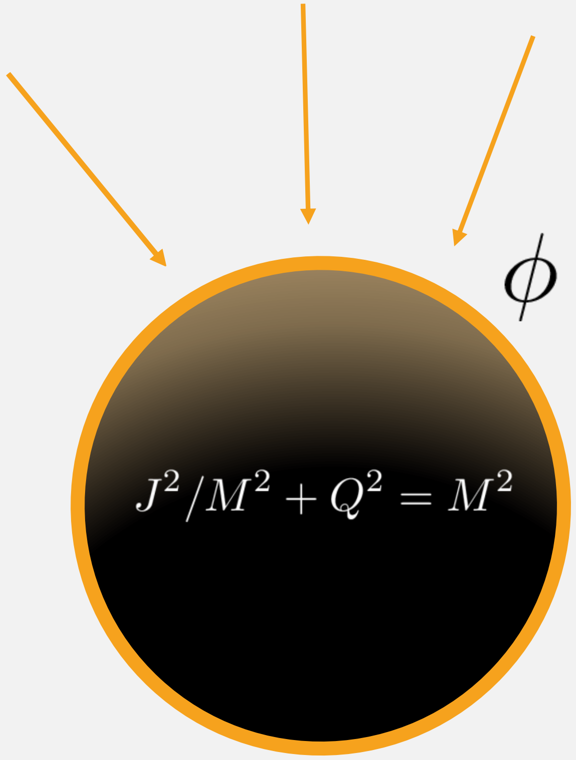


Image credit: Wikipedia

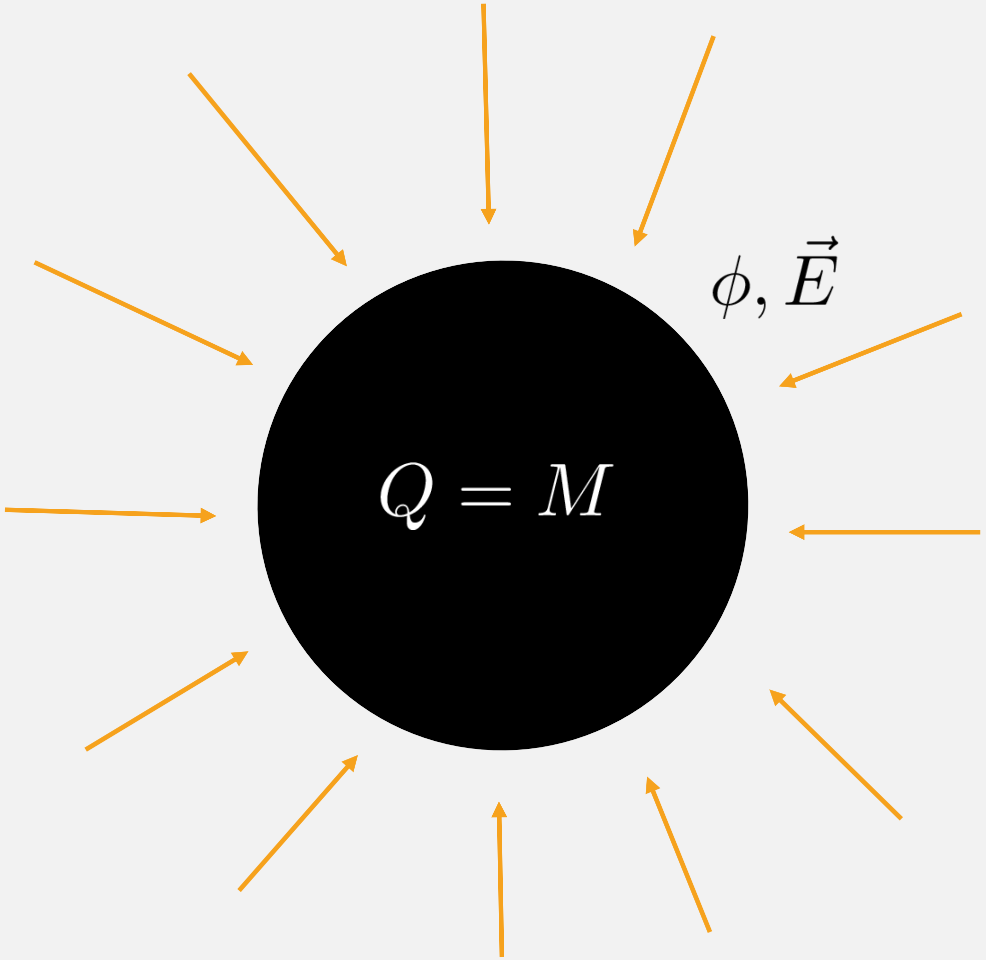
NON-LINEAR INSTABILITIES

- What happens if we evolve the matter *and* the spacetime dynamically (wave equation + Einstein's equations)?
  - o Murata, Reall, & Tanahashi (MRT; 2013) showed that metric backreaction can regulate the instability for scalar fields
  - o But can still trigger instability for finely tuned initial data ("dynamical extremal black holes")
- Can a similar phenomenon happen with *charged* scalar fields (i.e. coupled to electromagnetism)?



MOTIVATION

- Charged fields have greater potential for instability, superradiance, toy model for rotating matter
  - Linear case treated by Zimmerman (2017); Casals, Gralla, & Zimmerman (CGZ; 2016). Formation of extremal BH (Kehle & Unger 2022)
- Much harder since we now need to evolve three sets of equations:
  1. Wave equation (linear)
  2. Maxwell's equation (non-linear)
  3. Einstein's equations (non-linear)
- Set J=0 and work in spherical symmetry
- **Goal:** investigate nonlinear horizon instabilities of charged scalar fields on extremal Reissner-Nordström spacetime in spherical symmetry



OUTLINE

1. Numerical Scheme
2. Review of linear problem (fixed background metric)
   - Stronger Aretakis instability
   - Scattering results highlight importance of weakly damped mode at the onset of superradiance
3. Non-Linear Problem
   - Mode decomposition still triggers instability
   - Emergent instability for fine-tuned data, universality
   - Large curvatures visible from Scri?

SCALAR ELECTRODYNAMICS

- Can couple the scalar field to electromagnetism by making it complex:

$\phi \to \{\phi, \phi^*\}$ evolves via wave equation: $D_\mu D^\mu \phi = 0$

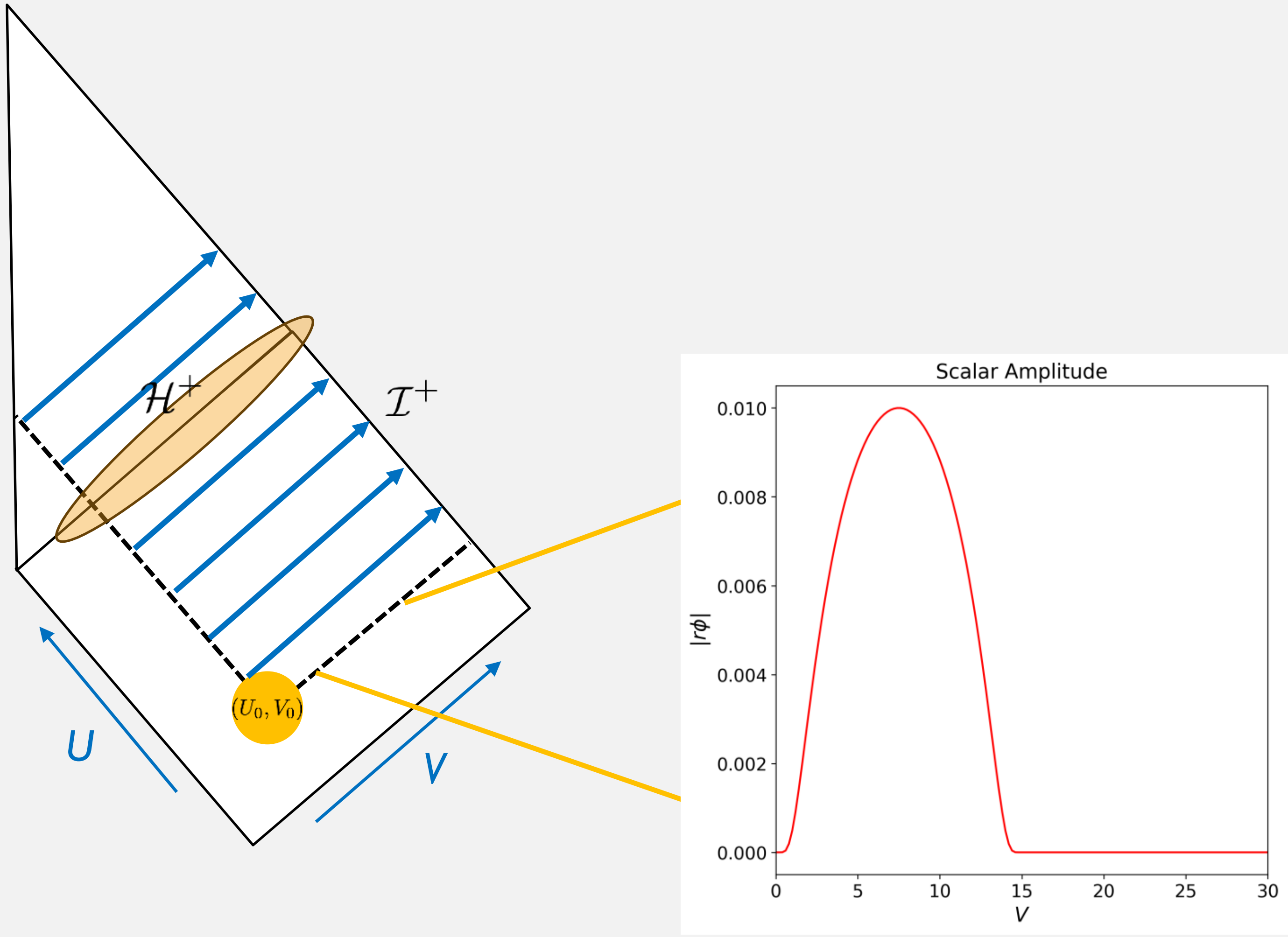
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ evolves via Maxwell's equations: $\nabla_\mu F^{\nu\mu} = 4\pi J^\mu$

Coupled via constant $\tilde{e}$

- Spherical symmetry and electromagnetic gauge symmetry reduce degrees of freedom to 2: $\{|\phi|, E^r\}$
- First step: Solve Wave+Maxwell system on fixed Reissner-Nordström background

NUMERICAL SCHEME: DOUBLE-NULL COORDINATES

1. Construct initial data: ingoing pulse of charged matter
2. Evolve data along null hypersurfaces using finite difference scheme
   - Static mesh refinement clusters resolution near event horizon
3. Examine fields along the event horizon



