Multiple Solutions for Variational Inequalities via Latent Variable Proximal Point and Deflation

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University of Oxford
Mathematical Institute

Oxford Mathematics

Variational inequalities (VI)

$$\min_{u\in K} J(u), \quad K \subsetneq U$$

$$\Downarrow$$

$$u \in K : \ \langle J'(u), v-u\rangle \geq 0$$

Practical Applications of VI

Requirements for an ideal solver

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- *Contact mechanics*
- *Fluid dynamics*
- *Option pricing, etc*

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- *Discretisation-independent*
- *Mesh-independent convergence*
- *Multiple solutions*

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- *Feasible solution* $\surd^*$
- *Discretisation-independent* $\surd$
- *Mesh-independent convergence* $\surd$
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Latent Variable Proximal Point

$$\alpha J'(u) + \mathcal{B}^*(\psi - \psi^{k-1}) = 0 \quad (1a)$$
$$\mathcal{B}u - \nabla R_x^*(\psi) = 0 \quad (1b)$$

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This Work

LVPP for multiple solutions.

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- **Voronoi–Bregman Proximal Point Iteration**: For a suitable finite set of initial guesses $\mathcal{U}^0 := \{u_i^0\}_{i=1}^{N_0} \subset U$, compute

$$\operatorname{argmin}_{u \in K} \left\{ J(u) + \alpha^{-1} \min_{u_i^{k-1} \in \mathcal{U}^{k-1}} \int_{\Omega_d} D_{R_x} \left(\mathcal{B}u | \mathcal{B}u_i^{k-1} \right) \mathrm{dm} \right\}, \quad (2)$$

with the optimality condition (in LVPP form)

- **Deflation**:

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with the optimality condition (in LVPP form)

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$$\mathcal{B}u-\nabla R_x^*(\psi)=0 \quad (3b)$$

- **Deflation**: Apply the deflation operator $\mathcal{M}_{2,1}(u;u_r)=\frac{1}{\|u-u_r\|_U^2}+1$ whenever a new solution (u_r,ψ_r) for (3) is found (by scaling the newton steps).

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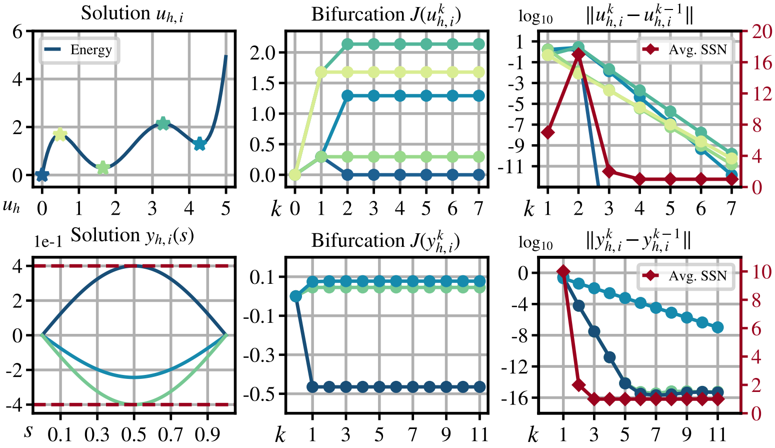

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THANKS FOR YOUR LISTENING !

