

Uncertainty in uncertainty, risk measures, and Rockafellian relaxation

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Collaborators and acknowledgements

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- Hugo Díaz (North Carolina State University)
- Alonso J. Bustos (Universidad de Concepción)
- Benjamín Venegas (Universidad de Concepción)



References:

- Antil, **C.**, Díaz & Royset; *SIAM J. Optim.*, to appear. (arXiv:2405.00176)
- Antil, Bustos, **C.** & Venegas; submitted. (arXiv:2604.00226)

Takeaways

1. Risk-averse optimization problems can be sensitive to corrupted probability distributions.
2. We can enhance robustness of risk-averse optimization to data corruption by minimizing a modified objective function that includes an additional perturbation variable.
3. Problem relaxation with CVaR_β allows insight into the sensitivity of optimal control strategies under changing levels of *both* problem corruption *and* risk-aversion.

1. **Motivation: distributional ambiguity**
2. Rockafellian relaxation
3. Connections with both “optimistic” and “robust” optimization
4. Numerical examples
5. Concluding remarks

Generic, risk-averse, stochastic PDECO

Consider:

$$\begin{aligned} \min_{z \in Z_{\text{ad}}} \varphi(z), \quad \varphi(z) &= f_0(z) + \text{CVaR}_{\beta}(g(s(\xi, z))) \\ &= f_0(z) + \inf_{\gamma \in \mathbb{R}} \left(\gamma + \mathbb{E}_{\mathbb{P}}[(g(s(\xi, z)) - \gamma)^+] \right) \end{aligned}$$

- s is solution operator to some PDE, depends on control $z \in Z_{\text{ad}}$
- g maps the PDE solution to some quantity of interest (QoI)
- f_0 is control penalty or regularization term
- $\beta \in [0, 1)$ is CVaR parameter

Stochastic optimal control with CVaR

- For gradient based optimization algorithms, CVaR_β is often mollified
- $(x)^+ = \max\{0, x\}$ replaced with a smooth approximation, e.g.:

$$(x)_\delta^+ = x + \delta \log(1 + e^{-x/\delta})$$

- Gives rise

$$\varphi^\delta(z, \gamma) := f_0(z) + \gamma + \frac{1}{1 - \beta} \mathbb{E}_{\mathbb{P}} [(g(s(\xi, z)) - \gamma)_\delta^+]$$

Kouri & Surowiec, *SIAM J. Optim.*, 2016.

Problem uncertainty

“Aleatoric uncertainty”: uncertainty may exist in the constraining PDE, e.g. forcing terms, coefficients in differential operator, boundary conditions, ...

$$\begin{aligned} \min_{z \in Z_{\text{ad}}} \varphi(z), \quad \varphi(z) &= f_0(z) + \text{CVaR}_\beta(g(s(\xi, z))) \\ &= f_0(z) + \inf_{\gamma \in \mathbb{R}} \left(\gamma + \mathbb{E}_{\mathbb{P}} [(g(s(\xi, z)) - \gamma)^+] \right) \end{aligned}$$

- Solution map s parameterized with random quantity ξ
- ξ belongs to sample space Ξ
- ξ drawn from some hypothesized probability measure \mathbb{P}

Epistemic uncertainty and data corruption

“Epistemic uncertainty”: inexact knowledge of the form of Ξ and/or \mathbb{P}

- If we hypothesize some \mathbb{P} , what if our ansatz is not quite right?
- If we estimate \mathbb{P} empirically, what if our measurements are prone to error, corruption, adversarial attacks, ...?

Epistemic uncertainty and data corruption

“*Epistemic uncertainty*”: inexact knowledge of the form of Ξ and/or \mathbb{P}

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- If we estimate \mathbb{P} empirically, what if our measurements are prone to error, corruption, adversarial attacks, . . . ?

“The use of [expectations and risk measures] is justified when the probability law \mathbb{P} is known but can lead to nonsensical, even dangerous, results if \mathbb{P} is unknown.”

Epistemic uncertainty and data corruption

We model data corruption in two ways.

- Let $(\Xi, \mathcal{A}, \mathbb{P})$ be a probability space and assume

$$\mathbb{E}_{\mathbb{P}}[X] = \int_{\Xi} X(\xi) \underbrace{\rho(\xi)}_{\frac{d\mathbb{P}}{d\mu} :=} d\mu(\xi)$$

where μ is a σ -finite measure

- Corruption type 1—corruption to probability density:

ρ replaced with ρ_{ϵ}

- Corruption type 2—corruption to support of distribution:

ξ replaced with $\eta_{\epsilon}(\xi)$

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2. **Rockafellian relaxation**
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Rockafellian relaxation

How to account for distributional uncertainty? Use *problem relaxation*...

Rockafellian relaxation

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- Main idea: embed (a potentially corrupted) stochastic optimal control problem in a family of perturbed problems
- In practice: minimize a bivariate objective function of the control variable(s) z (and γ) and additional perturbation variable t

$$\varphi = \varphi(z, \gamma, t)$$

- Based on ideas from R. T. Rockafellar

Rockafellar, "Convex Analysis", 1970

Royset & J-B Wetts, "An Optimization Primer", 2021

Royset, Chen, & Eckstrand, *Maths. Oper. Res.*, 2024

Antil, C., Díaz & Royset, *SIAM J. Optim.*, to appear.

Tian & Royset, *arXiv:2507.15801*, 2025.