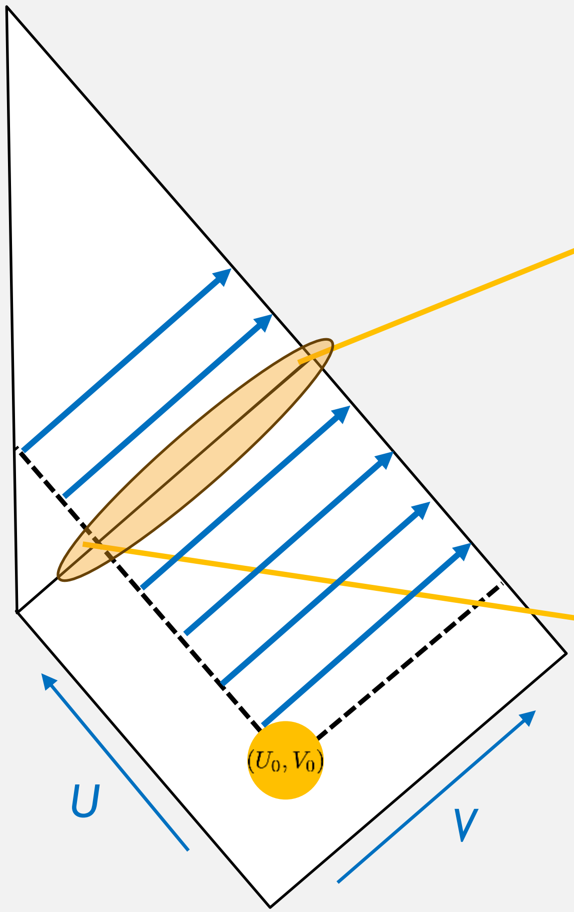
RESULTS ON FIXED BACKGROUND

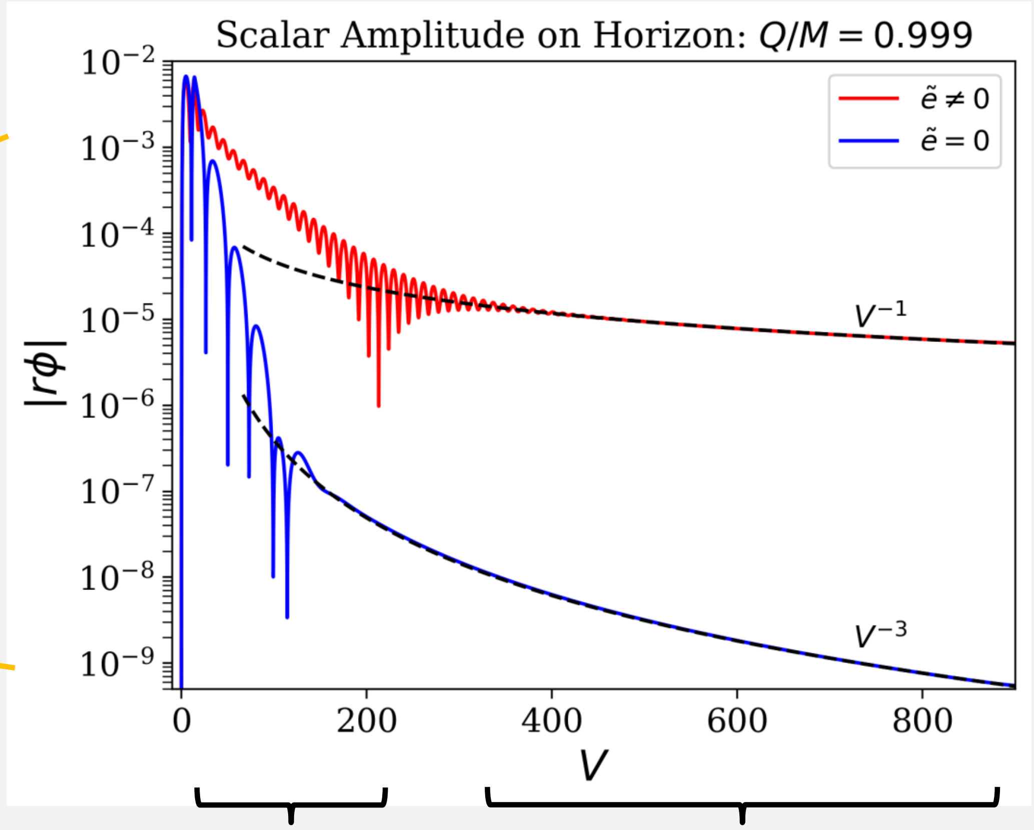
Quantities of Interest

Scalar amplitude $|\phi|$, energy density $\rho_E \sim (\partial_r |\phi|)^2$, charge density $\rho_Q \sim \partial_r (r^2 E^r)$

- Behavior of fields treated analytically for fixed metric + fixed electromagnetic potential in Zimmerman (2017)
- Incorporate Maxwell's equations in Gelles & Pretorius (2025)

RESULTS – SCALAR AMPLITUDE





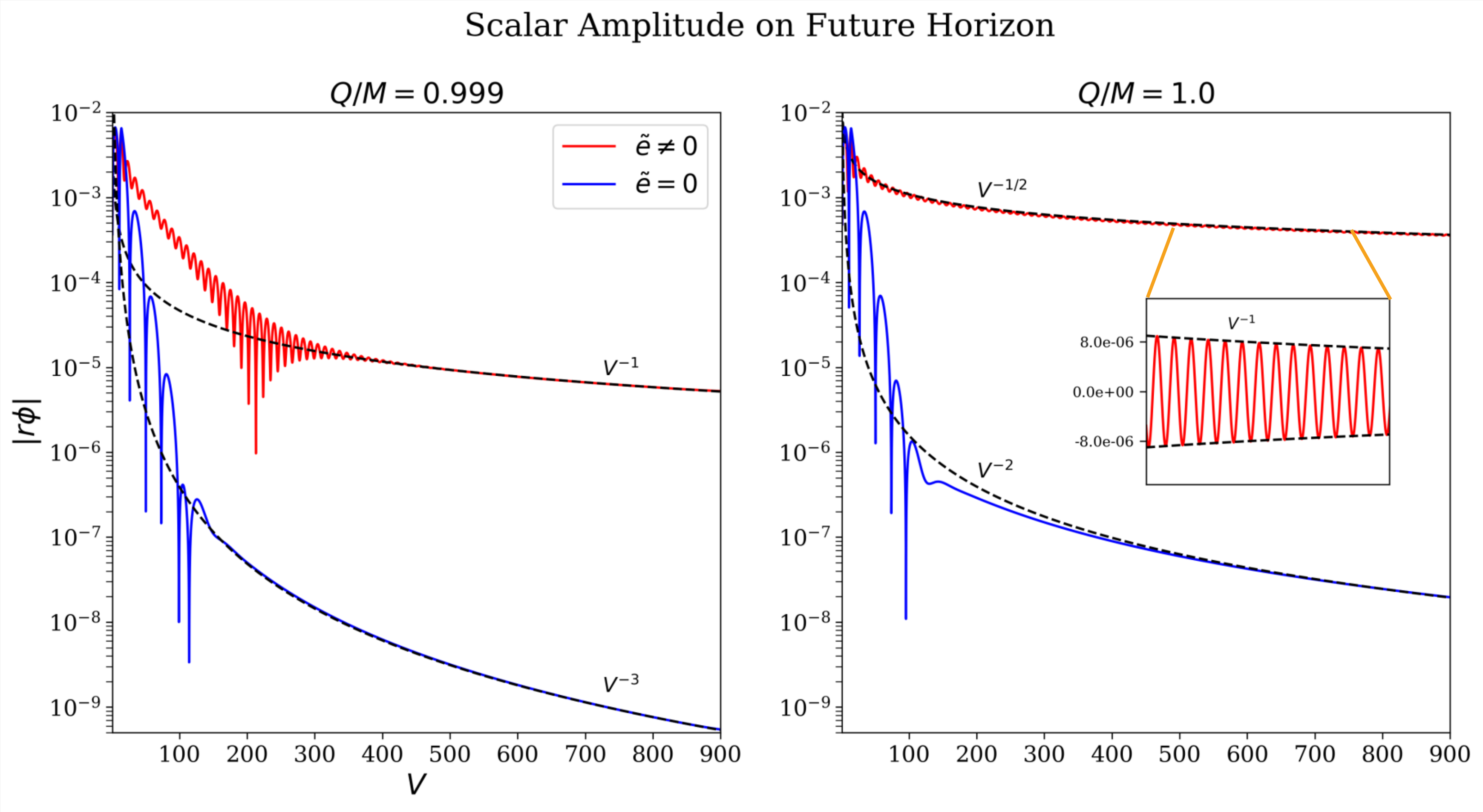
Quasinormal Modes

$$\phi \sim e^{i\omega V}$$

Power-law tail

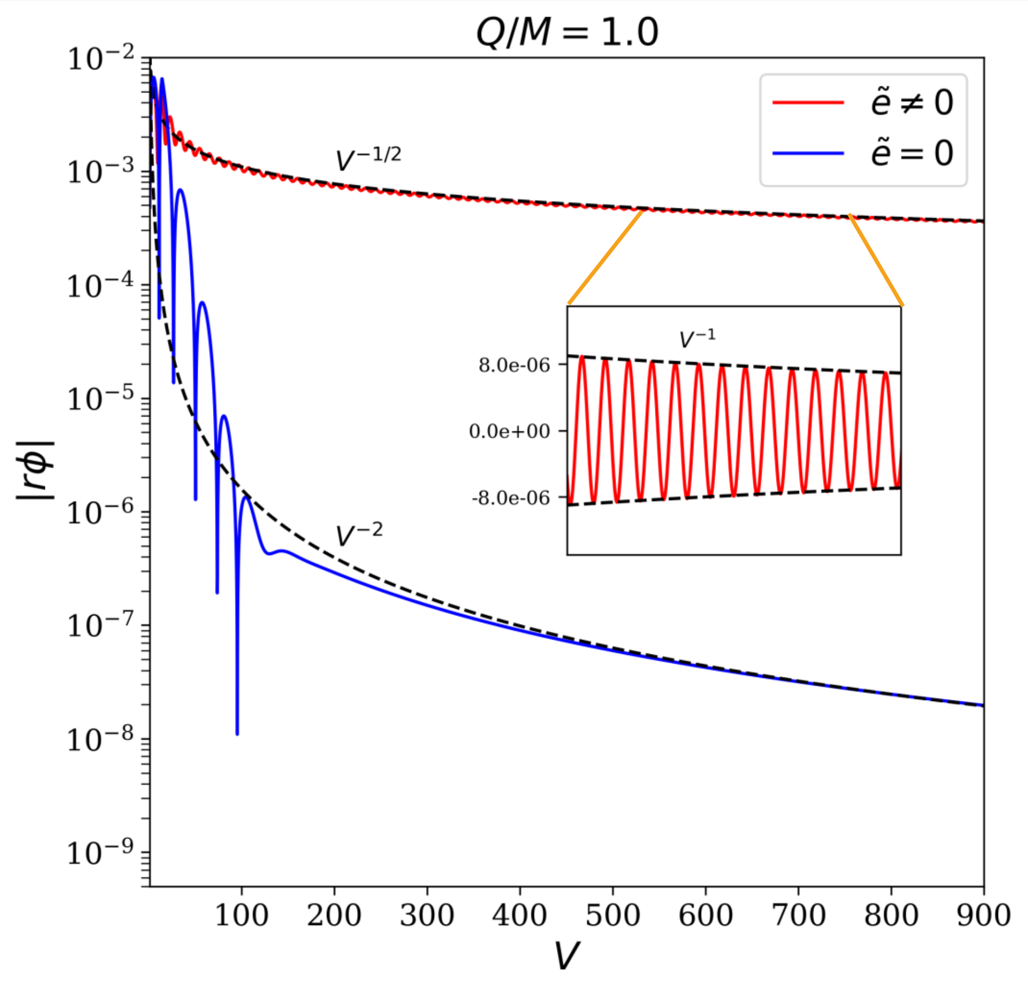
$$\phi \sim V^{-3}$$

RESULTS – SCALAR AMPLITUDE



Tail exponents for neutral and charged fields computed in, e.g.: Price (1972), Hod & Piran (1998), Lucietti+ (2012), Angelopoulos, Aretakis, & Gajic (2018), Gajic & van de Moortel (2024)

LONG-LIVED OSCILLATIONS



- As $Q \to M$ a set of "Nearly-Zero-Damped" (NZD) modes emerges:

$$\omega_{\rm NZD} = \frac{\tilde{e}Q}{r_+} + \frac{i\kappa}{2} + \mathcal{O}(\kappa^2)$$