Oxford Mathematics

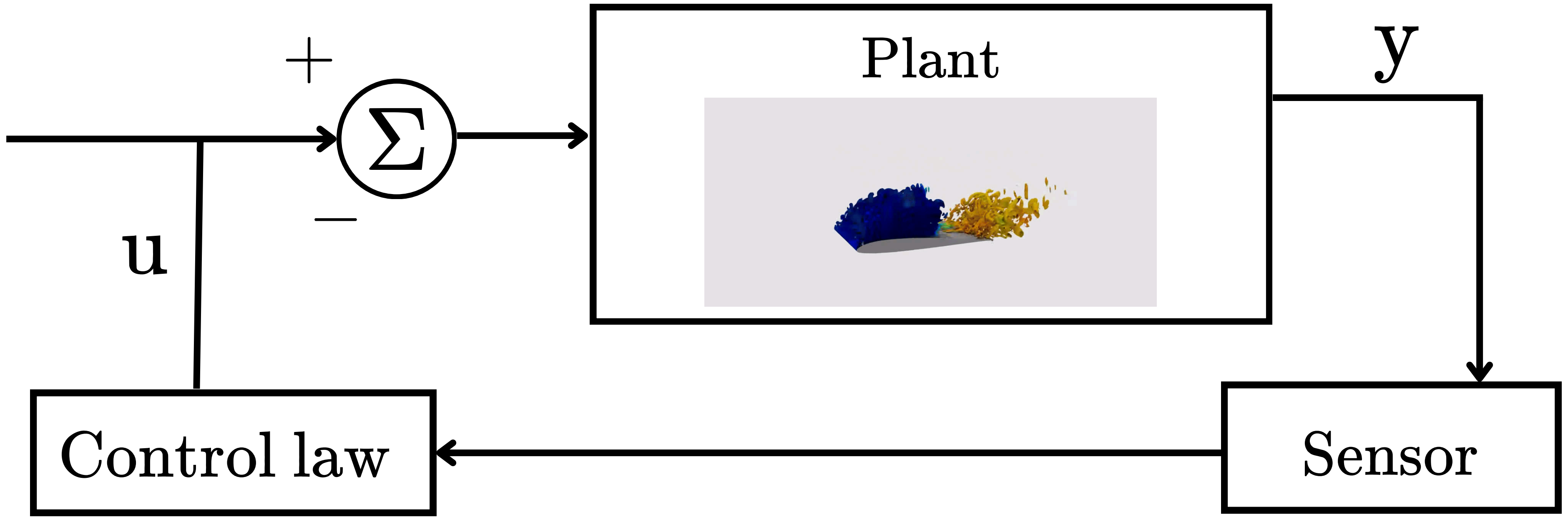
Data-Driven Model Reduction for Optimal Control via Trajectory Optimization and Covariance Balancing.

Presented by :