Rockafellian relaxation

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Let's examine the main idea for a toy problem...

Toy example: linear stochastic program

- Let $g(x, \xi) = \xi x + \frac{1}{2}(1 - x)$
- Consider stochastic optimization problem

$$\min_{x \in [0,1]} \varphi(x), \quad \varphi(x) = \mathbb{E}_{\mathbb{P}} [g(x, \xi)]$$

where \mathbb{P} assigns $\xi = 0$ with probability 1

- Has unique minimizer $x^* = 1$, and $\varphi(x^*) = 0$

Toy example: linear stochastic program

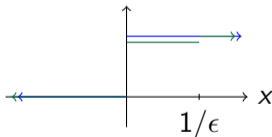
- Now consider perturbed problem:

$$\min_{x \in [0,1]} \varphi_\epsilon(x), \quad \varphi_\epsilon(x) = \mathbb{E}_{\mathbb{P}_\epsilon} [g(x, \xi_\epsilon)]$$

where \mathbb{P}_ϵ assigns $\xi_\epsilon = 0$ with probability $1 - \epsilon$, $\xi_\epsilon = 1/\epsilon$ with probability ϵ

- Has unique minimizer $x_\epsilon^* = 0$, and $\varphi_\epsilon(x_\epsilon^*) = 1/2$

$$\mathbb{P}_\epsilon[\xi_\epsilon \leq x], \quad \mathbb{P}[\xi \leq x]$$



- $\xi_\epsilon \rightarrow \xi$ in probability, however, $x_\epsilon^* \not\rightarrow x^*$ as $\epsilon \rightarrow 0$... **no convergence of minimizers!**

Toy example: Rockafellian relaxation

- Denote corrupted probabilities $p_{\epsilon,1} = 1 - \epsilon$, $p_{\epsilon,2} = \epsilon$
- Let Δ denote set of probability vectors in \mathbb{R}^2
- For some $\theta_\epsilon > 0$, define bivariate “Rockafellian” objective function $\Phi_\epsilon : [0, 1] \times \mathbb{R}^2 \rightarrow \bar{\mathbb{R}}$:

$$\Phi_\epsilon(x, t) = \frac{1}{2}(1-x) + 0 \cdot (p_{\epsilon,1} + t_1)x + 1/\epsilon \cdot (p_{\epsilon,2} + t_2)x + \frac{\theta_\epsilon}{2}(t_1^2 + t_2^2) + \iota_\Delta(p_\epsilon + t)$$

Elementary calculus \longrightarrow global minimum of Φ_ϵ at $x_\epsilon^* = 1$, $t^* = (\epsilon, -\epsilon)$, which *recovers the minimizer to the uncorrupted problem!* (Caveat: need $\theta_\epsilon < 4/\epsilon^2$)

Rockafellian relaxation

Returning to the PDE case...

Rockafellian relaxation

Returning to the PDE case...

- Define a Rockafellian for the uncorrupted objective function:

$$\Phi^\delta(z, \gamma, t_1, t_2) = f_0(z) + \gamma + \iota_{\{(0,0)\}}(t_1, t_2) + \left(\frac{1}{1-\beta}\right) \int_{\Xi} g(s(\xi + t_1(\xi), z)) - \gamma)_\delta^+(\rho(\xi) + t_2(\xi)) d\mu(\xi).$$

- The Rockafellian is “*anchored*” at $(t_1, t_2) = (0, 0)$
- *Notice:* minimizing Φ^δ is equivalent to minimizing φ^δ (since $(t_1, t_2) = (0, 0)$ is the only choice)

Rockafellian relaxation

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- The Rockafellian is “*anchored*” at $(t_1, t_2) = (0, 0)$
- *Notice:* minimizing Φ^δ is equivalent to minimizing φ^δ (since $(t_1, t_2) = (0, 0)$ is the only choice)

Note that the choice of a Rockafellian (and its anchor) is generally not unique.

Rockafellian relaxation

Also define a Rockafellian for the *corrupted* objective:

$$\begin{aligned} \Phi_{\epsilon}^{\delta}(z, \gamma, \mathbf{t}_1, \mathbf{t}_2) &= f_0(z) + \gamma + \theta_{\epsilon}(\|\mathbf{t}_1\|_{T_1}^q + \|\mathbf{t}_2\|_{T_2}^{q'}) + \iota_{\mathbb{P}}(\rho_{\epsilon} + \mathbf{t}_2) \\ &\quad \left(\frac{1}{1-\beta}\right) \int_{\Xi} g(s(\eta_{\epsilon}(\xi) + \mathbf{t}_1(\xi), z)) - \gamma)_{\delta}^{+}(\rho_{\epsilon}(\xi) + \mathbf{t}_2(\xi)) d\mu(\xi) \end{aligned}$$

where $\theta_{\epsilon} > 0$ and $T_1 = L^q(\Xi; \Xi)$, $T_2 = L^{q'}(\Xi; \mathbb{R})$, and $1/q + 1/q' = 1$.

Overview of theoretical results

Under suitable assumptions on s , g , f_0 , Z_{ad} , and Ξ , we have:

- Existence of optimizers for smoothed Rockafellian $\Phi_\epsilon^\delta(z, \gamma, t)$
- $\Phi_\epsilon^\delta \xrightarrow{\Gamma} \Phi^\delta$ as $\epsilon \downarrow 0$ (for $\delta \geq 0$):
- Optimality conditions for $\Phi_\epsilon^\delta(z, \gamma, t)$

In each case, Ξ can be infinite-dimensional and $\mu(\Xi) = \infty$ is possible.