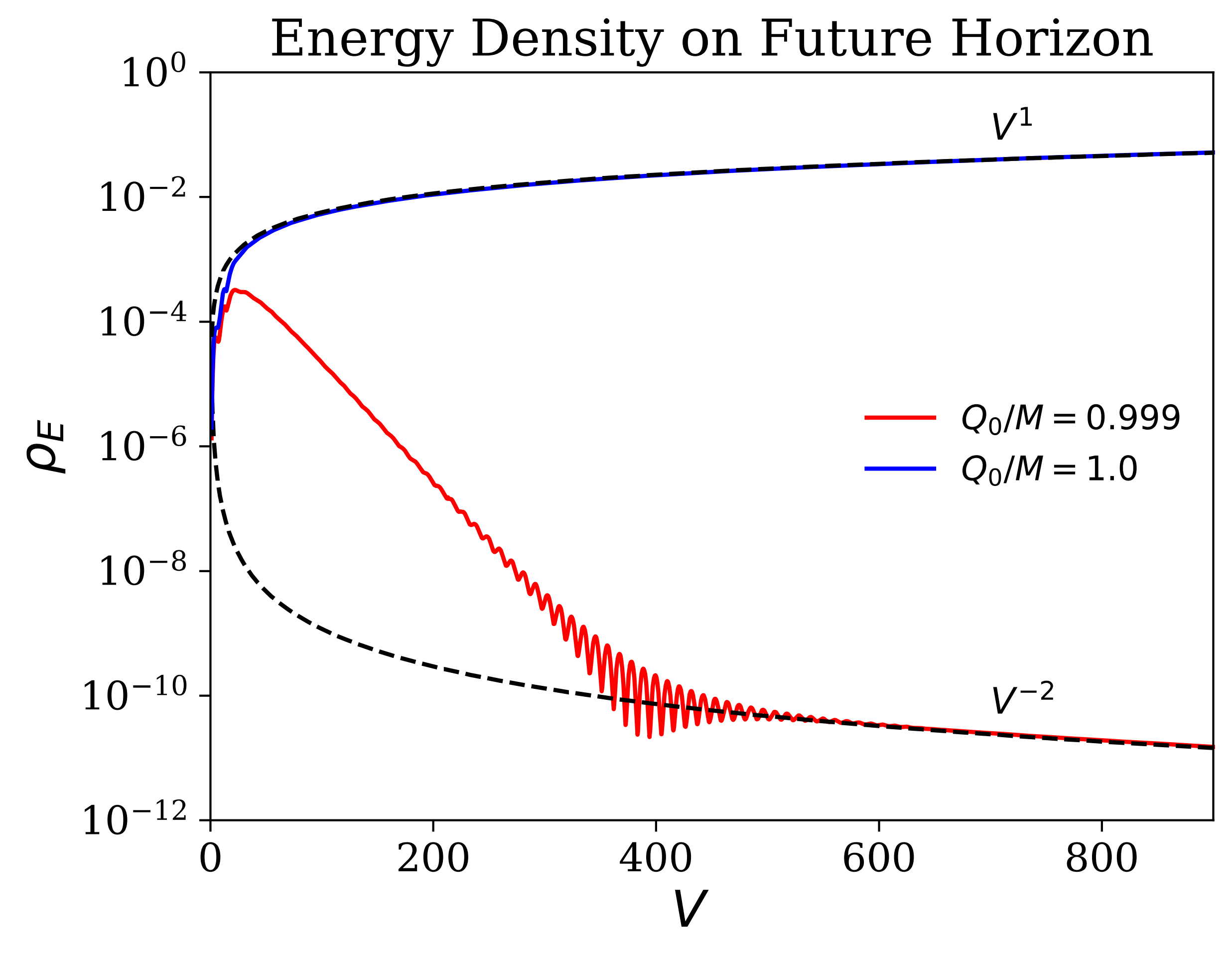
- Since $\mathrm{Im}(\omega_{\rm NZD}) \to 0$ at extremality, this mode is very long-lived
  - Shows up as oscillations in the power-law tail → instability?
  - Importance of NZD discussed in (Hod 2010; CGZ 2016; Zimmerman 2017)

RESULTS – ENERGY DENSITY

- Energy density: $\rho_E \sim (\partial_r|\phi|)^2$
- For sufficiently charged matter, energy density grows as ~V on the extremal horizon
- Stands in contrast to the uncharged Aretakis instability:

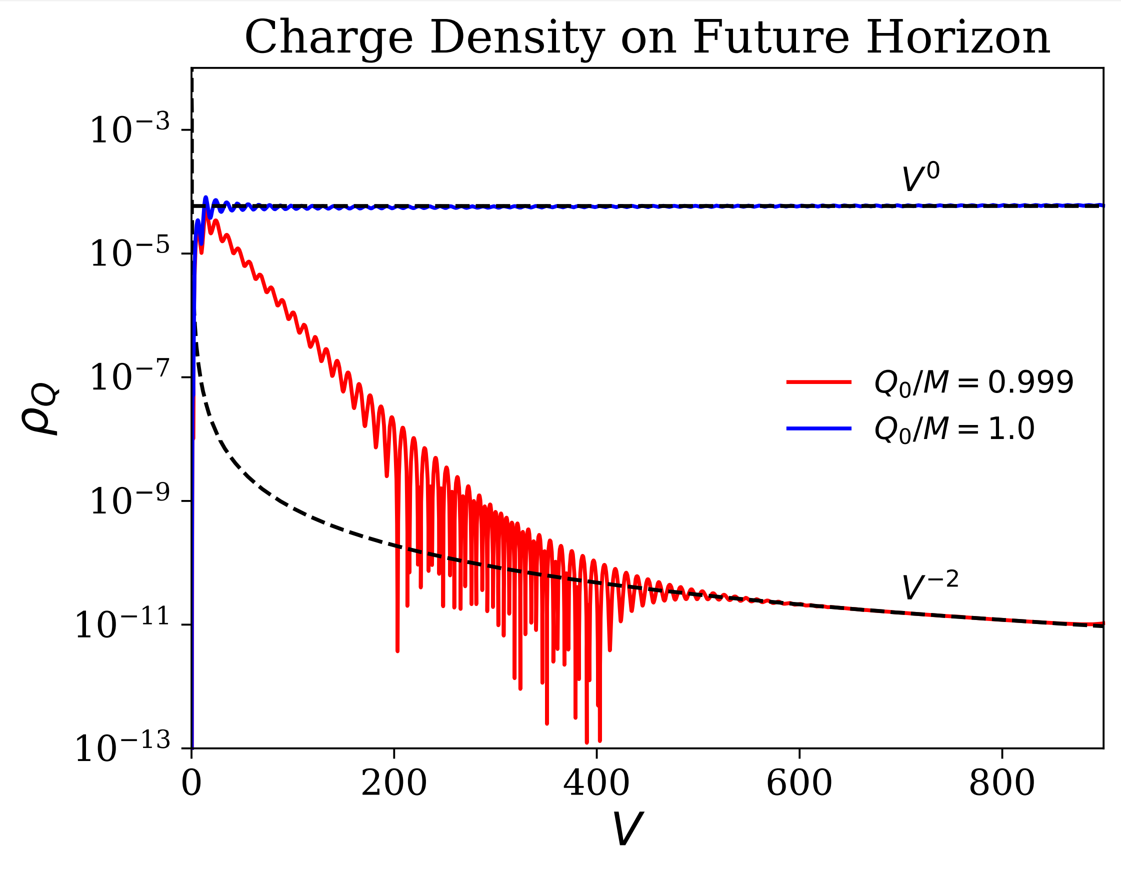
$$\rho_E \sim (\partial_r \phi)^2 \sim \text{const}$$

- Consistent with linearized, analytic results (Zimmerman 2017; CGZ 2016)



RESULTS – CHARGE DENSITY

- Charge density: $\rho_Q \sim \partial_r(r^2 E^r)$
- For sufficiently charged matter, charge density remains *constant* on the extremal horizon
- Represents an instability induced by dynamics of Maxwell's equations
  - A new (asymptotic) constant emerges in the presence of electromagnetic charge
- Charge accumulates on the extremal event horizon



PHYSICAL EXPLANATION

- Enhancement of charged Aretakis instability comes from the presence of a nearly-zero-damped (NZD) mode (Hod 2010; Zimmerman 2017; Richartz, Herdeiro, & Berti 2017):

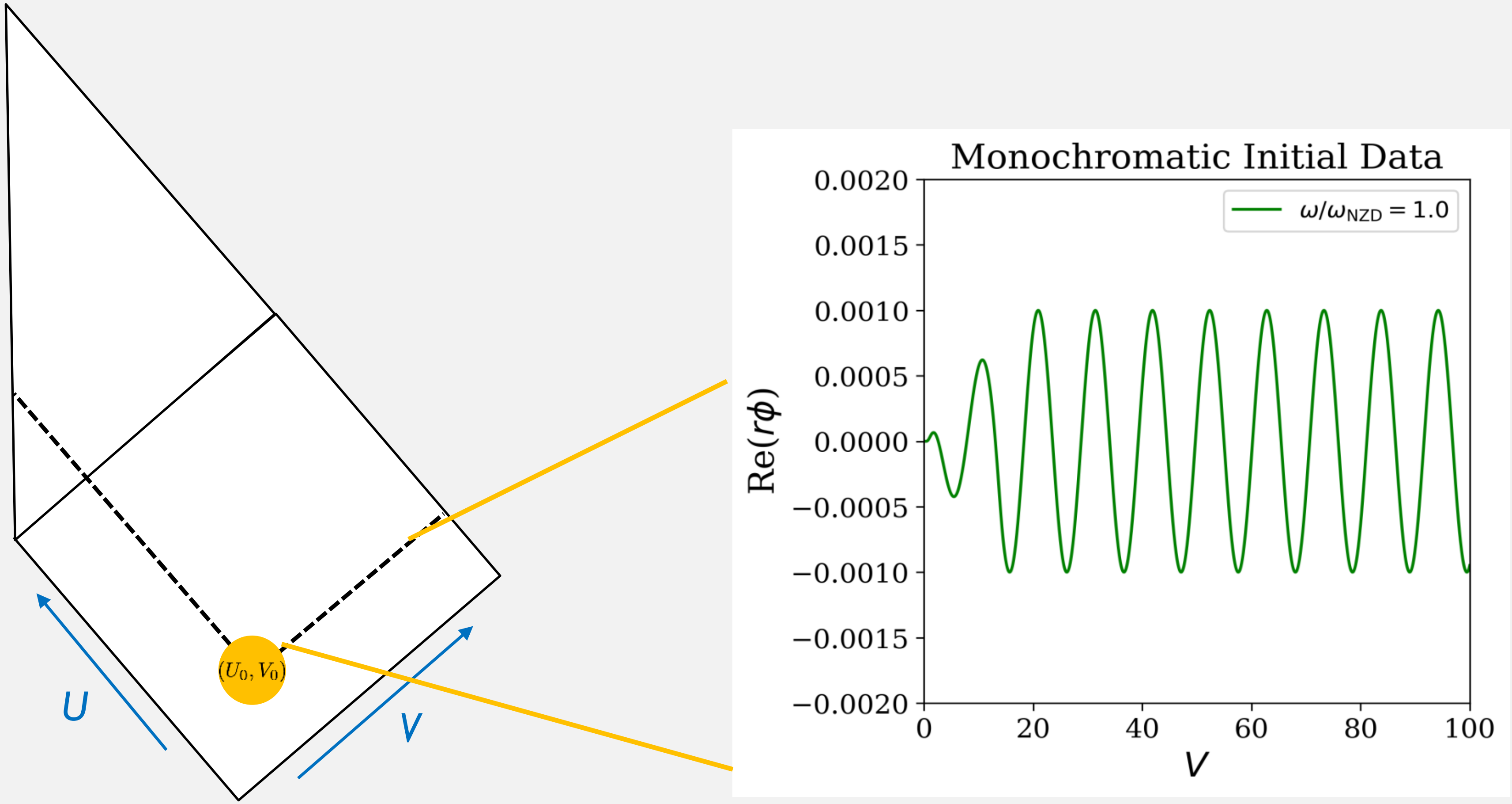
$$\omega_{\rm NZD} = \frac{\tilde{e}Q}{r_+} + \frac{i\kappa}{2} + \mathcal{O}(\kappa^2)$$

- At extremality, NZD mode coincides with onset of charged superradiance
  - Shuts off energy loss and prevents ρ_Q from decaying
- Can confirm by constructing *monochromatic* initial data at this frequency

$\omega < \omega_{\rm NZD}$	$\omega > \omega_{\rm NZD}$
Extract charge from black hole	Inject charge into black hole

