Space-time convergence of a physics-compatible discrete scheme for optimal control of Maxwell's equations

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LDRD
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Collaborators

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Thanks to: Eric Cyr (Sandia National Labs)

Soon to appear on arXiv:
Structure-preserving optimal control of Maxwell's equations with applications to source cloaking

Foundation:
Antil (2026), *Well-posedness and approximation of weak solutions to time-dependent Maxwell's equations with L^2 data*, `https://arxiv.org/abs/2510.20752`

Motivation

- Sandia LDRD project ACEM: Active Currents for Electromagnetic Interference Resilience.
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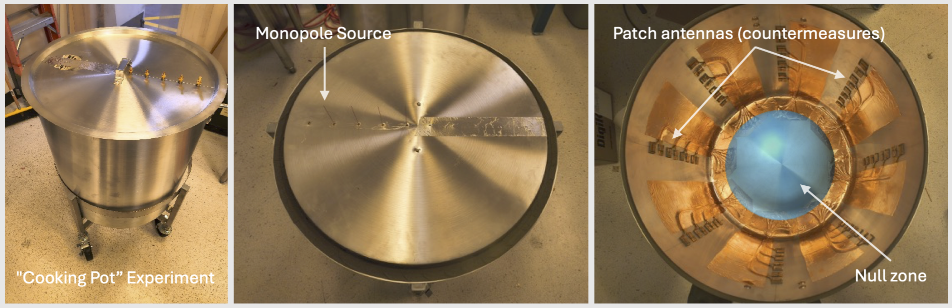
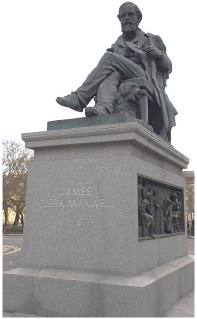
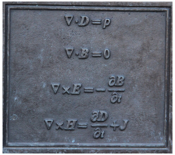


Where to start? Maxwell's equations.



https://commons.wikimedia.org/wiki/File:Statue_of_James_Clerk_Maxwell,_Edinburgh_-_geograph.org.uk_-_2174925.jpg



https://commons.wikimedia.org/wiki/File:James_clerk_maxwell_statue_rear_equations.jpg

- Electric field E, electric displacement D, magnetic flux density B, magnetic field H, free charge density ρ, and source current J:

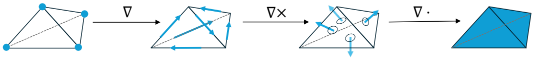
$\operatorname{div} D = \rho$	Gauss's law (electricity)
$\operatorname{div} B = 0$	Gauss's law for magnetism
$\operatorname{curl} E = -\partial B/\partial t$	Faraday's law of induction
$\operatorname{curl} H = J + \partial D/\partial t$	Ampère's law (with displacement)

  Constitutive relations: $D = \varepsilon E$ and $B = \mu H$, with ε the electric permittivity and μ the magnetic permeability (tensors).
- Gauss's law (electricity): Free electric charges make electric fields. In an equivalent integral form, the total electric flux through a closed surface is proportional to the enclosed electric charge.
- Gauss's law for magnetism: In an equivalent integral form, the net magnetic charge enclosed by a surface is zero. I.e., there are no magnetic monopoles.
- Numerical solution methods must identically preserve these laws.