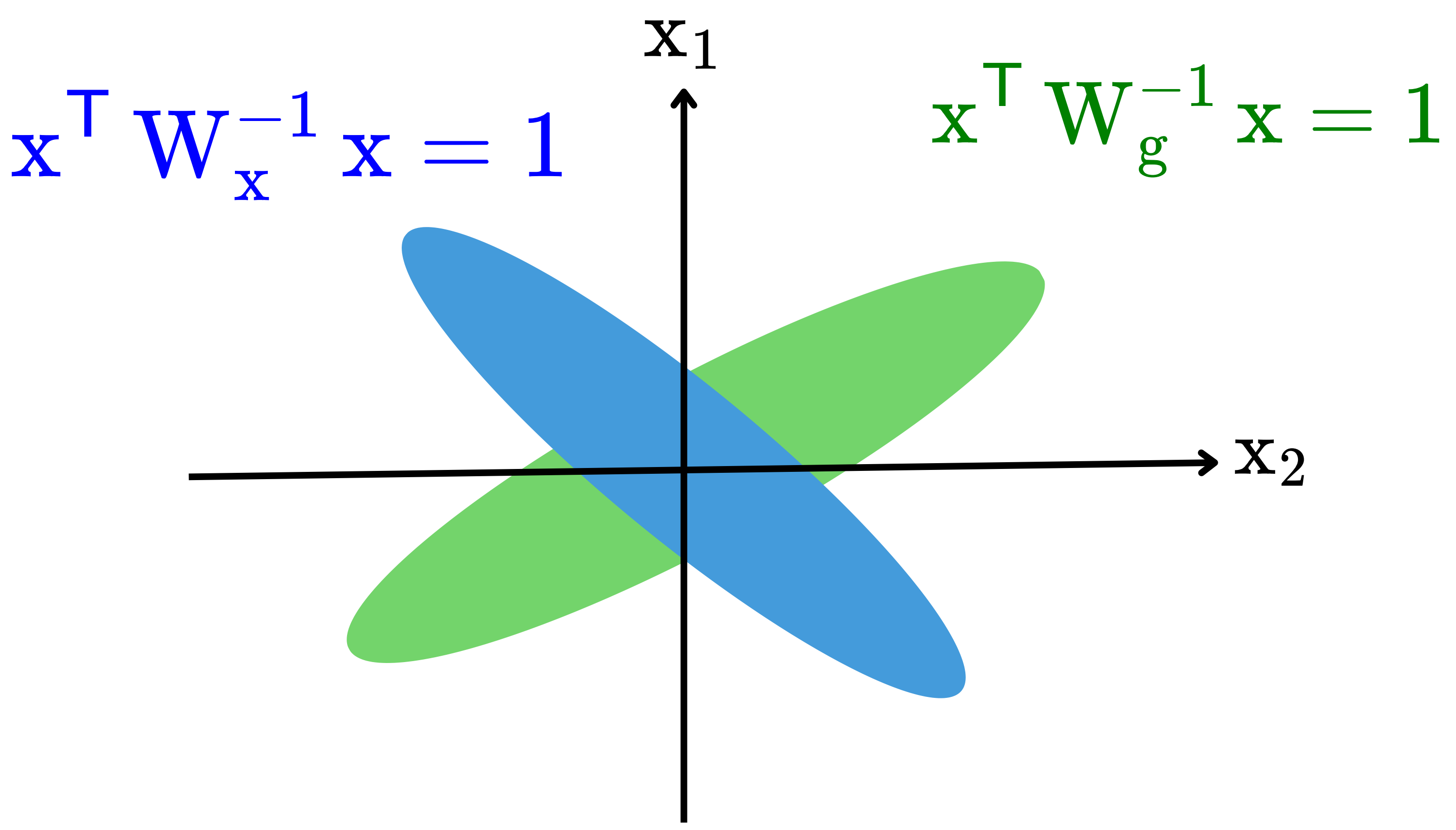
Ameh Emmanuel

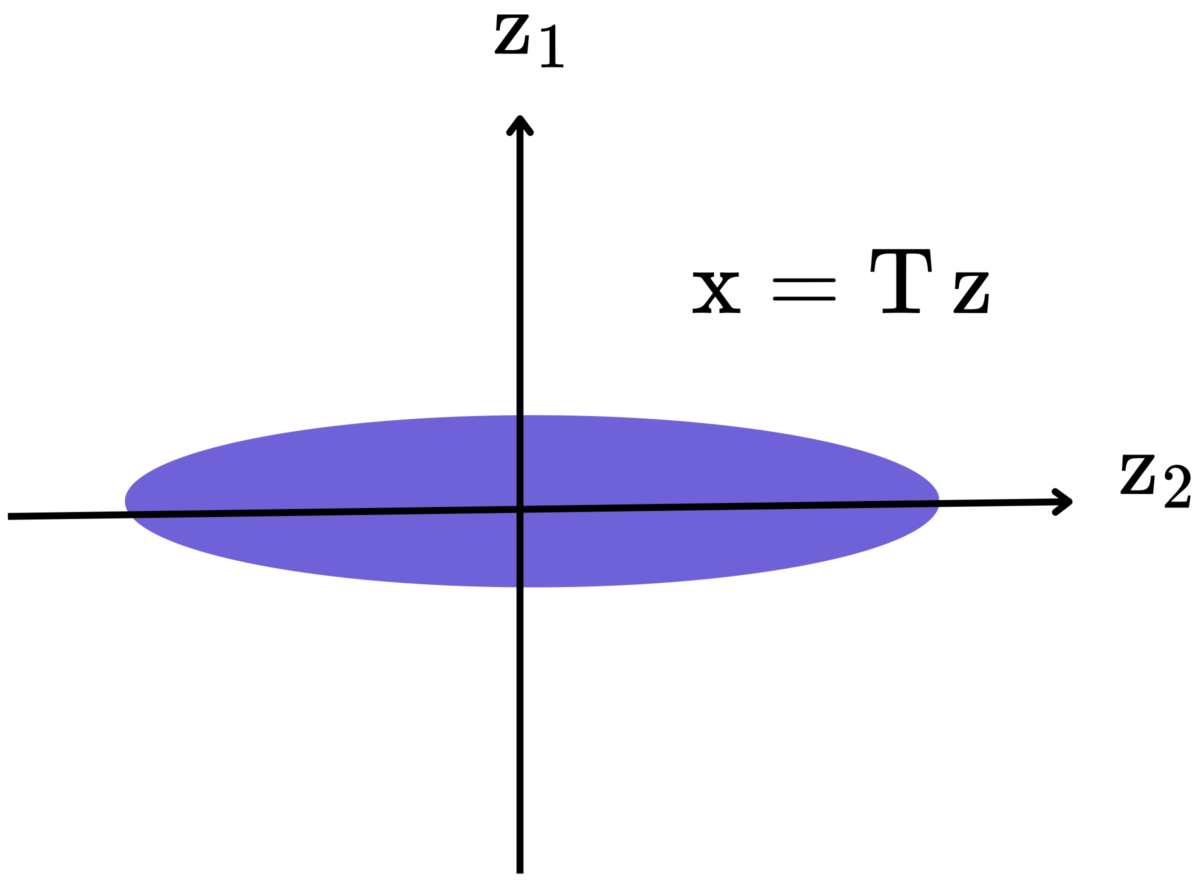
Our goal : ▶ Optimal control for high dimensional fluid flows in highly nonlinear regimes!