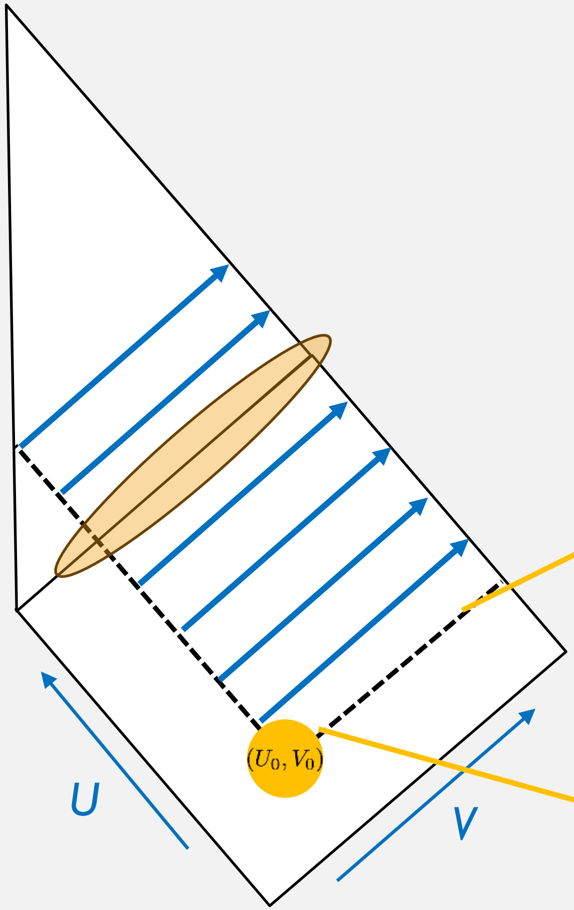
MONOCHROMATIC INITIAL DATA

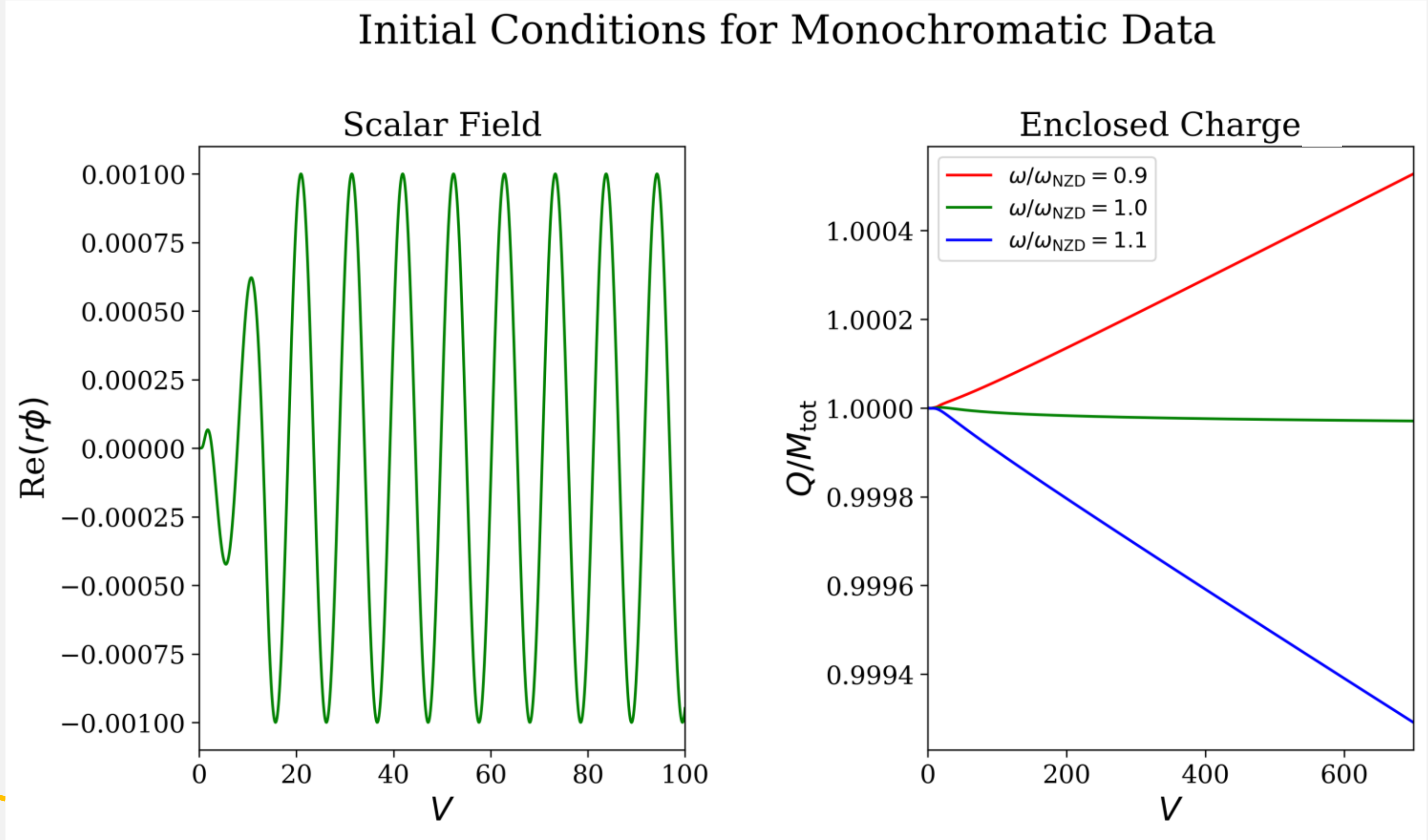
- Construct monochromatic initial data with $r\phi = e^{i\omega V}$



MONOCHROMATIC INITIAL DATA

- Ratio of $\omega/\omega_{\mathrm{NZD}}$ directly controls the *charge-to-mass* ratio of the monochromatic initial data

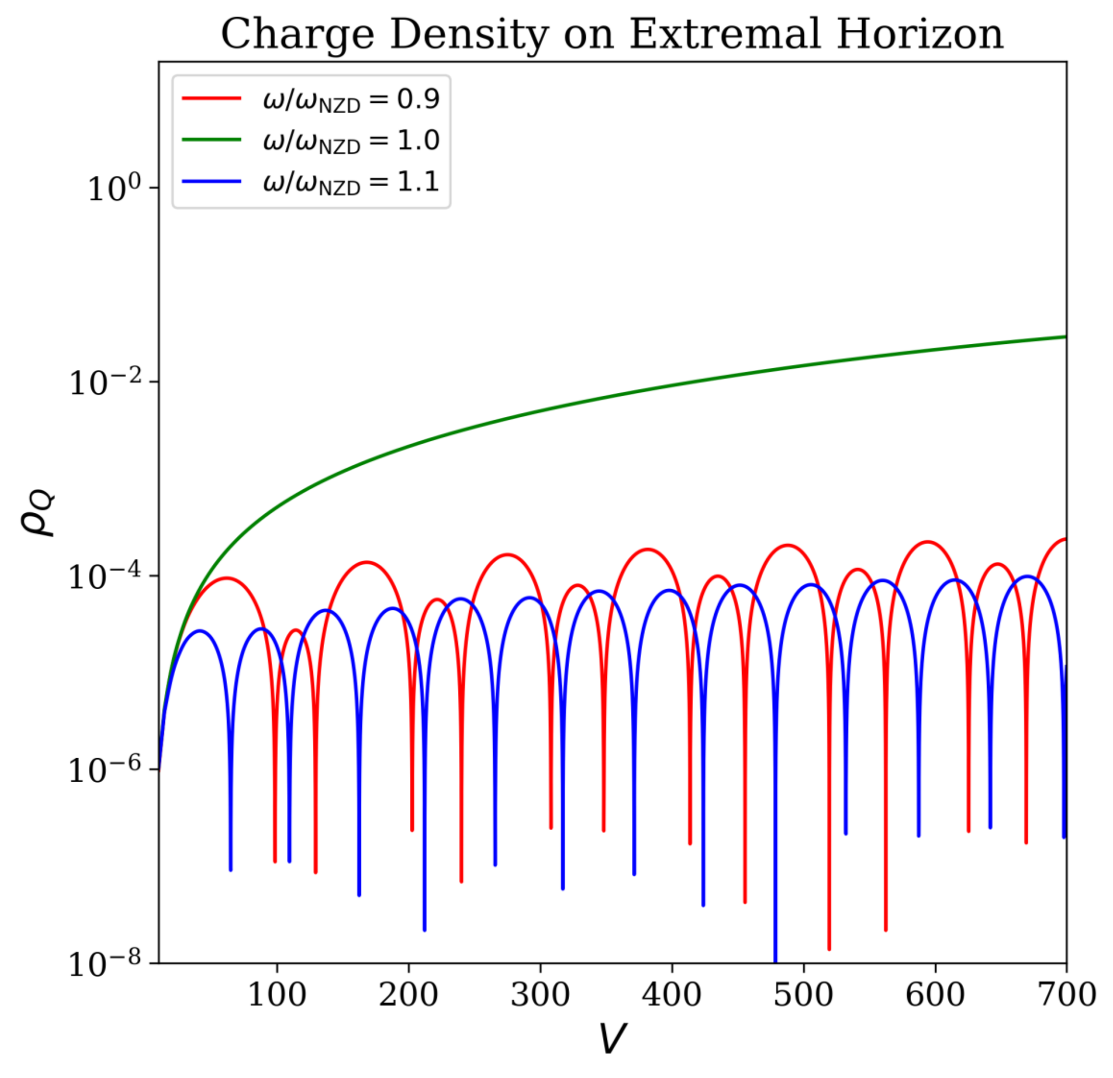




MONOCHROMATIC RESULTS

- Initial data with NZD frequency triggers the strongest instability
- Due to confluence of three effects:
  1. Weakly damped mode
  2. Onset of superradiance
  3. Charge-to-mass ratio of one

Key Question! Does this mode explanation persist in the presence of Einstein's equations?

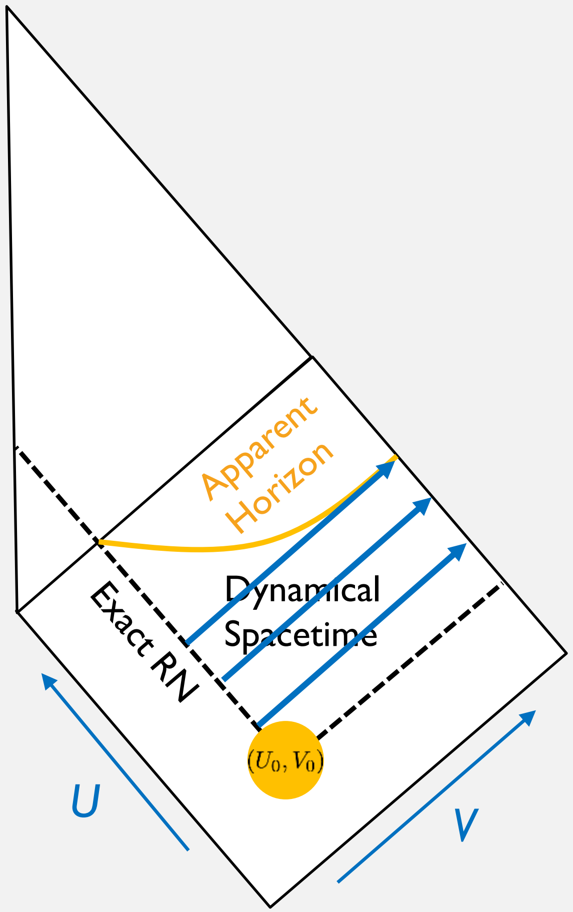


INCORPORATING EINSTEIN'S EQUATIONS

- Einstein's equations: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$
  - In spherical symmetry, two independent metric components: $ds^2 = -2g_{UV}dUdV + r^2 d\Omega^2$
- Four quantities in total: $|\phi|$ (Wave), Q/M (Einstein + Maxwell), r, g_{UV} (Einstein)

<u>Numerical Scheme:</u>

- Same as before but with dynamical refinement around apparent horizon: $\Delta U \propto g^{UV}|_{\mathcal{I}^+}$

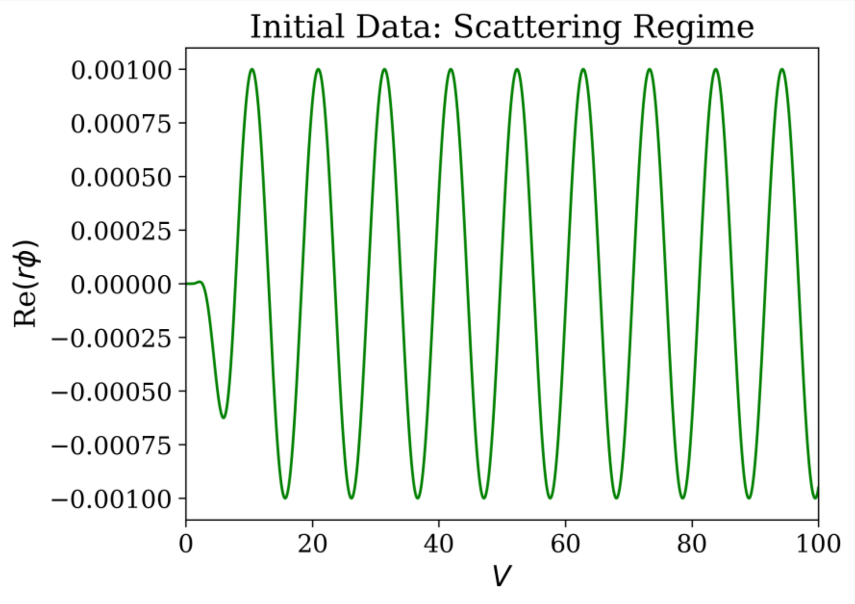


NONLINEAR NZD'S

With Einstein's equations, response to monochromatic initial data splits into two regimes:

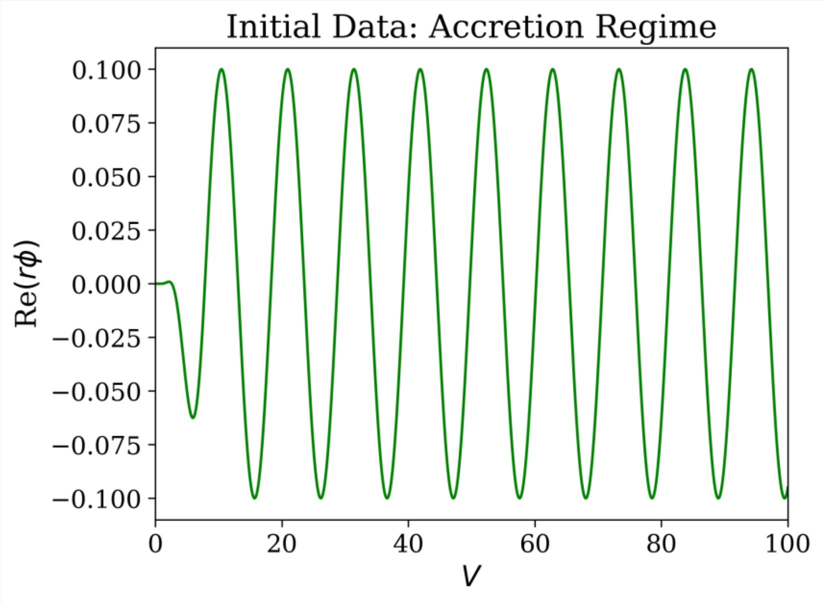
Regime 1: Scattering (Low Amplitude)

- Get back growth from linear problem, but it's transient



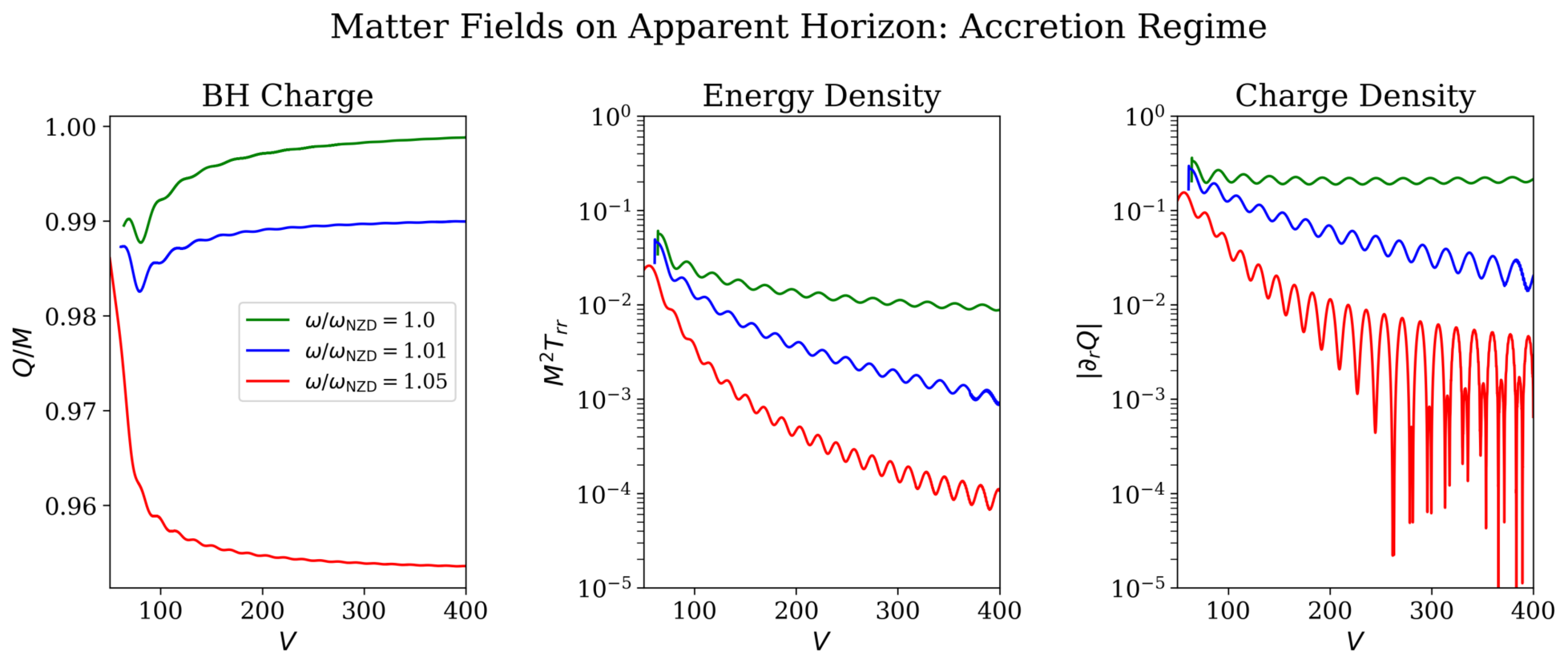
Regime 2: Accretion (High Amplitude)

- Unstable behavior can be drawn out by accreting to extremality



RESULTS: ACCRETION REGIME

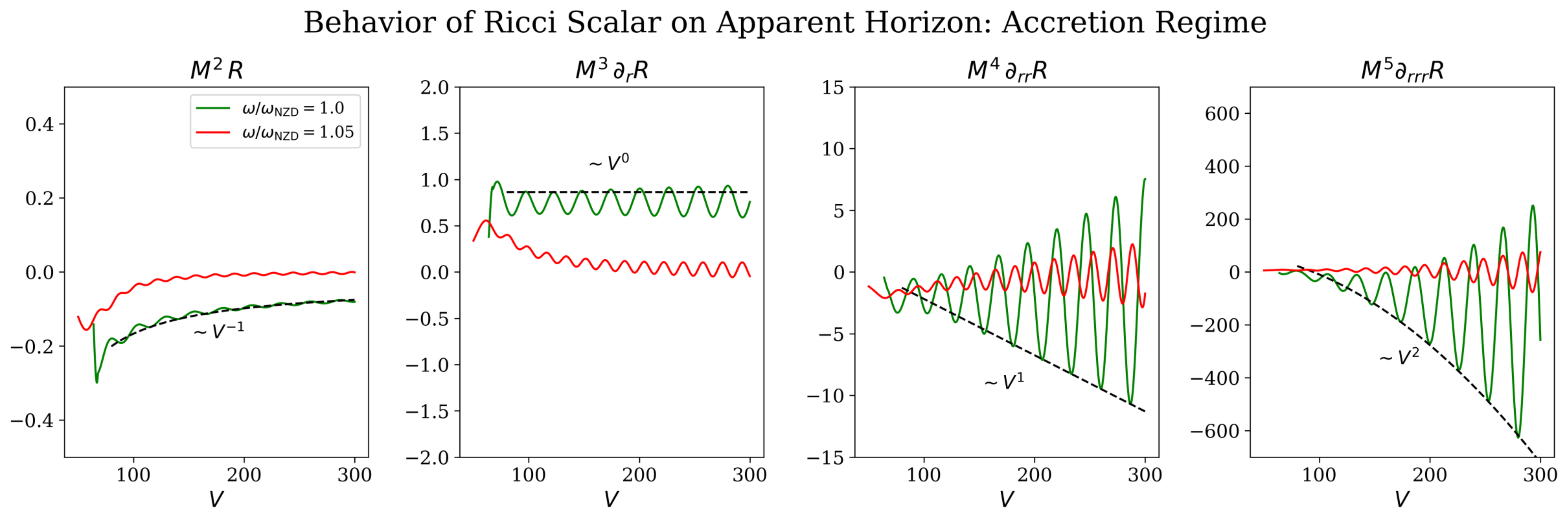
- Since NZD has a charge-to-mass ratio of one, it pushes the black hole *towards* extremality



- Energy decay weakens and charge becomes constant

RESULTS: ACCRETION REGIME

- Curvature scalars might grow in response to matter: $R = -8\pi T \propto -8\pi\partial_r|\phi|\partial_V|\phi| + \frac{Q_{,r}Q_{,V}}{8\pi\tilde{e}|\phi|^2 r^4}$



- Ricci scalar decays, but its *gradients* grow → results agree with naive expectations from stress tensor in linear theory

RESULTS: ACCRETION REGIME

Implications

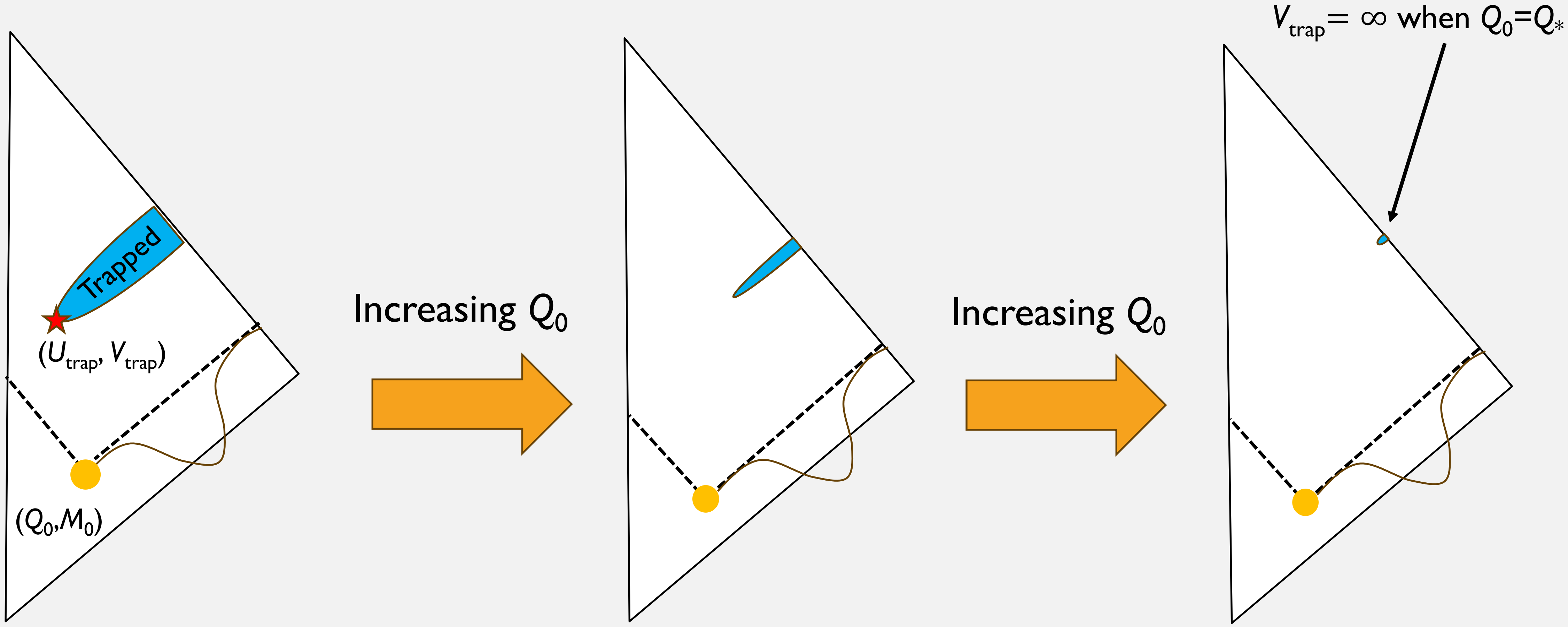
- When backreaction is large, NZD switches from driving superradiant scattering to driving accretion towards extremality
  - Still creates divergent transverse derivatives → shows up in curvature scalars
  - Suggests nonlinear instability on horizon

Key Question: Can the effects of this mode in the nonlinear regime be isolated in compactly-supported data?