Notes on compatible or structure-preserving methods

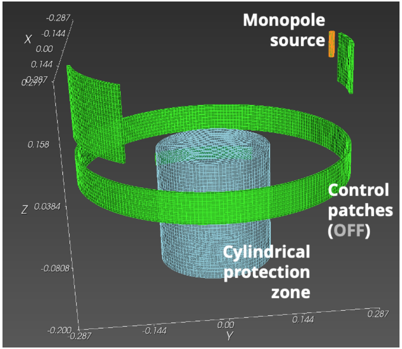
- Collocated methods, with all discrete values at one node or point, lead to numerical artifacts that result in spurious oscillations and drift in quantities like field divergence.
- Yee scheme ("FDTD"):
  - Yee (1966), *Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media.*
  - Staggered grid for electric and magnetic fields in space and time.
  - Preserves divergences, symplectic (energy/phase preserving) time integrator.
- Nédélec edge elements:
  - Nédélec (1980), *Mixed finite elements in* $\mathbb{R}^3$.
  - Discrete values are placed along the edges, not at nodes.
  - Enforces tangential continuity, in the $H(\mathrm{curl})$ space, for electric fields.
- Bossavit and connection to Whitney forms:
  - Bossavit (1988), *Whitney forms: A class of finite elements for three-dimensional computations in electromagnetism.*
  - Connection to the Whitney complex, which discretizes the exact de Rham sequence:

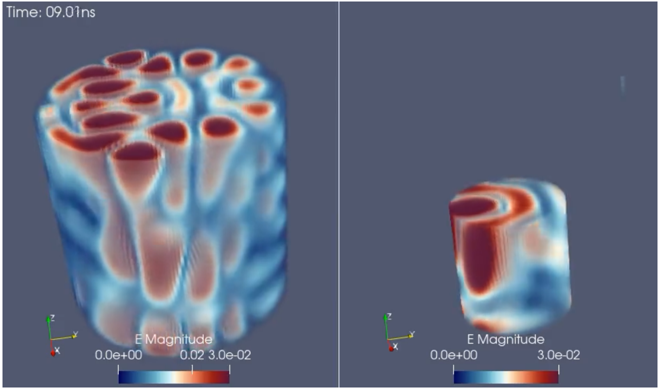
$$H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0$$



Structure-preserving simulation of the “cooking pot”

- Source: Sinusoid in time, windowed by a Gaussian (ramps up and down gently).
- Center frequency is 2 GHz, with a 1 GHz bandwidth in the full-width at half-maximum sense.
- Controls are off, i.e., simple wave propagation in a cavity with reflecting walls.
- The “cooking pot” cylinder is approximately 55 cm in diameter.





Optimal control formulation

We seek a structure-preserving solution framework for the optimization problem

$$\min_z \left\{ J(E,B,z) := \frac{1}{2}\int_0^T \left(\|E - E_d\|^2_{L^2(\Omega_{\mathrm{obs}})^3} + \|B - B_d\|^2_{L^2(\Omega_{\mathrm{obs}})^3} \right) dt + \frac{1}{2}\|z\|^2_Z \right\}$$

subject to Maxwell's equations on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ for a fixed $T > 0$,

$$\begin{cases} \varepsilon \partial_t E - \operatorname{curl}\left(\mu^{-1} B\right) + \sigma E = \chi_{\Omega_{\mathrm{src}}} I_{\mathrm{src}} + \chi_{\Omega_{\mathrm{ctrl}}} \operatorname{curl} z & \text{in } \Omega \times (0,T), \\ \partial_t B + \operatorname{curl} E = 0 & \text{in } \Omega \times (0,T), \\ E \times \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ E(0) = E_0, \quad B(0) = B_0 & \text{in } \Omega, \end{cases} \tag{1}$$

including the Gauss law for electricity, $\partial_t \operatorname{div}(\varepsilon E) + \operatorname{div}(\sigma E - \chi_{\Omega_{\mathrm{src}}} I_{\mathrm{src}}) = 0$, and the Gauss law for magnetism, $\operatorname{div} B(\cdot, t) = 0$ assuming $\operatorname{div} B_0 = 0$.

Note: To ensure conservation of the control charge density, we consider the control current density $\operatorname{curl} z$ instead of z. Recall that $\operatorname{div}\operatorname{curl} z = 0$.