Problem : ▶ Solve the Hamilton--Jacobi--Bellman PDE for $V(x) : \mathbb{R}^n \to \mathbb{R}$, the value function.

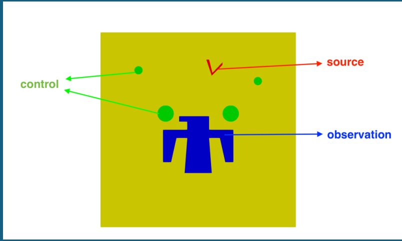


Approach : ▶ Obtain state W_x and gradient data W_g for $V(x)$ along optimal trajectories





Balancing provides a projection P for approximating the value function $\approx V(Px)$ in a subspace!



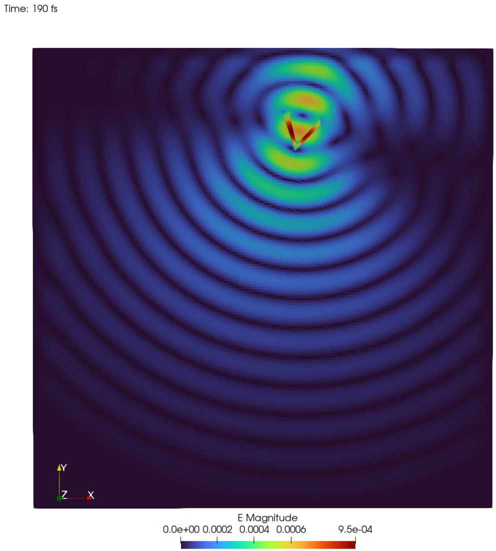
Goal: Apply current density $\mathbf{J}$ in Ω_{ctrl} to hide a source current $\mathbf{J}_{\text{src}}$ from observation region Ω_{obs}

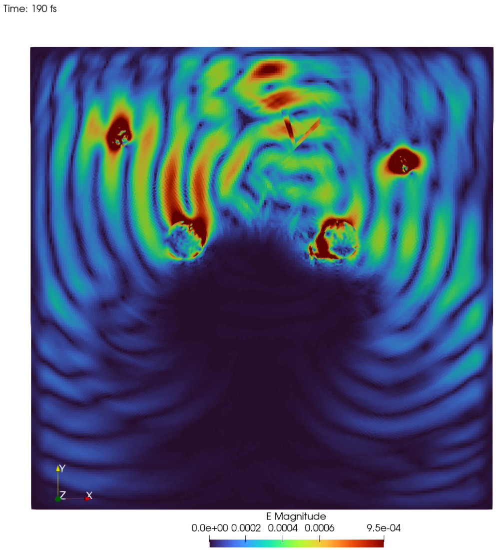
$$\text{Minimize} \quad \frac{1}{2}\int_0^T \int_{\Omega_{\text{obs}}} \left(\epsilon|\mathbf{E}-\mathbf{E}_d|^2 + \mu^{-1}|\mathbf{B}-\mathbf{B}_d|^2\right)\,dx\,dt + \int_0^T\int_{\Omega_{\text{ctrl}}} R(\mathbf{J})\,dx\,dt$$

subject to

$$\begin{aligned}
\epsilon\frac{\partial \mathbf{E}}{\partial t} - \text{curl}\left(\mu^{-1}\mathbf{B}\right) + \sigma\mathbf{E} &= \mathbf{I}_{\text{src}} + \chi_{\text{ctrl}}\mathbf{J} && \text{in } \Omega\times(0,T), \quad \Omega\subset\mathbb{R}^3\\
\frac{\partial \mathbf{B}}{\partial t} + \text{curl}\,\mathbf{E} &= \mathbf{0} && \text{in } \Omega\times(0,T),\\
\mathbf{E}\times\mathbf{n} &= \mathbf{0} && \text{in } \partial\Omega\times(0,T),\\
\mathbf{E}(\cdot,0) &= \mathbf{E}_0 && \text{in } \Omega,\\
\mathbf{B}(\cdot,0) &= \mathbf{B}_0 && \text{in } \Omega.
\end{aligned}$$

Find out more at the poster session!