- Turn to "dynamical extremal black holes" (MRT 2013)

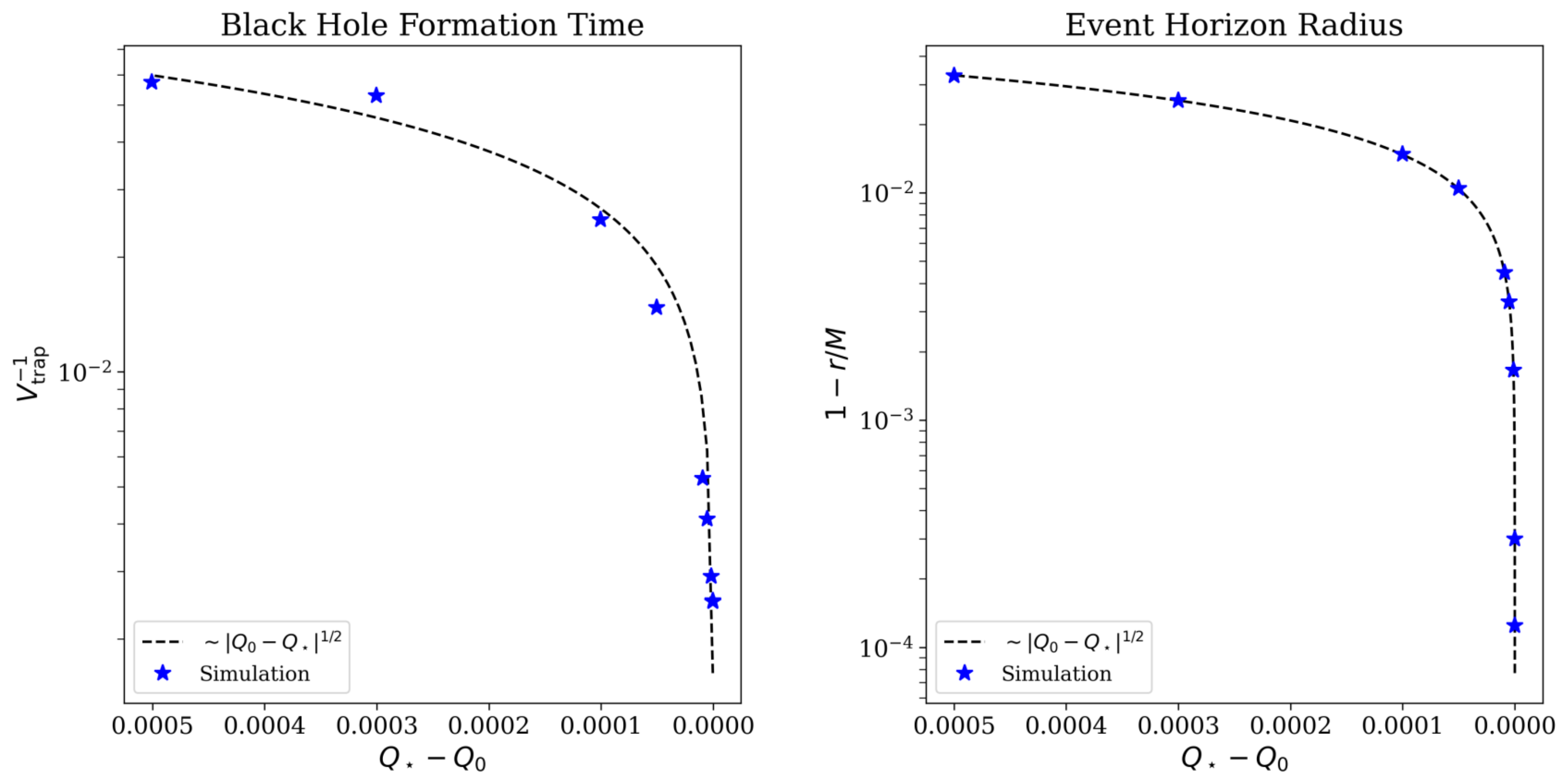
DYNAMICAL EXTREMAL BLACK HOLES

- Begin with super-extremal spacetime (no trapped surfaces) and tune background charge Q_0 (MRT; see Kehle & Unger 2024 for similar fine-tuning arguments)



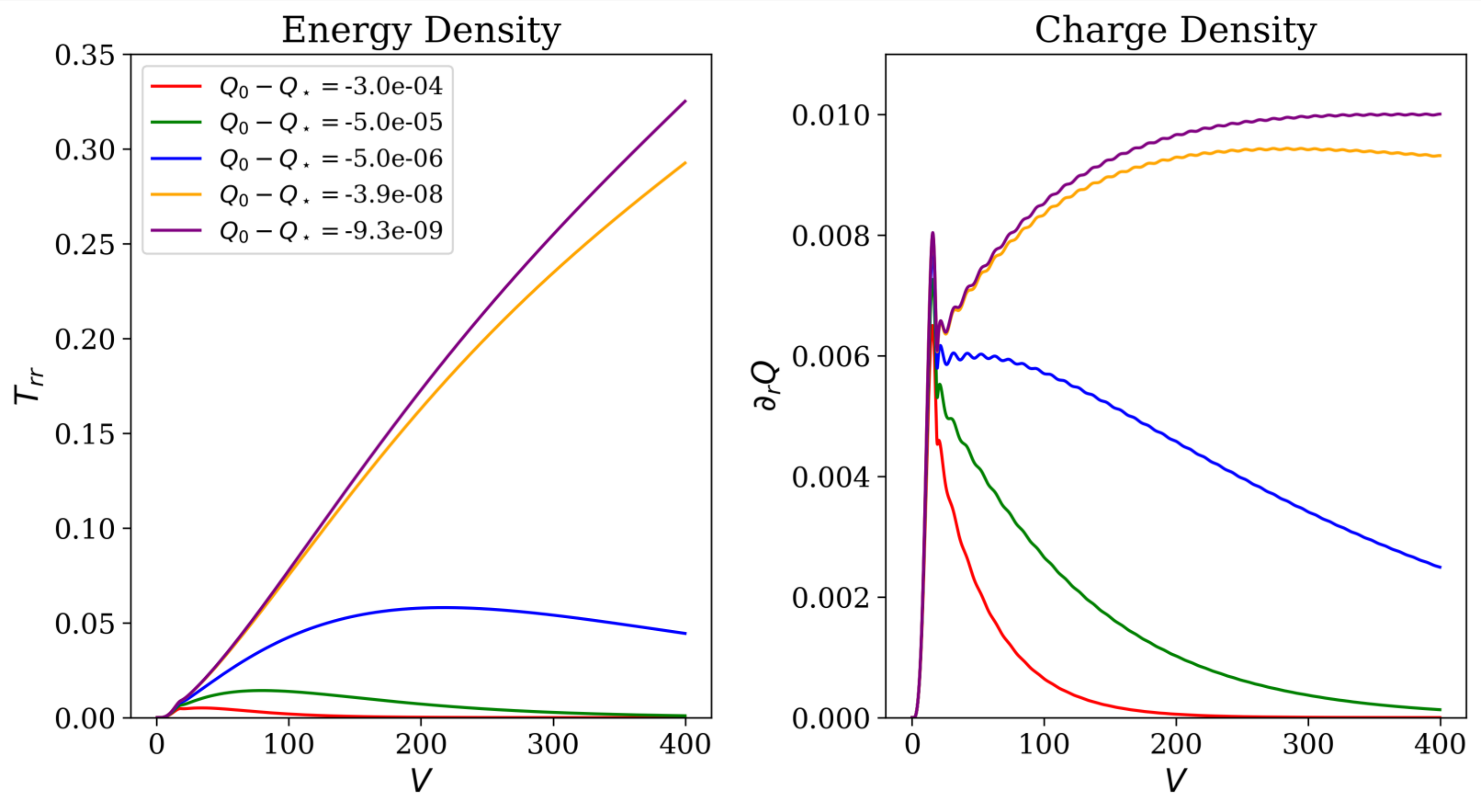
UNIVERSALITY

- As $V_{\text{trap}} \to \infty$, resultant black hole approaches black hole with $r = Q = M$
  - Universal critical exponent is ½ (consistent with MRT 2013; Gralla & Zimmerman 2018; East 2025): $V_{\text{trap}} \sim |Q_* - Q_0|^{-1/2}$ and $r/M - 1 \sim |Q_* - Q_0|^{1/2}$



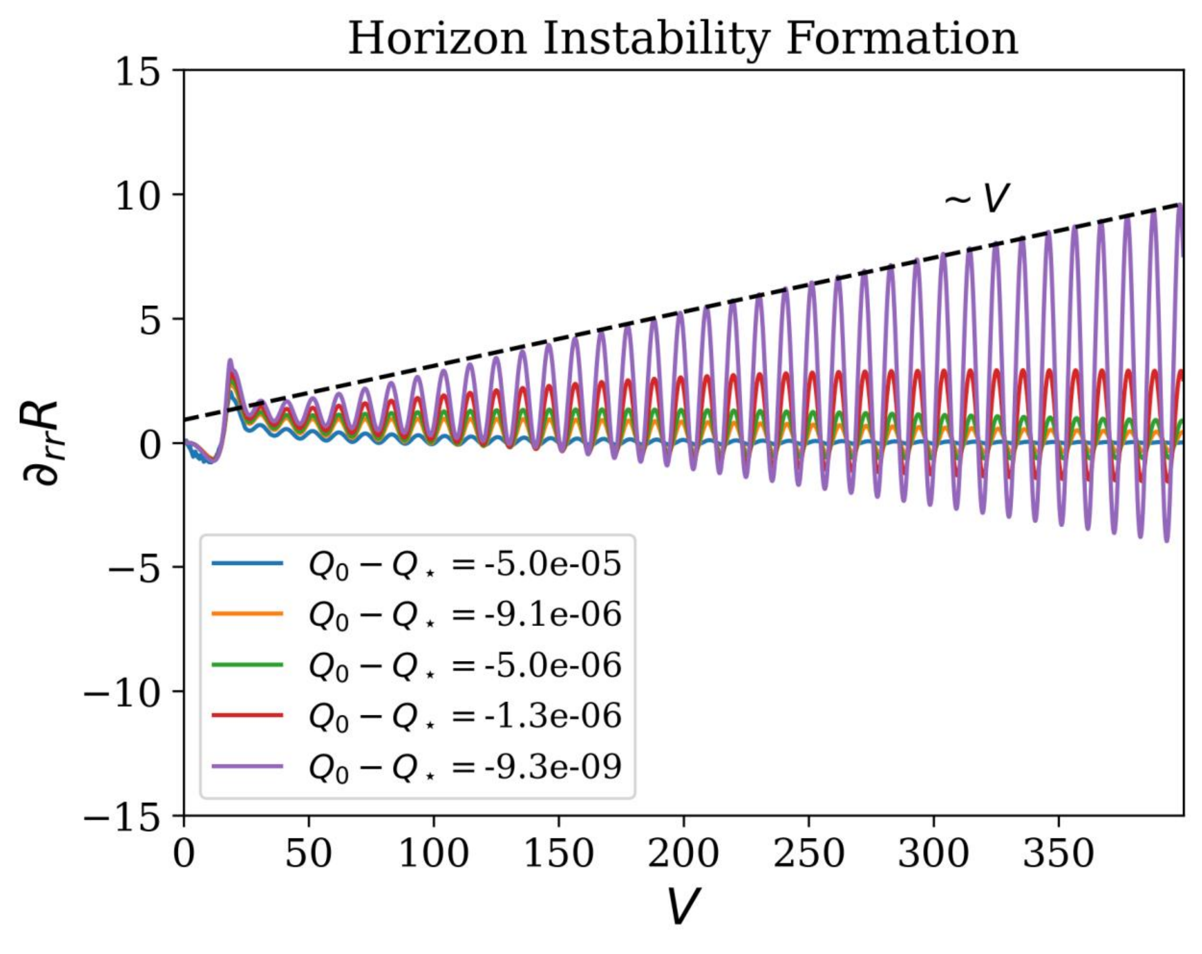
NON-LINEAR ARETAKIS INSTABILITY

- Dynamical extremal black hole shows full-blown Aretakis instability
  - Like MRT results but with actual energy blowup
- Backreaction does *not* regulate instabilities because $T^{\mu}_{\mu}, T_{\mu\nu}T^{\mu\nu}$ remain finite
  - On horizon, energy density balanced by radial pressure and momentum
  - But derivatives of these stresses do diverge…