Functional setting with low regularity assumptions, Antil (2026)

- Uniformly elliptic $\varepsilon, \mu \in L^\infty(\Omega)^{3\times 3}$ and nonnegative $\sigma \in L^\infty(\Omega)^{3\times 3}$.
- For $I_{\text{src}} \in L^2(0,T;L^2(\Omega_{\text{src}})^3)$ and $(E_0, B_0) \in L^2(0,T;L^2(\Omega)^3) \times L^2(0,T;L^2(\Omega)^3)$, the weak solution class is

$$E \in H^1(0,T;H_0(\text{curl};\Omega)^*) \cap C([0,T];L^2(\Omega)^3), \quad B \in H^1(0,T;H(\text{curl};\Omega)^*) \cap C([0,T];L^2(\Omega)^3),$$

  with the perfect electrical conductor (PEC) boundary condition $E \times \nu = 0$ on $\partial\Omega$.
- To discretize E and B in space, we use de Rham-compatible Nédélec and Raviart-Thomas (RT) spaces $\mathcal{N}_h^0 \subset H_0(\text{curl};\Omega)$ and $\mathcal{RT}_h \subset H(\text{div};\Omega)$, respectively, with $\text{curl}\,\mathcal{N}_h^0 \subset \mathcal{RT}_h$, which help us enforce Gauss laws and energy identities.
- Optimization problem: We use the spaces

$$Z := L^2(0,T;H(\text{curl};\Omega_{\text{ctrl}})), \qquad Y := L^2(0,T;L^2(\Omega_{\text{obs}})^3) \times L^2(0,T;L^2(\Omega_{\text{obs}})^3),$$

  for the control z and the target data E_d and B_d, respectively.
- For $\alpha_1 > 0$ and $\alpha_2 > 0$, the control norm is given by

$$\|z\|_Z^2 := \int_0^T \left(\alpha_1 \|z(t)\|^2_{L^2(\Omega_{\text{ctrl}})^3} + \alpha_2 \|\,\text{curl}\, z(t)\|^2_{L^2(\Omega_{\text{ctrl}})^3}\right) dt.$$

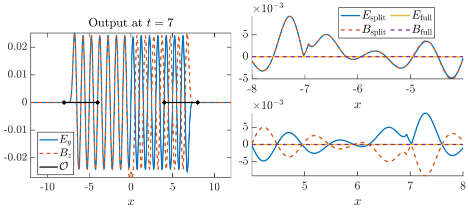
Contributions and talk outline

1. Structure-preserving *fully discrete* forward scheme based on Crank–Nicolson (CN) time stepping, coupled with the Nédélec/RT finite element discretization in space. We prove
   - per-step energy balance;
   - stability for nonnegative σ;
   - preservation of discrete Gauss laws; and
   - convergence of fully discrete solutions to the unique weak solution with rough problem data.
2. For every control $z \in Z$, the Maxwell system admits a unique weak solution (E, B). Optimal controls exist for convex quadratic objective functionals and closed convex admissible sets.
3. Gradient derivation via the adjoint Maxwell system in dual energy spaces:
   - the reduced objective functional is Gâteaux differentiable in z;
   - necessary and, with convexity assumptions, sufficient optimality conditions;
   - a structure-preserving discrete adjoint scheme.
4. Consistent discrete gradient of the reduced objective functional, enabling a structure-preserving discrete optimization scheme.
5. Convergence of discrete optimal controls as mesh size and time step tend to zero.
6. Numerical solution of the source-cloaking model problem (with $E_d = 0$ and $B_d = 0$).

Prior work

The only work that treats optimal control of the *fully time-dependent* Maxwell system is Bommer, Yousept (2016). Differences:

- our tracking objective functional is formulated over the full time horizon $(0, T)$, to enforce cloaking for all times, while B/Y consider a terminal-time target;
- we do not use a space-time separable control $z(x,t) = u(x)a(t)$—a separable control cannot effectively cloak typical sources $I_{\mathrm{src}}(x,t)$;
- we use the curl z control for charge conservation, without auxiliary smoothness assumptions, while B/Y use an explicit divergence-free constraint on the control and additional time regularity; and
- we provide a convergence analysis for the widely accepted CN-Nédélec/RT discretization.