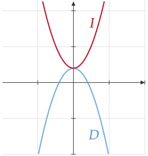


Duality Framework for Flux Constrained Flow

- **Dual problem** $\partial\Omega = \Gamma_N \dot\cup \Gamma_D \dot\cup \Gamma_C$ find $\boldsymbol{u} \in H(\mathrm{div};\Omega)$ such that maximize

$$D(\boldsymbol{v}) := -\frac{1}{2}\|\boldsymbol{K}^{-\frac{1}{2}}(\cdot)\boldsymbol{v}\|^2_{L^2(\Omega)} - \langle p_D, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_D}$$
$$- I^{\Omega}_{\{f\}}(\mathrm{div}\,\boldsymbol{v}) - I^{\Gamma_C}_{+}(\beta - \boldsymbol{v}\cdot\boldsymbol{n}) - I^{\Gamma_C}_{+}(\boldsymbol{v}\cdot\boldsymbol{n} - \alpha) - I^{\Gamma_N}_{\{g\}}(\boldsymbol{v}\cdot\boldsymbol{n}),$$

$$\inf_{x\in X}\Big\{\underbrace{F(x) + G(\Lambda x)}_{I(x)}\Big\} = \sup_{y\in Y^*}\Big\{\underbrace{-F^*(\Lambda^* y) - G^*(-y)}_{D(y)}\Big\}.$$

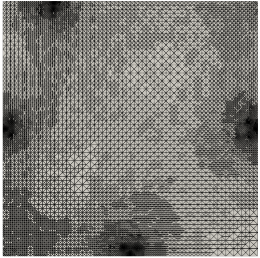


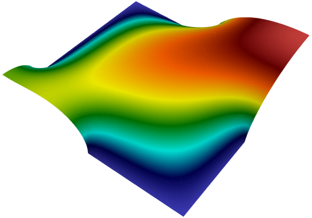
- **Primal Problem** find $p \in H^1(\Omega)$ such that minimize

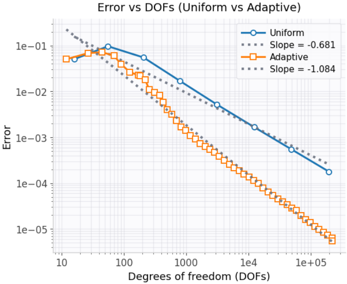
$$I(q) := \frac{1}{2}\|\boldsymbol{K}^{\frac{1}{2}}\nabla q\|^2_{2,\Omega} - (f,q)_\Omega + \langle \tfrac{\alpha+\beta}{2}, q\rangle_{\Gamma_C} + \|(\tfrac{\beta-\alpha}{2})q\|_{1,\Gamma_C} + \langle g, q\rangle_{\Gamma_N} + I^{\Gamma_D}_{\{p_D\}}(q).$$

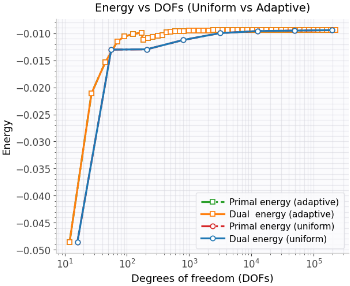
Same at the Discrete Level!

Numerical Example: A Posteriori Analysis

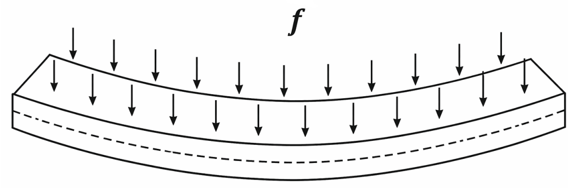








Nonlinear Kirchhoff plate model



- $\Omega \subset \mathbb{R}^2$ be a polygonal open bounded domain with a Lipschitz boundary Γ.
- Deformation of mid-surface $\boldsymbol{y} : \Omega \to \mathbb{R}^3$
- Dirichlet boundary conditions $\boldsymbol{y} = \boldsymbol{y}_D$ and $(\nabla \boldsymbol{y})\boldsymbol{n} = \boldsymbol{G}_D$ on Γ_D
- Quadratic energy:

$$E(\boldsymbol{y}) := \frac{1}{2} \int_\Omega \nabla^2 \boldsymbol{y} : \nabla^2 \boldsymbol{y} - \int_\Omega \mathbf{f} \cdot \boldsymbol{y}$$