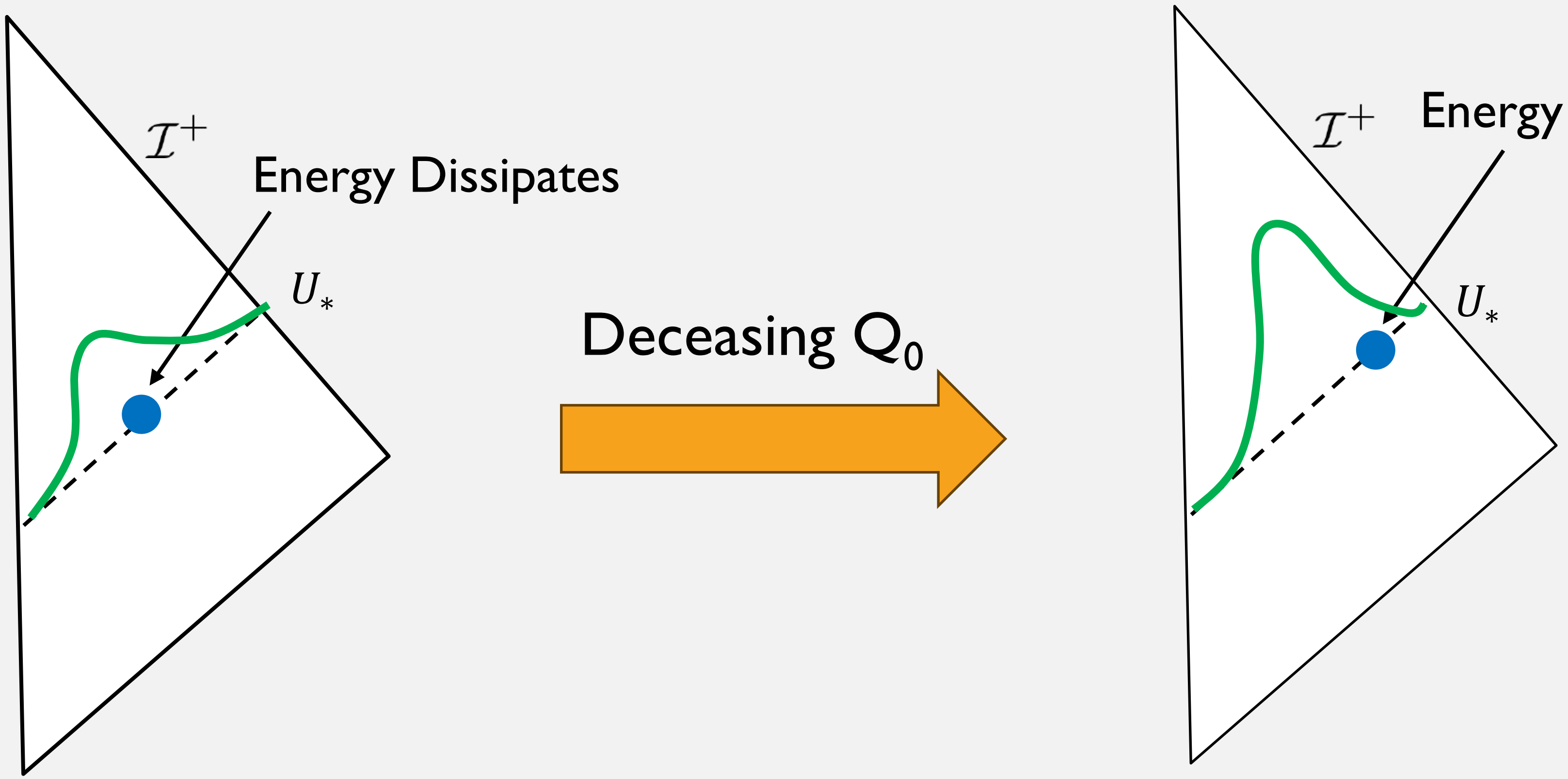
CURVATURE SCALARS

- Derivatives of Ricci scalar blow up at the same rate as the accretion problem
- Oscillations at linear value of NZD!



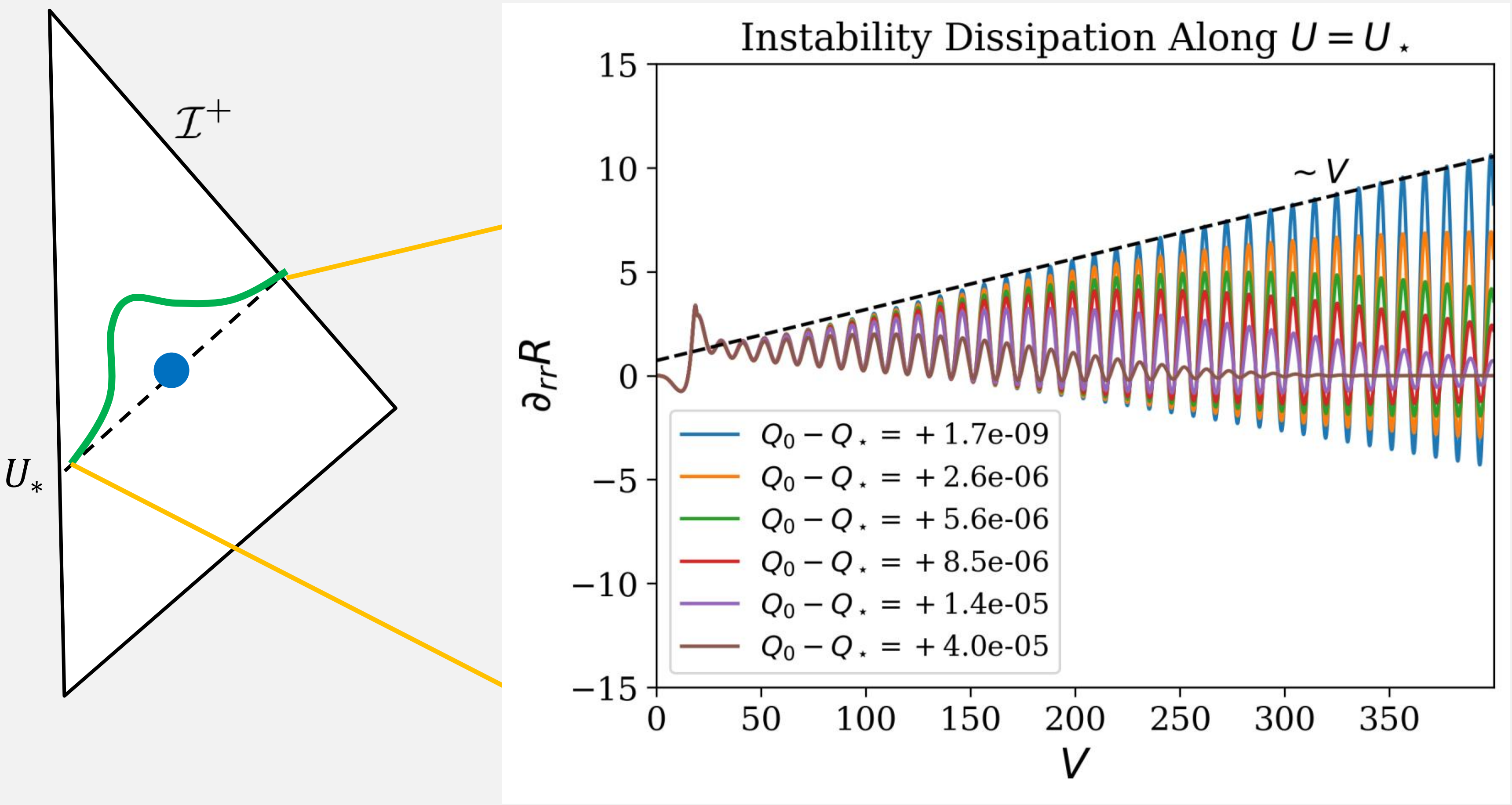
WHAT ABOUT $Q_0 > Q_*$?

- Can find critical behavior when $Q_0 > Q_*$ (no trapped surfaces) side by tracking gradients along location of "would-be" horizon: $U = U_*$



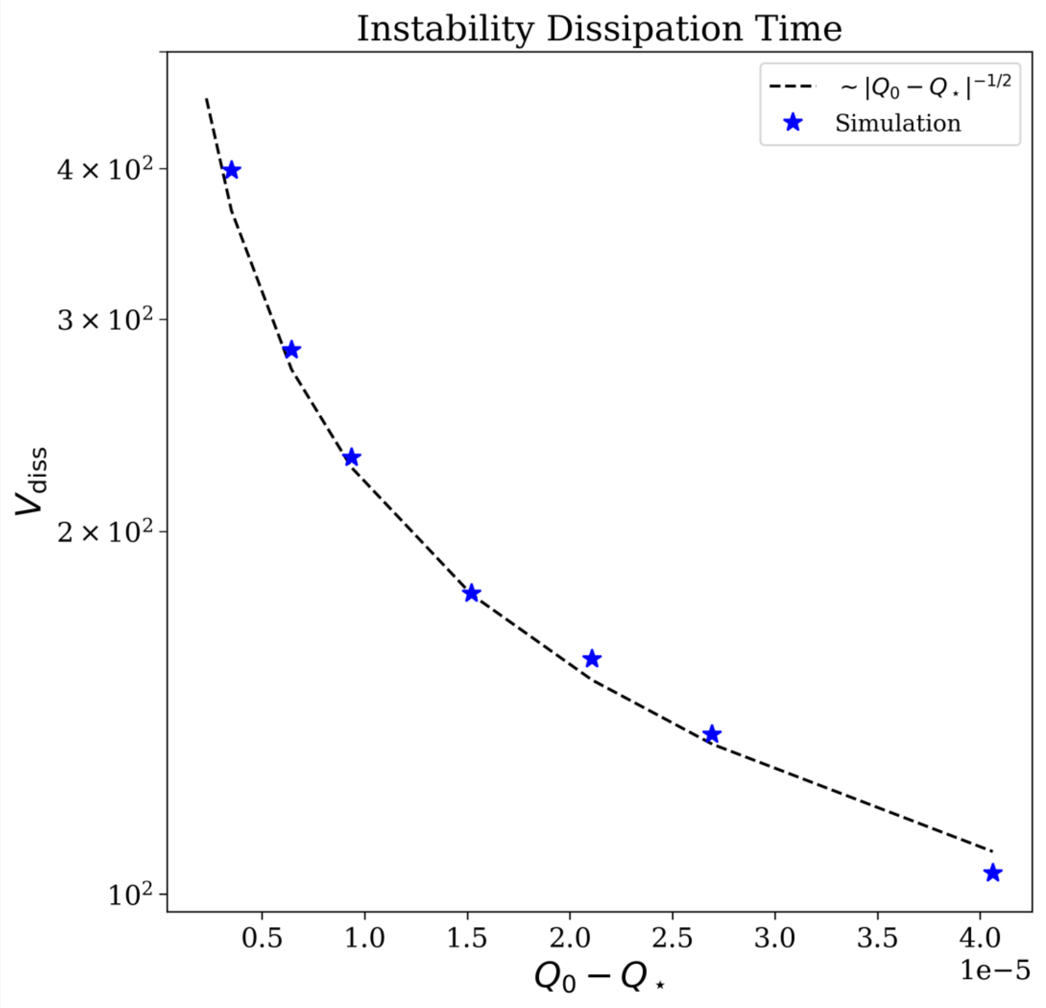
INSTABILITY DISSIPATION: $Q_0 > Q_*$

- Radial derivatives of Ricci scalar grow (and oscillate at NZD frequency) and dissipate