Electric and magnetic field components, propagating transversely along the x-axis, consistent with a 1D realization of the source cloaking problem with $E_d = 0$ and $B_d = 0$. Left: Uncontrolled wave propagation, with solid horizontal lines indicating observation regions $\mathcal{O} = \Omega_{\mathrm{obs}}$. Right: Controlled dynamics in the observation regions. Optimal control in space-time factored form generates fields labeled by 'split'. Our control generates fields labeled by 'full'.

Notation and background

- Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $T > 0$ be fixed. We first study the forward Maxwell problem with spatially varying material tensors,

$$\varepsilon(x)\,\partial_t E - \operatorname{curl}\left(\mu^{-1}(x)B\right) + \sigma(x)\,E = f(x,t) \qquad \text{in } \Omega \times (0,T), \tag{2a}$$

$$\partial_t B + \operatorname{curl} E = 0 \qquad \text{in } \Omega \times (0,T). \tag{2b}$$

The system is equipped with initial data and PEC boundary conditions,

$$E(0,x) = E_0(x), \qquad B(0,x) = B_0(x) \quad \text{in } \Omega, \qquad E \times \nu = 0 \quad \text{on } \partial\Omega \times (0,T).$$

- Recall the standard Sobolev spaces

$$\begin{aligned}
H(\operatorname{div};\Omega) &:= \{v \in L^2(\Omega)^3 : \ \operatorname{div} v \in L^2(\Omega)\},\\
H_0(\operatorname{div};\Omega) &:= \{v \in H(\operatorname{div};\Omega) : \ \gamma_\nu(v) = v \cdot \nu = 0 \text{ on } \partial\Omega\},\\
H(\operatorname{div}^0;\Omega) &:= \{v \in H(\operatorname{div};\Omega) : \ \operatorname{div} v = 0 \text{ a.e. in } \Omega\},\\
H(\operatorname{curl};\Omega) &:= \{v \in L^2(\Omega)^3 : \ \operatorname{curl} v \in L^2(\Omega)^3\},\\
H_0(\operatorname{curl};\Omega) &:= \{v \in H(\operatorname{curl};\Omega) : \ \gamma_\tau(v) = v \times \nu = 0 \text{ on } \partial\Omega\},
\end{aligned}$$

where γ_ν and γ_τ are the normal and tangential trace operators.

Notation and background

Definition 1 (Weak solution)

We call (E, B) *a weak solution of the forward problem* (2) *if*

$$E \in H^1(0,T;H_0(\mathrm{curl};\Omega)^*) \cap C([0,T];L^2(\Omega)^3), \qquad B \in H^1(0,T;H(\mathrm{curl};\Omega)^*) \cap C([0,T];L^2(\Omega)^3),$$

and, for t a.e. in $(0, T)$,

$$\langle \varepsilon\, \partial_t E,\, \psi\rangle_{H_0(\mathrm{curl};\Omega)} + (\sigma E, \psi) - (\mu^{-1} B, \mathrm{curl}\,\psi) = (f, \psi), \qquad \forall\, \psi \in H_0(\mathrm{curl};\Omega), \tag{3a}$$

$$\langle \partial_t B,\, \phi\rangle_{H(\mathrm{curl};\Omega)} + (E, \mathrm{curl}\,\phi) = 0, \qquad \forall\, \phi \in H(\mathrm{curl};\Omega), \tag{3b}$$

with initial conditions $E(0) = E_0$, $B(0) = B_0$ *in* $L^2(\Omega)^3$. *If* $\mathrm{div}\, B_0 = 0$ *in* $L^2(\Omega)$, $B \in L^\infty(0,T;H(\mathrm{div}^0;\Omega))$.