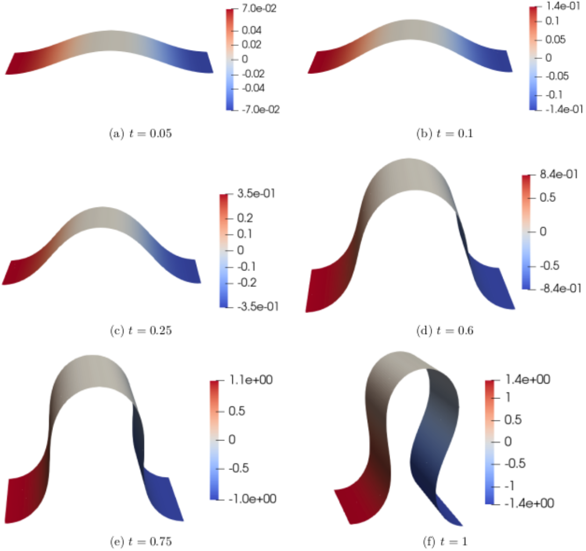
- Non-convex admissible set:

$$\mathbb{A} := \{\boldsymbol{w} \in H^2(\Omega)^3; \nabla \boldsymbol{w}^{\mathsf{T}} \nabla \boldsymbol{w} = I \text{ a.e. in } \Omega, \boldsymbol{w} = \boldsymbol{y}_D \text{ and } (\nabla \boldsymbol{w})\boldsymbol{n} = \boldsymbol{G}_D \text{ on } \Gamma_D\}.$$

- PDE constrained minimization problem:

$$\inf_{\boldsymbol{y} \in \mathbb{A}} E(\boldsymbol{y}).$$

Buckling test

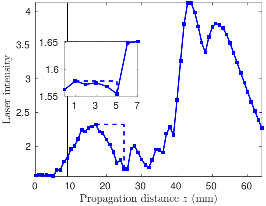


(a) $t = 0.05$ (b) $t = 0.1$ (c) $t = 0.25$ (d) $t = 0.6$ (e) $t = 0.75$ (f) $t = 1$

PPG [FM and Keith, 2026]

Structure-exploiting optimization

Many problems have the form: $\min f(y) = h(S(y))$ where S is a simulation/system, h is known.

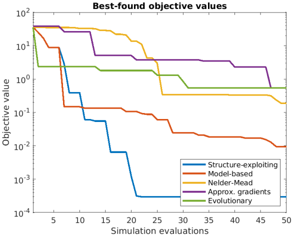


- Intensity of laser-plasma accelerator (LBNL):

$$\underset{y}{\text{minimize}} \max_{i \in \Theta_1(y)} v(z_i; y) - \min_{i \in I} v(z_i; y)$$

- Normalized emittance (SLAC):

$$\underset{y}{\text{minimize}} \sqrt{\langle x(y)^2 \rangle \langle p_x(y)^2 \rangle - \langle x p_x(y) \rangle^2}$$

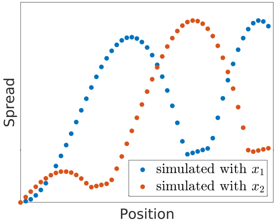


- Plasma beam loading (UCLA):

$$\underset{y}{\text{minimize}} \sum_{i=1}^{m} \left[F_i(y) - \frac{1}{m} \sum_{j=1}^{m} F_j(y) \right]^2 - \alpha \left(\frac{1}{m} \sum_{j=1}^{m} F_j(y) \right)^2$$

- “Spread” example:

$$\underset{y}{\text{minimize}} \max_{i} F_i(y)$$



Jeffrey Larson jmlarson@anl.gov

Multidimensional Structure-Preserving FDTD Methods for Maxwell's Equations in Kerr-Debye-Lorentz Media

Emmanuel E. Oguadimma, Vrushali A. Bokil, Nathan L. Gibson

Department of Mathematics, Oregon State University

Simulation-Based Optimization with Applications
April 14, 2026

Oregon State University

This research was conducted at Oregon State University and is partially funded by NSF-DMS grant # 2012882.

Problem

- Nonlinear optical media can exhibit effects such as self-focusing, harmonic generation, and soliton-like propagation.
- We model these with a 3D nonlinear dispersive Maxwell model, including delayed Kerr response and Lorentz polarization.
- In order to perform material design optimization or uncertainty quantification, we need robust and efficient numerical methods for the forward simulations.
- Additionally, standard schemes can lose key physical structure.