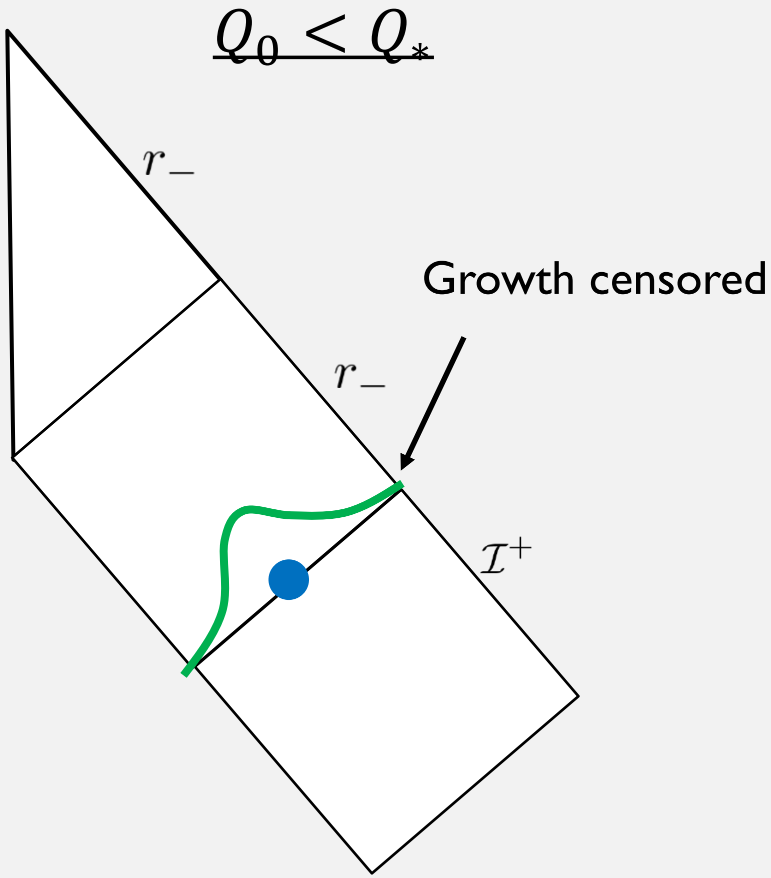


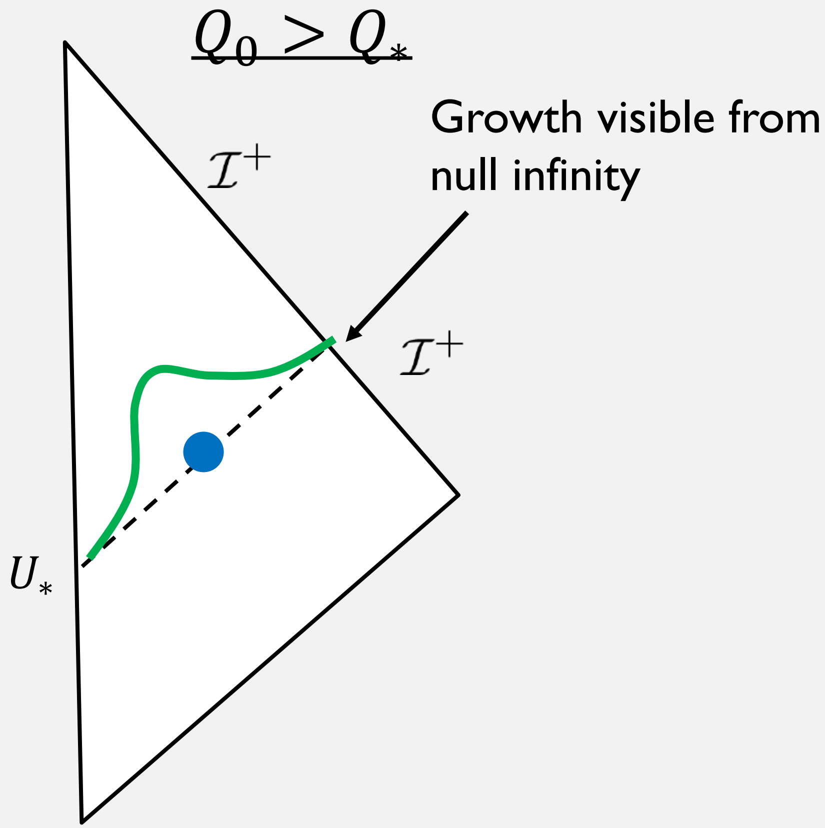
UNIVERSALITY FOR $Q_0 > Q_*$

- Define dissipation timescale as time for which radial derivatives of *R* stop growing
  - Scales as $V_{\text{diss}} \sim (Q_0 - Q_*)^{-1/2}$



SUB-CRITICAL VS. SUPER-CRITICAL

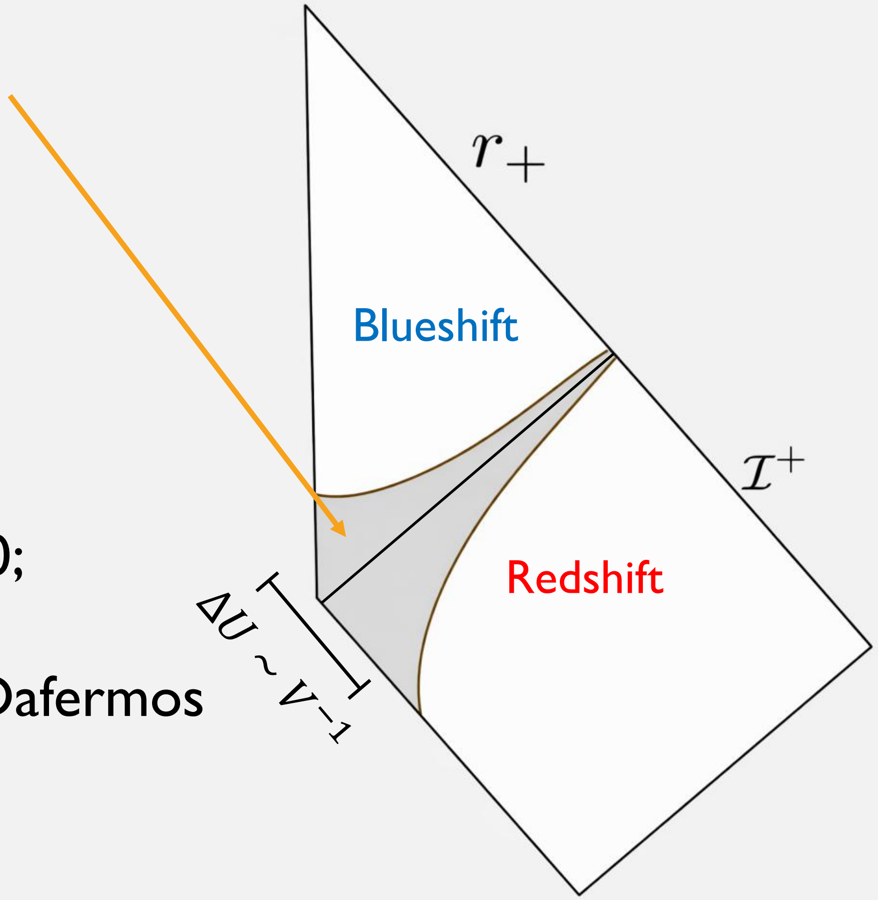




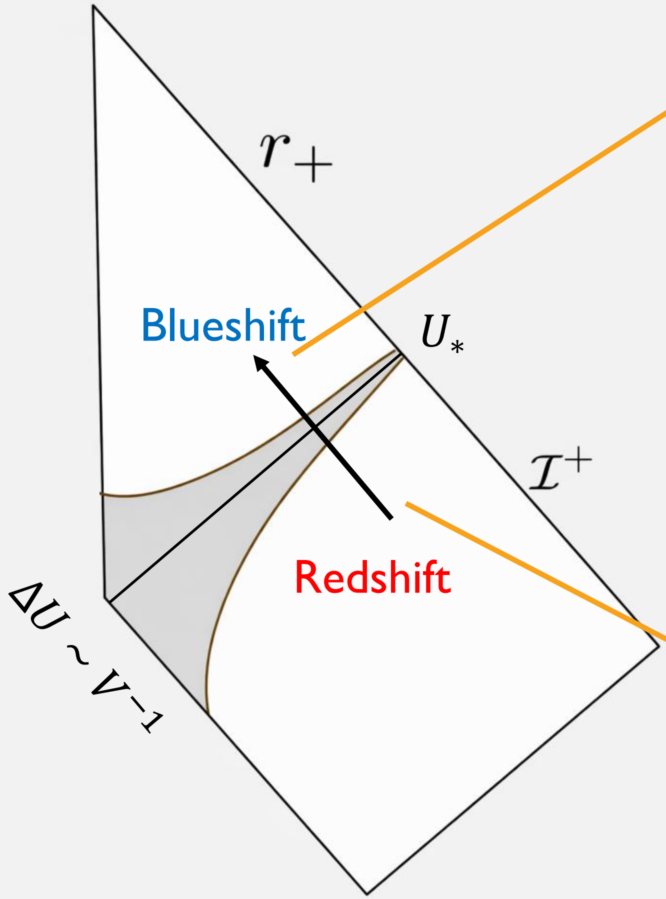
- To understand whether large curvatures are visible from Scri, we need to understand how *localized* the instability is around the horizon

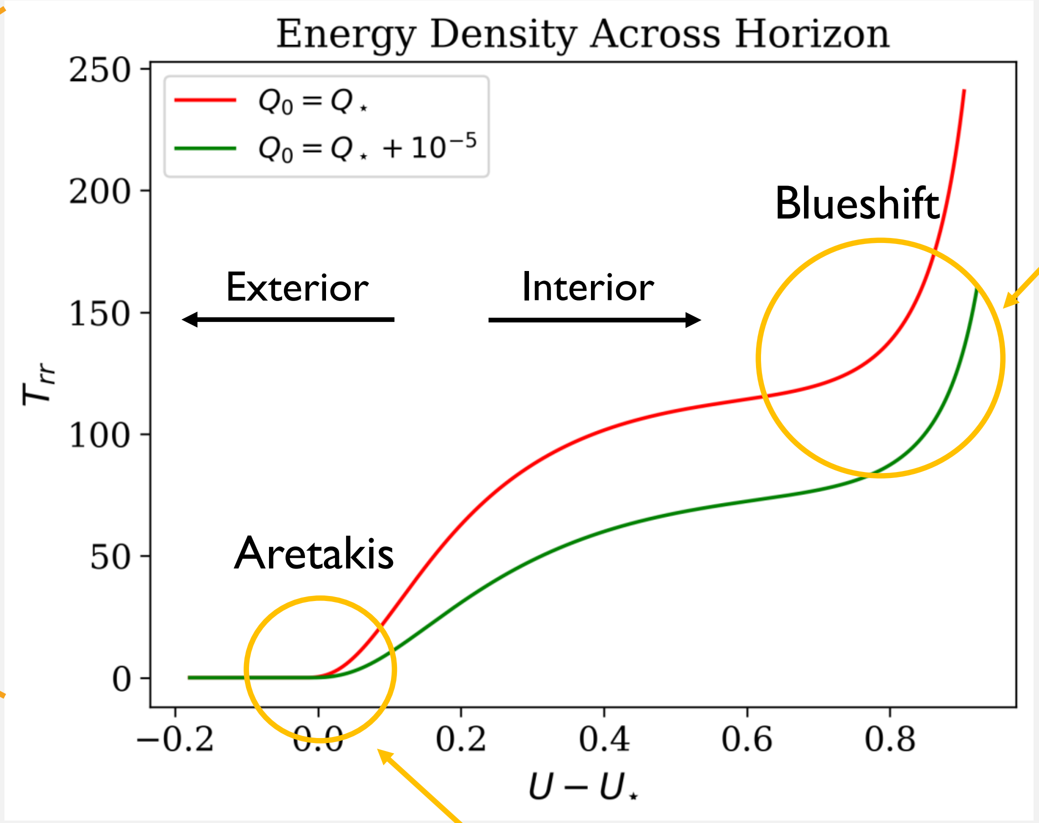
INSTABILITY LOCALIZATION

- Aretakis instability persists in region with no redshift: $\frac{\partial U}{\partial r} \sim 1$
  - Only exists within a small patch of the extremal horizon
- Further in exterior, *r* redshifts: $\frac{\partial U}{\partial r} \to 0$
  - Causes energy decay
- Further in interior, *r* blueshifts: $\frac{\partial U}{\partial r} \to \infty$
  - Causes energy growth
  - A *different* instability of extremal BH interiors (Marolf 2010; Garfinkle 2011)
    - Sub-extremal analog at inner horizon (Penrose 1968; Dafermos 2005; Marolf & Ori 2011)
    - Coincides with the Aretakis instability at late times



ENERGY GROWTH IN INTERIOR





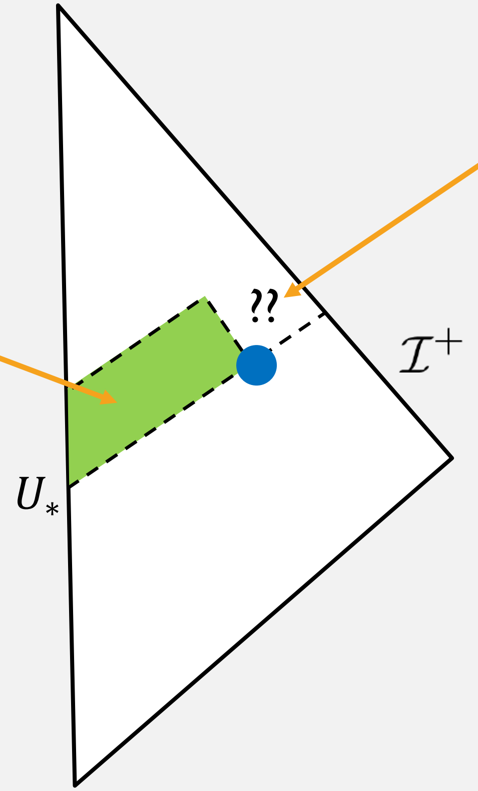
Curvature scalars *not* bounded

Curvature scalars bounded

** Even with no trapped surfaces, similar divergences appear in the "would-be" interior!

IMPLICATIONS: $Q_0 > Q_*$

Curvature scalars *growing* in "would-be" interior… Could in principle be visible from null infinity!



Black hole formation at late times could censor large curvature, but it would have to form *arbitrarily* close to U_* in the limit as $Q_0 \to Q_*$

CONCLUSION

- Extremal black hole horizons develop divergent energy density in presence of charged perturbations
  - Behavior present at linear level (fixed metric) and non-linear level (dynamical ERN)
  - Behavior due NZD mode. Remains a resonance via accretion
- Universality is observed in fine-tuning of non-linear perturbations
  - By tuning to critical parameter from above, can potentially generate large curvature visible from Scri

<u>Future Steps</u>

- Continue to explore critical phenomena
  - Universality of horizon "hairs"?
  - Visibility from null infinity?
  - Extensions to collapse problem
- Breaking spherical symmetry