However, for this talk we assume $\mu(\Xi) < \infty$ and Ξ finite dimensional.

Technical assumptions

Let \mathcal{U} be a Banach spaces, \mathcal{Z} a reflexive Banach space, and Z_{ad} a convex subset of \mathcal{Z} .

Conditions on solution operator:

- *Measurability:* $s(\cdot, z) : \Xi \rightarrow \mathcal{U}$ is \mathcal{A} -measurable for all $z \in \mathcal{Z}$
- *Continuity:* If both $\xi_\epsilon \rightarrow \xi$ in Ξ and $z_\epsilon \rightarrow z$ in \mathcal{Z} , then $s(\xi_\epsilon, z_\epsilon) \rightarrow s(\xi, z)$ in \mathcal{U}

Conditions on control regularizer and QoI map:

- f_0 is proper and sequentially lower semicontinuous
- g is sequentially lower semicontinuous

Technical assumptions, continued

Conditions on sample and measure space:

- The set Ξ is equipped with a norm $\|\cdot\|_{\Xi}$
- The measure μ is σ -finite on (Ξ, \mathcal{A})
- *Compact embeddings:* $\exists (W_{1,\epsilon})_{\epsilon}$ and $(W_{2,\epsilon})_{\epsilon}$ that compactly embed into $L^q(\Xi_{\epsilon}; \Xi)$ and $L^{q'}(\Xi_{\epsilon}; \mathbb{R})$, respectively for each $\epsilon > 0$.

Finally, a growth condition on $g \circ s$:

- There exists some monotonic increasing $\varrho : [0, \infty) \rightarrow [0, \infty)$ such that for all $z \in \mathcal{Z}$ and $\xi \in \Xi$ (a.s.):

$$|(g \circ s)(\xi, z)| \leq \varrho(\|z\|_{\mathcal{Z}}) \|\xi\|_{\Xi}$$

Existence theorem

Under the stated assumptions, we have:

Theorem

Fix $\epsilon > 0$ and $\delta > 0$. Suppose that either f_0 is coercive or that Z_{ad} is bounded. Then there exists

$$(z^*, \gamma^*, t_1^*, t_2^*) \in Z_{\text{ad}} \times \mathbb{R} \times L^q(\Xi; \Xi) \times L^{q'}(\Xi; \mathbb{R})$$

such that

$$\Phi_\epsilon^\delta(z^*, \gamma^*, t_1^*, t_2^*) \leq \Phi_\epsilon^\delta(z, \gamma, t_1, t_2)$$

for all $(z, \gamma, t_1, t_2) \in Z_{\text{ad}} \times \mathbb{R} \times L^q(\Xi; \Xi) \times L^{q'}(\Xi; \mathbb{R})$.

Weak-strong Γ -convergence

Definition

The sequence $(\phi_\epsilon)_\epsilon$ weak-strong Γ -converges to ϕ , denoted by $\phi_\epsilon \xrightarrow{\Gamma} \phi$, if the following conditions hold:

1. For all $(x, y) \in X \times Y$, there exists a sequence $(x_\epsilon, y_\epsilon)_{\epsilon > 0}$ in $X \times Y$ converging strongly to (x, y) such that

$$\limsup_{\epsilon \downarrow 0} \phi_\epsilon(x_\epsilon, y_\epsilon) \leq \phi(x, y).$$

2. For all sequences $x_\epsilon \rightarrow x \in X$ and $y_\epsilon \rightarrow y \in Y$, there holds

$$\liminf_{\epsilon \downarrow 0} \phi_\epsilon(x_\epsilon, y_\epsilon) \geq \phi(x, y).$$

Convergence theorem

Theorem

Let $\rho \in P \cap T_2$, and consider corruption sequences $(\eta_\epsilon)_\epsilon$ and $(\rho_\epsilon)_\epsilon$ such that

$$\eta_\epsilon \in T_{1,\epsilon} \quad \text{and} \quad \rho_\epsilon \in P \cap T_{2,\epsilon}$$

for all $\epsilon > 0$. Assuming that

$$\theta_\epsilon \rightarrow \infty \quad \text{and} \quad \theta_\epsilon (\|\rho_\epsilon - \rho\|_{T_1}^q + \|\eta_\epsilon - I\|_{T_2}^{q'}) \rightarrow 0$$

as $\epsilon \downarrow 0$, then $\Phi_\epsilon^\delta \xrightarrow{\Gamma} \Phi^\delta$ as $\epsilon \downarrow 0$.

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$$\theta_\epsilon \rightarrow \infty \quad \text{and} \quad \theta_\epsilon (\|\rho_\epsilon - \rho\|_{T_1}^q + \|\eta_\epsilon - I\|_{T_2}^{q'}) \rightarrow 0$$

as $\epsilon \downarrow 0$, then $\Phi_\epsilon^\delta \xrightarrow{\Gamma} \Phi^\delta$ as $\epsilon \downarrow 0$.

- **Note:** as the corruption levels get smaller, θ_ϵ must grow to ensure convergence.
- Reminder: Γ -convergence preserves convergence of minimizers!