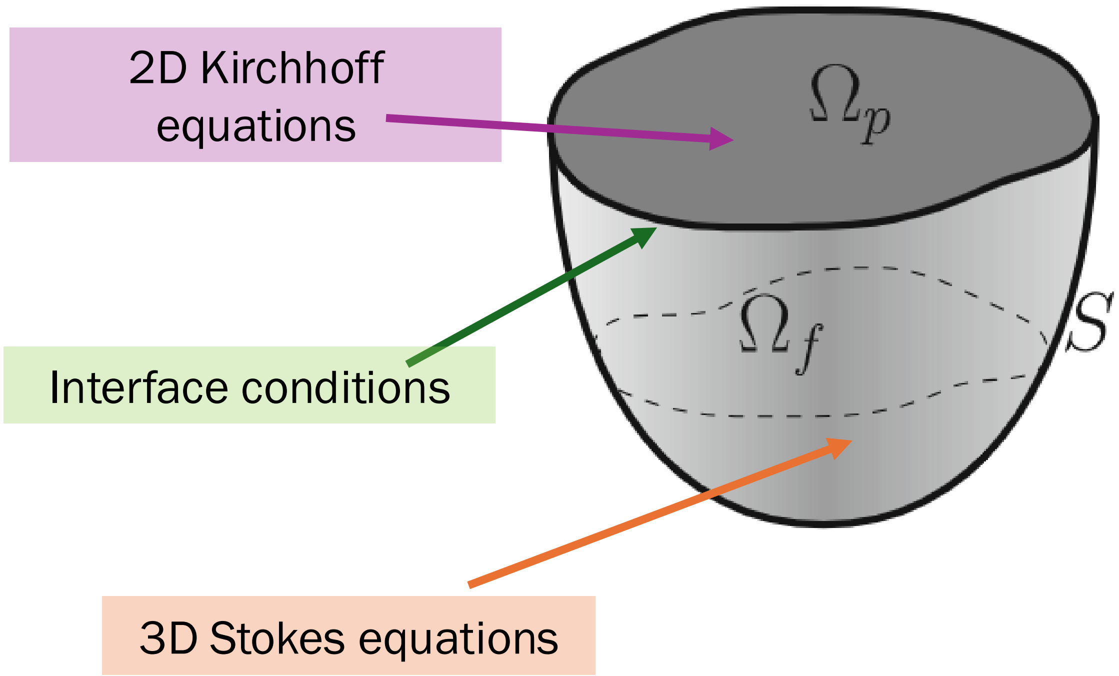
Method

- 3D FDTD discretization with a modified exponential update for the Kerr response.
- Newton–GMRES solve with matrix-free Jacobian–vector products.

Main Results

- Preserves nonnegativity of the Kerr susceptibility.
- Discrete energy decay.
- Discrete divergence preservation.
- Second-order accuracy.
- Captures nonlinear optical phenomena.

Schur complement domain decomposition for 3D-2D FSI



Mixed FEM + Lagrange Multiplier

↓

Saddle-point system

Key idea: Reduce coupled system to an interface problem

Algorithm:

Solve the interface problem (p, g).

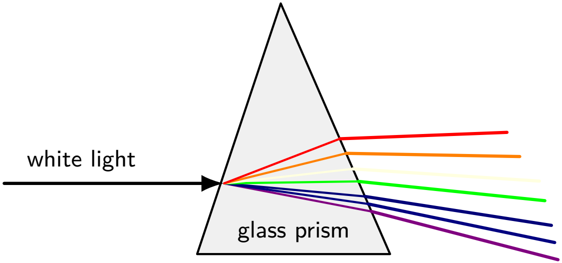
Fluid solve ($\boldsymbol{u}$)

Plate solve (w)

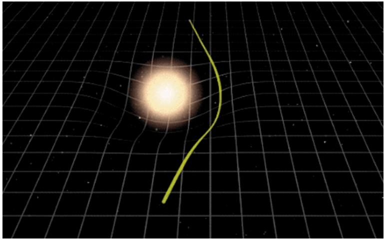
■ No sub-iterations ■ Strongly coupled

■ Parallelizable ■ Modular

Polarization-Induced Beam Bending



Refraction (prism)