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Connections to both DOO and DRO

Rockafellian minimization problem:

$$\min_{z \in Z_{\text{ad}}} \min_{(t_1, t_2) \in T_1 \times T_2} \left\{ f_0(z) + \theta_\epsilon (\|t_1\|_{T_1}^q + \|t_2\|_{T_2}^{q'}) + \iota_P(\rho_\epsilon + t_2) \right. \\ \left. + \inf_{\gamma \in \mathbb{R}} \left(\gamma + \frac{1}{1-\beta} \int_{\Xi} g(s(\eta_\epsilon(\xi) + t_1(\xi), z)) - \gamma \right)_\delta^+(\rho_\epsilon + t_2) d\mu \right\}$$

- **“Min-min”** formulation is an example of Distributionally Optimistic Optimization (DOO)
- DOO has connections to robust statistics
- This aspect of the method is well suited for “adversarial” data regime

Blanchet et al., *NeuRIPS*, 2024.

Jiang & Xie, *SIAM J. Optim.*, 2024.

Blanchet et al., *Statistical Sci.*, 2025.

Connections to both DOO and DRO

On the other hand...

- Distributionally Robust Optimization (DRO) is inherent to the method
- CVaR_β is coherent risk measure \implies for fixed (t_1, t_2) we solve:

Artzner et. al, *Math. Finance*, 1999.

$$\min_{z \in Z_{\text{ad}}} \left\{ f_0(z) + \theta_\epsilon (\|t_1\|_{T_1}^q + \|t_2\|_{T_2}^{q'}) + \iota_{\mathbb{P}}(\rho_\epsilon + t_2) \right. \\ \left. + \sup_{\theta \in \Theta_\epsilon} \left(\int_{\Xi} \theta(\xi) g(s(\eta_\epsilon(\xi) + t_1(\xi), z)) (\rho_\epsilon + t_2) d\mu \right) \right\}$$

where risk envelope

$$\Theta_\epsilon = \left\{ \vartheta \in L^\infty(\Xi, \mathbb{P}) \mid \int_{\Xi} \vartheta (\rho_\epsilon + t_2) d\mu = 1, 0 \leq \vartheta \leq (1 - \beta)^{-1} \mathbb{P} \text{ a.s.} \right\}$$

- **“Min-max”** DRO can yield better out-of-sample performance, especially for tail events

Connections to both DOO and DRO

- The combination of DRO and DOO can yield a compromise between the two methods
- Recent effort to combine the two
 - Blanchet et al., *NeuRIPS*, 2024.
 - Jiang & Xie, *SIAM J. Optim.*, 2024.
 - Blanchet et al., *Statistical Sci.*, 2025.
- Rockafellian relaxation allows insight into sensitivity of optimal control to perturbations in θ_ϵ and β

$$\min_{z \in Z_{\text{ad}}} \min_{(t_1, t_2) \in T_1 \times T_2} \left\{ f_0(z) + \theta_\epsilon (\|t_1\|_{T_1}^q + \|t_2\|_{T_2}^{q'}) + \nu_{\mathbb{P}}(\rho_\epsilon + t_2) \right. \\ \left. + \inf_{\gamma \in \mathbb{R}} \left(\gamma + \frac{1}{1 - \beta} \int_{\Xi} g(s(\eta_\epsilon(\xi) + t_1(\xi), z)) - \gamma \right)_\delta^+(\rho_\epsilon + t_2) d\mu \right\}$$

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Numerical example 1: risk-neutral case

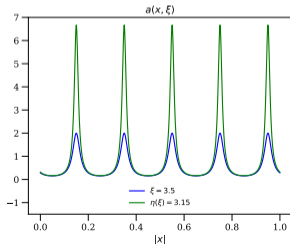
- For $x \in \mathbb{R}^2$, consider diffusion coefficients:

$$a(x, \xi) = \frac{1}{\xi + 3 \sin(10\pi|x|)} \quad (\text{uncorrupted})$$

vs.

$$a(x, \xi) = \frac{1}{\eta(\xi) + 3 \sin(10\pi|x|)} \quad (\text{corrupted})$$

where $\mathbb{P}[\xi = 3.5] = 1$ and $\eta(\xi) \sim \text{unif}(3.5 - \delta, 3.5 + \delta)$ for $\delta \in (0, 0.5)$



Numerical example 1: risk-neutral case

Consider:

- Target function $u_*(x) = 1$ on $\Omega = B_1(0)$ (unit ball in \mathbb{R}^2)
- Objective function:

$$\varphi(z) = \underbrace{\frac{\alpha}{2} \|z\|_{L^2(\Omega)}^2}_{f_0(z)=} + \mathbb{E}_{\mathbb{P}} \left[\underbrace{\frac{1}{2} \|s(\eta(\xi), z) - u_*\|_{L^2(\Omega)}^2}_{g(s(\eta(\xi), z))=} \right]$$

where $u(\cdot, \xi) := s(\eta(\xi), z) \in H_0^1(\Omega)$ solves

$$\int_{\Omega} a(x, \eta(\xi)) \nabla u(x, \xi) \cdot \nabla v(x) \, dx = \int_{\Omega} z(x) v(x) \, dx, \quad \forall v \in H_0^1(\Omega)$$

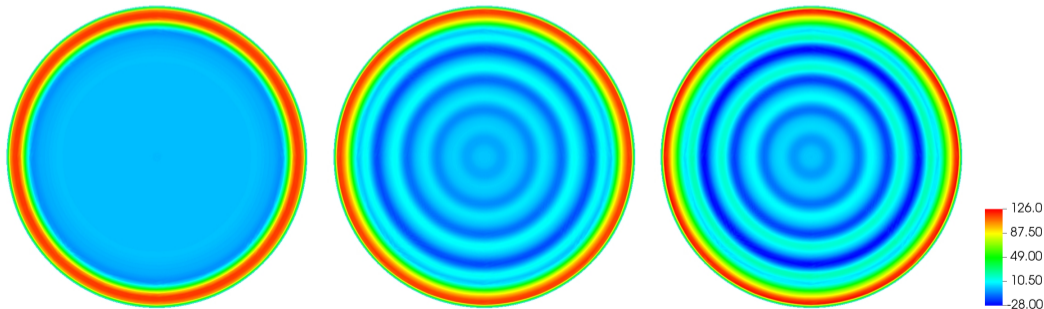
with $\alpha = 10^{-5}$

- Stochastic collocation discretization: Q1 finite elements in space (deal.II),
 $N_P = 8$ GQ nodes for $\mathbb{E}[\cdot]$

Numerical example 1: risk-neutral case

Optimal controls $z(x)$:

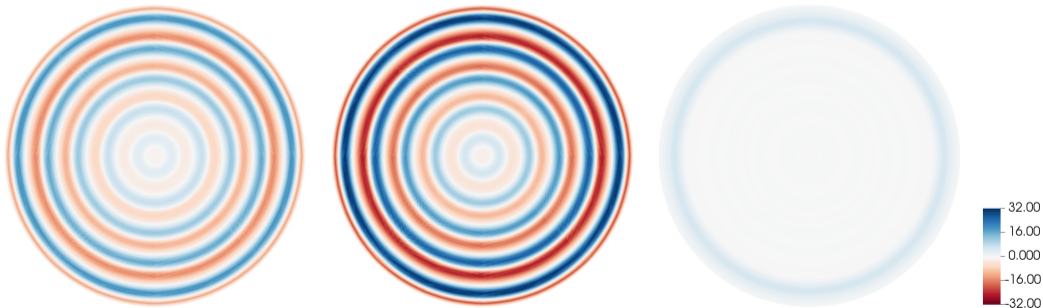
- Left: uncorrupted
- Middle: corrupted, $\delta = 0.3$
- Right: corrupted, $\delta = 0.4$



Numerical example 1: risk-neutral case

Pointwise error in optimal controls:

- Left: corrupted ($\delta = 0.3$)
- Middle: corrupted ($\delta = 0.4$)
- Right: Rockafellian ($\delta = 0.4$)



Numerical example 1: risk-neutral case

Corruption level δ	$E_{\text{rel}}(z_{\text{Rock}}^*)$	$\mathcal{E}_{\text{ratio}}$	$\mathcal{V}_{\text{ratio}}$
0.3	$3.04 \cdot 10^{-1}$	6.31	$9.626 \cdot 10^2$
0.4	$3.12 \cdot 10^{-1}$	8.72	$7.612 \cdot 10^2$

- Here define relative L^2 error and ratio of L^2 errors

$$E_{\text{rel}}(z) := \frac{\|z - z_{\text{true}}^*\|_{L^2(\Omega)}}{\|z_{\text{true}}^*\|_{L^2(\Omega)}}, \quad \mathcal{E}_{\text{ratio}} := \frac{E_{\text{rel}}(z_{\text{corrupted}}^*)}{E_{\text{rel}}(z_{\text{Rock}}^*)}.$$

- Define variance in L^2 norm and the ratio of variances

$$\text{Var}(u) := \mathbb{E}[(\|u(\cdot, \xi)\|_{L^2(\Omega)} - \mathbb{E}[\|u(\cdot, \xi)\|_{L^2(\Omega)}])^2], \quad \mathcal{V}_{\text{ratio}} := \frac{\text{Var}(u_{\text{corrupted}})}{\text{Var}(u_{\text{Rock}})}$$

Numerical example 1: risk-neutral case

Effect of parameter θ for $\delta = 0.4$:

θ	$E_{\text{rel}}(z_{\text{Rock}}^*)$	$\mathcal{E}_{\text{ratio}}$	$\mathcal{V}_{\text{ratio}}$
10^{-2}	$4.27 \cdot 10^{-2}$	7.30	$1.427 \cdot 10^4$
10^{-1}	$3.58 \cdot 10^{-2}$	8.72	$1.624 \cdot 10^2$
1	$7.43 \cdot 10^{-2}$	4.20	$4.075 \cdot 10^0$

- Larger θ : less variance reduction
- Smaller θ : larger variance reduction

Numerical example 2: risk-averse case

Consider 1D BVP version of the previous elliptic problem ($\Omega = (0, 1)$) with CVaR_β and:

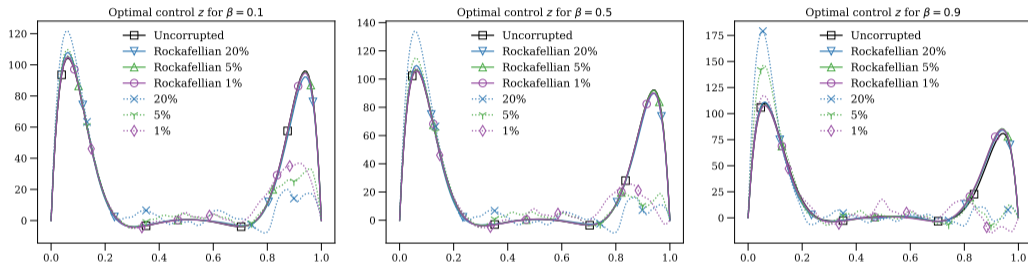
- Diffusion coefficient

$$a(x, \xi) = \exp\left(\sum_{k=1}^d \sqrt{\lambda_k} \sin(x/\sqrt{\lambda_k}) \xi_k\right), \quad \lambda_k = \frac{4}{(2k-1)^2 \pi^2}$$

where $\xi_k \sim \mathcal{N}(0, 1)$ and $d = 50$

- Monte Carlo approximation of $\mathbb{E}[\cdot]$ with $N = 5000$ samples
- Corrupt a fraction of the samples by $\xi \mapsto 5 \cdot \xi$
- CVaR_β smoothing parameter $\delta = 10^{-3}$

Numerical example 2: risk-averse case



- Corrupting effect of outlier data increases as β increases
- Rockafellian controls can minimize outliers' influence

Numerical example 2: risk-averse case

Quantitative results at fixed $\theta = 10^{-1}$, changing β :

Corruption level	$E_{\text{rel}}(z_{\text{Rock}}^*)$	$\mathcal{E}_{\text{ratio}}$	Corrupted deleted	Clean deleted
$\beta = 0.1$				
1%	$7.50 \cdot 10^{-2}$	48.0	14/50=28%	1/4950=0.02%
20%	$3.55 \cdot 10^{-1}$	13.7	352/1000=35.2%	0/4000=0.0%
$\beta = 0.5$				
1%	$1.36 \cdot 10^{-1}$	35.4	21/50=42%	11/4950=0.22%
20%	$2.83 \cdot 10^{-1}$	18.6	508/1000=50.8%	9/4000=0.25%
$\beta = 0.9$				
1%	$4.92 \cdot 10^{-1}$	11.7	33/50=66%	199/4950=4.02%
20%	$4.51 \cdot 10^{-1}$	13.2	693/1000=69.3%	139/4000=3.48%

Numerical example 2: risk-averse case

Quantitative results at fixed $\beta = 0.9$ and 10% corruption, changing θ :

θ	$E_{\text{rel}}(z_{\text{Rock}}^*)$	$\mathcal{E}_{\text{ratio}}$	Corrupted deleted	Clean deleted
1	$1.75 \cdot 10^0$	3.12	93/500=18.6%	0/4500=0.0%
10^{-1}	$4.93 \cdot 10^{-1}$	11.1	359/500=71.8%	179/4500=3.98%
10^{-2}	$1.18 \cdot 10^0$	4.62	451/500=90.2%	1853/4500=41.2%
10^{-3}	$9.70 \cdot 10^{-1}$	5.64	445/500=89.0%	2038/4500=45.3%

- Larger θ : increased confidence in data \rightarrow fewer data points removed
- Smaller θ : lower confidence in data \rightarrow more data points removed (including “clean” points)

Numerical example 3: risk-averse case

Consider 2D stochastic, elliptic advection-diffusion equation on $\Omega = B_1(0)$ with CVaR_β and:

- PDE constraint:

$$\int_{\Omega} \nabla u(x, \xi) \cdot \nabla w(x) dx + \int_{\Omega} v(\xi) \cdot \nabla u(x) w(x) dx = \int_{\Omega} z(x) w(x) dx, \quad \forall w \in H_0^1(\Omega)$$

- Random advection field:

$$v(\xi) = \left\langle \xi \cos(\xi\pi/v_{\max} - \pi/2), \xi \sin(\xi\pi/v_{\max} - \pi/2) \right\rangle$$

where $v_{\max} = 20$

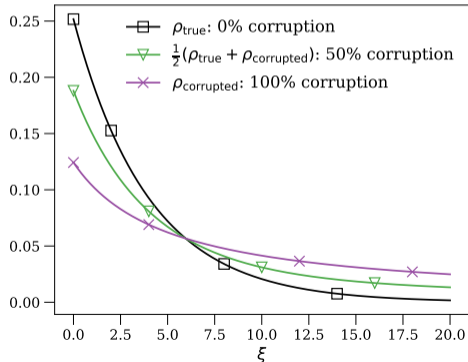
- Stochastic collocation discretization: Q1 finite elements in space (deal.II), $N_P = 15$ GQ nodes for $\mathbb{E}[\cdot]$

Numerical example 3: risk-averse case

Random parameter ξ in advection field

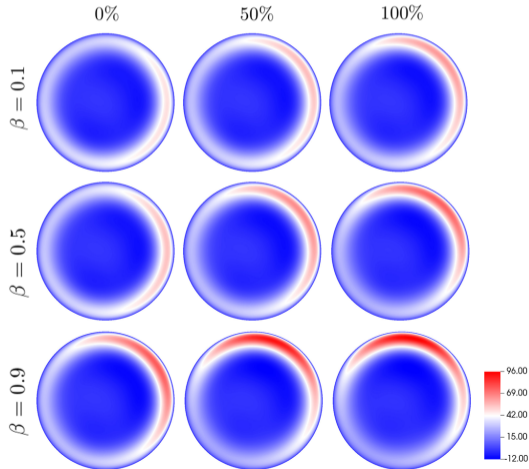
$$v(\xi) = \left\langle \xi \cos \left(\xi \pi / v_{\max} - \pi / 2 \right), \xi \sin \left(\xi \pi / v_{\max} - \pi / 2 \right) \right\rangle$$

sampled from one of three possible density functions:



Numerical example 3: risk-averse case

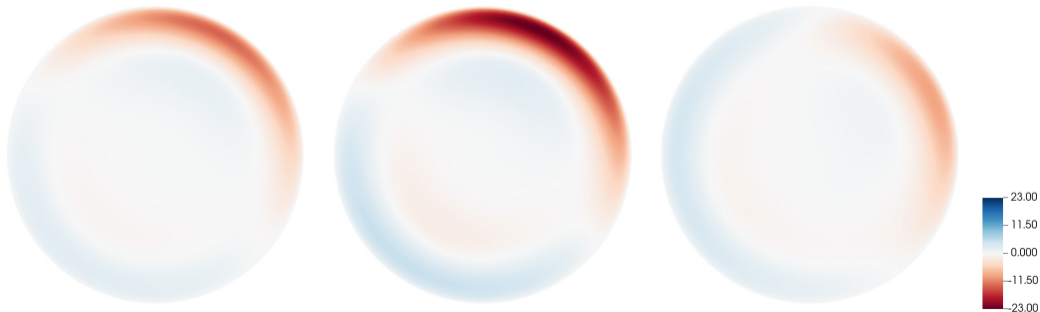
- Optimal controls *without* any Rockafellian relaxation
- Increasing β has similar qualitative effect as increasing corruption



Numerical example 3: risk-averse case

Pointwise error for $\beta = 0.1$ in optimal controls:

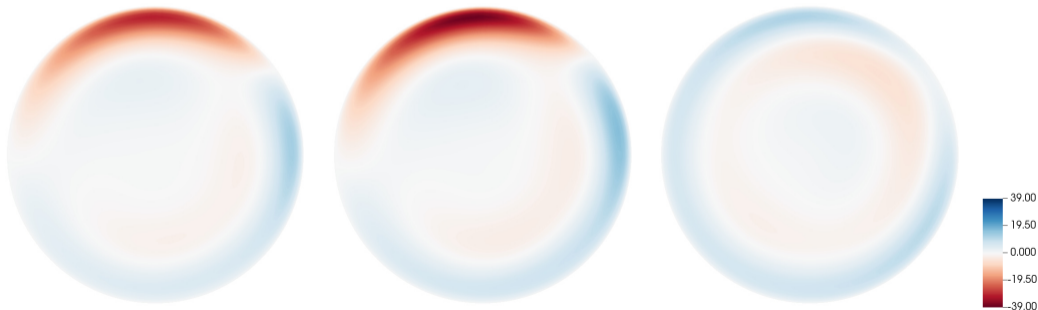
- Left: corrupted (50 %)
- Middle: corrupted (100 %)
- Right: Rockafellian (100 %)



Numerical example 3: risk-averse case

Pointwise error for $\beta = 0.9$ in optimal controls:

- Left: corrupted (50 %)
- Middle: corrupted (100 %)
- Right: Rockafellian (100 %)



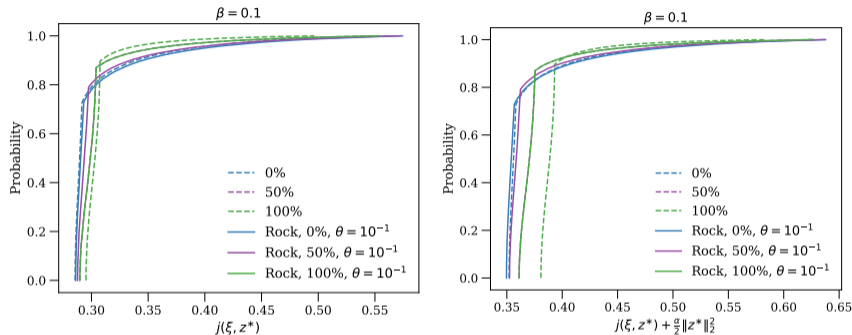
Numerical example 3: risk-averse case

- Define random variable from objective function at the optimal control z^* :

$$j(\xi, z^*) := \frac{1}{2} \|s(\xi, z^*) - u_\star\|_{L^2(\Omega)}^2,$$

- Consider the cumulative distribution function (CDF) of $j(\xi, z^*)$
- Generate CDF numerically from Sobol samples $\{\xi_i\}_{i=1}^N$ from ρ_{true} , where $N = 2^{17} = 131\,072$

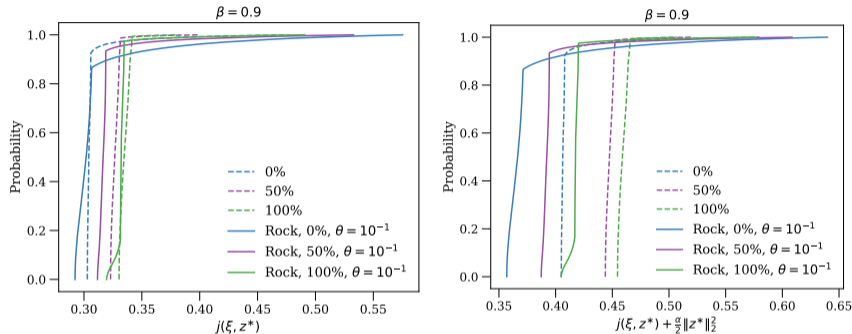
Numerical example 3: risk-averse case



For fixed corruption level, Rockafellian controls are less risk-averse than non-Rockafellian counterparts:

- At lower quantiles $j(\xi, z^*)$ is *smaller* for Rockafellian controls
- At higher quantiles $j(\xi, z^*)$ is *larger* for Rockafellian controls

Numerical example 3: risk-averse case



For fixed corruption level, Rockafellian controls are less risk-averse than non-Rockafellian counterparts:

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Numerical example 3: risk-averse case

Q: What happens when Rockafellian relaxation is used with $\beta = 0.9$ and 0% corruption? Consider quantiles of the total control cost $j(\xi, z^*) + \frac{\alpha}{2} \|z^*\|_{L^2(\Omega)}^2$:

β	Corruption level	θ	Min	Max	0.5	0.75	0.9	0.99
0.1	0%	–	0.3520	0.6287	0.3560	0.3605	0.4059	0.5613
0.9	0%	10^{-1}	0.3569	0.6398	0.3665	0.3703	0.3866	0.5656
0.9	0%	–	0.4050	0.5798	0.4062	0.4076	0.4079	0.4871

- Rockafellian control similar to control for $\beta = 0.1$ with 0%
- **Note:** At $\theta = 1$, we recover the control for $\beta = 0.9$ with 0%

Takeaways

1. Risk-averse optimization problems can be sensitive to corrupted probability distributions.
2. We can enhance robustness of risk-averse optimization to data corruption by minimizing a modified objective function that includes an additional perturbation variable.
3. Problem relaxation with CVaR_β allows insight into the sensitivity of optimal control strategies under changing levels of *both* problem corruption *and* risk-aversion.

1. Motivation: distributional ambiguity
2. Rockafellian relaxation
3. Connections with both “optimistic” and “robust” optimization
4. Numerical examples
5. **Concluding remarks**

Philosophy

- In applications, the level of corruption is unknown... (this is the whole point!)
- Best practice: if possible, solve optimization problem for a handful of β and θ values

Royset & J-B Wets (*An Optimization Primer*) [on Rockafellian relaxation]: “Thus, a problem is viewed as an instance from a family of perturbed problems, which reflects the **practically important fact that optimization technology is more a tool for identifying possibilities than an oracle for producing a definite answer.**”

Ongoing and future work

In the future, Rockafellian relaxation can play a role in:

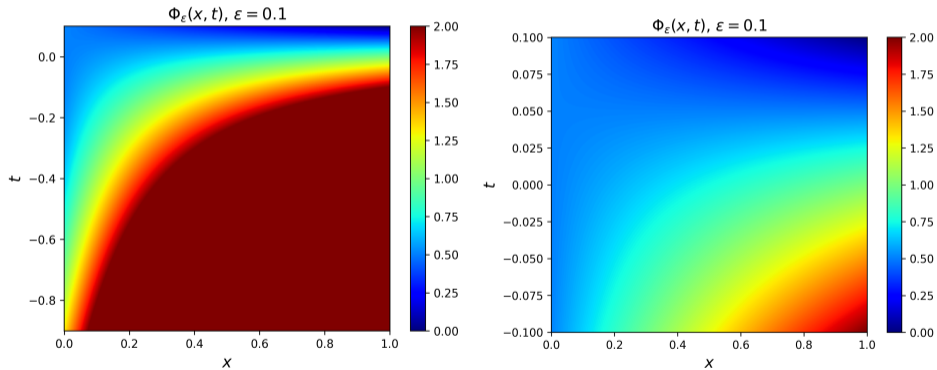
- State constrained optimization problems
- Chance-constrained optimization problems
- Applications: structural weakness identification (Digital Twins), topology optimization, trajectory optimization

Thank you for listening!

Appendix slides

Rockafellian relaxation: toy example

Full (left) and zoomed-in (right) optimization landscapes ($t \equiv t_1$):



Numerical example 3: risk-averse case

	β	Corruption level	$E_{\text{rel}}(z_{\text{Rock}}^*)$	$\mathcal{E}_{\text{ratio}}$
	$\theta = 10^{-1}$			
$E_{\text{rel}}(z) := \frac{\ z - z_{\text{true}}^*\ _{L^2(\Omega)}}{\ z_{\text{true}}^*\ _{L^2(\Omega)}}$	0.1	50%	$4.74 \cdot 10^{-2}$	3.05
	0.9	50%	$2.06 \cdot 10^{-1}$	1.22
	0.1	100%	$1.07 \cdot 10^{-1}$	2.39
	0.9	100%	$1.50 \cdot 10^{-1}$	2.09
	$\theta = 10^{-2}$			
$\mathcal{E}_{\text{ratio}} := \frac{E_{\text{rel}}(z_{\text{corrupted}}^*)}{E_{\text{rel}}(z_{\text{Rock}}^*)}$	0.1	50%	$1.64 \cdot 10^{-1}$	0.883
	0.9	50%	$2.06 \cdot 10^{-1}$	1.22
	0.1	100%	$1.34 \cdot 10^{-1}$	1.92
	0.9	100%	$2.12 \cdot 10^{-1}$	1.49

Not shown: Error for $\theta = 1 \dots$ this is same as $z_{\text{corrupted}}^*$!