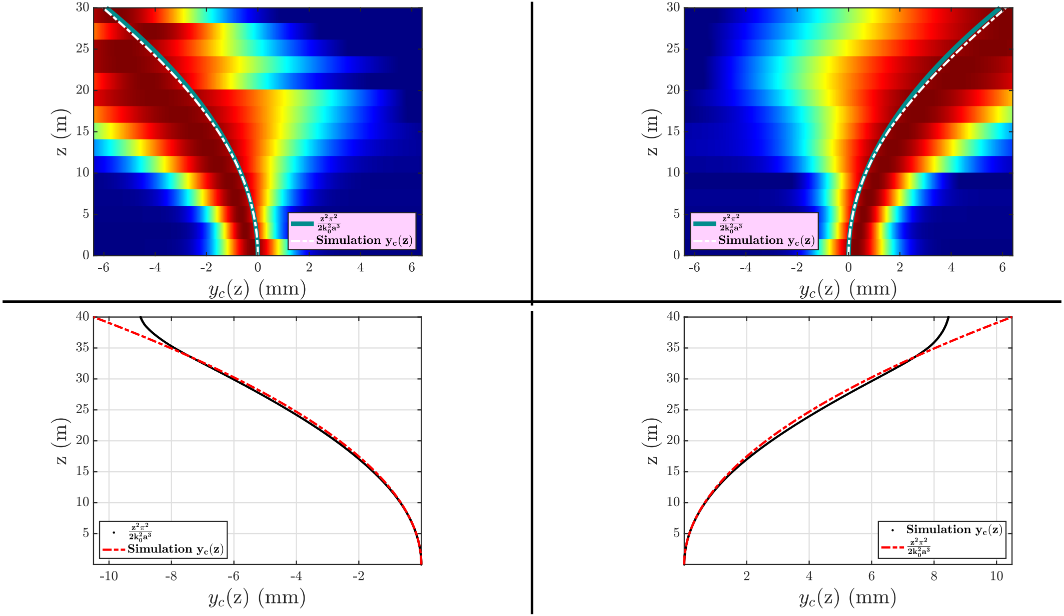


Gravity (NASA)

Two classical ways to bend light.

- **New Approach – Vector beam bending**: polarization structure + medium response.
- **Reduced model:** a coupled **hyperbolic PDE system** (transport + Hamilton-Jacobi).
- **Goal:** long-distance stable simulation + experimental validation.

Numerical vs. analytical vs. experimental background: $Z = 30$m, $Z = 40$m



Inf-dim Spherical-radial Decomposition for Probabilistic Functions

Kewei Wang, Georg Stadler
Courant Institute, NYU

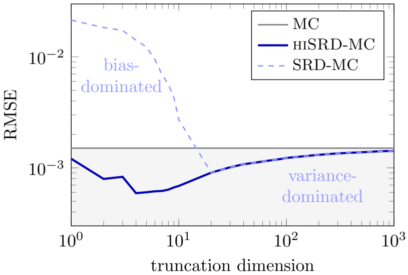
Stochastic optimization problem:

$$\min_{u \in U_{\mathrm{ad}}} \quad \mathcal{J}(u) \quad \text{s.t.} \quad \varphi(u) \geq p$$

Goal: Estimate probability function in inf-dim:

$$\varphi(u) := \mathbb{P}(g(u, \xi) \leq 0)$$

Such constraints arise in stochastic PDE-constrained optimal controls and Gaussian process regressions.



Want:

- Differentiability
- No truncation bias
- Low variance

Hybrid Infinite-Dimensional SRD (hiSRD)

Hybrid SRD for centered elliptical ξ:

$$\xi = \xi_K + \xi_R = \tau L_K \boldsymbol{\nu}_K + \xi_R,$$

where $\tau \sim \chi_K$, $\boldsymbol{\nu}_K \sim \mathcal{U}(\mathbb{S}^{K-1})$.

Evaluation and approximation of $\varphi(u)$:

$$\varphi(u) = \int_{\mathscr{H}_R} \int_{\boldsymbol{v} \in \mathbb{S}^{K-1}} e(u, \boldsymbol{v}, z) d\mu_{\mathcal{U}}(\boldsymbol{v}) d\mu_{\xi_R}(z)$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} e(u, \boldsymbol{v}_i, z_i).$$

$$e(u, \boldsymbol{v}, z) := \mu_{\chi_K}\left(\{r \geq 0 : g(u, z + r L_K \boldsymbol{v}) \leq 0\}\right)$$

$\{\boldsymbol{v}_i\}_{i=1}^N$ and $\{z_i\}_{i=1}^N$ are independently sampled from $\mathcal{U}(\mathbb{S}^{K-1})$ and ξ_R.