Theorem 1 (Existence, uniqueness, and stability, from Antil (2026))

There exists a unique weak solution (E, B) *of the forward problem* (2) *as in Definition 1. The solution satisfies*

$$\begin{aligned} &\|\partial_t E\|_{L^2(0,T;H_0(\mathrm{curl};\Omega)^*)} + \|\partial_t B\|_{L^2(0,T;H(\mathrm{curl};\Omega)^*)} + \|E\|_{C([0,T];L^2(\Omega)^3)} + \|B\|_{C([0,T];L^2(\Omega)^3)} \\ &\qquad \le C\Big(\|f\|_{L^2(0,T;L^2(\Omega)^3)} + \|E_0\|_{L^2(\Omega)^3} + \|B_0\|_{L^2(\Omega)^3}\Big), \end{aligned} \tag{4}$$

where $C > 0$ *depends only on* T *and the bounds/ellipticity of* ε, μ, σ.

Spatial discretization

Let $\{\mathcal{T}_h\}_h$ denote a family of shape-regular meshes of Ω, and fix a polynomial degree $k \geq 0$.

- Nédélec edge space, conforming to $H(\text{curl})$.

$$\mathcal{N}_h := \left\{ v_h \in H(\text{curl};\Omega) \;:\; v_h|_K \in \mathcal{N}_k(K) \text{ for all } K \in \mathcal{T}_h \right\},$$

  where $\mathcal{N}_k(K)$ refers to the local Nédélec space of the first kind with polynomial degree k. Moreover, we define

$$\mathcal{N}_h^0 := \mathcal{N}_h \cap H_0(\text{curl};\Omega).$$

- Raviart–Thomas face space, conforming to $H(\text{div})$.

$$\mathcal{RT}_h := \left\{ w_h \in H(\text{div};\Omega) \;:\; w_h|_K \in \mathcal{RT}_k(K) \text{ for all } K \in \mathcal{T}_h \right\}.$$

  $\mathcal{RT}_k(K)$ refers to the local Raviart–Thomas space with polynomial degree k.

- Elementwise polynomial scalars.

$$\mathcal{Q}_h := \left\{ q_h \in L^2(\Omega) \;:\; q_h|_K \in \mathbb{P}_k(K) \text{ for all } K \in \mathcal{T}_h \right\}.$$

Recall the discrete de Rham sequence: $\mathcal{P}_h \xrightarrow{\nabla} \mathcal{N}_h^0 \xrightarrow{\text{curl}} \mathcal{RT}_h \xrightarrow{\text{div}} \mathcal{Q}_h \to 0.$

Time discretization

Let N_t denote the number of time steps, $\Delta t := T/N_t$, and $t^n := n\,\Delta t$ for $0 \le n \le N_t$. Denote the time grid by $\mathcal{T}_t := \{t^n\}_{n=0}^{N_t}$. Set $g^n := g(t^n)$ and $g^{\Delta t} := \{g^n\}_{n=0}^{N_t}$. For $n = 0, \ldots, N_t - 1$ we use the backward difference and the midpoint average

$$\delta_t g^{n+1} := \frac{g^{n+1} - g^n}{\Delta t}, \qquad g^{n+\frac{1}{2}} := \frac{g^{n+1} + g^n}{2}.$$

Define the Crank–Nicolson midpoint value on $I_n = (t^n, t^{n+1})$ by the cell average

$$f^{\,n+\frac{1}{2}} := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(s)\, ds, \qquad n = 0, \ldots, N_t - 1.$$

Definition 2 (Fully discrete CN step)

Given $(E_h^n, B_h^n) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$*, find* $(E_h^{n+1}, B_h^{n+1}) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$ *such that for all* $(\psi_h, \phi_h) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$*,*

$$(\varepsilon\, \delta_t E_h^{n+1}, \psi_h) + (\sigma\, E_h^{n+\frac{1}{2}}, \psi_h) - (\mu^{-1} B_h^{n+\frac{1}{2}}, \operatorname{curl}\, \psi_h) = (f^{\,n+\frac{1}{2}}, \psi_h), \tag{5a}$$

$$(\delta_t B_h^{n+1}, \phi_h) + (\operatorname{curl}\, E_h^{n+\frac{1}{2}}, \phi_h) = 0, \tag{5b}$$

for $n = 0, \ldots, N_t - 1$.

Discrete and continuous weak forms differ structurally

Contrast the CN step, with $(E_h^n, B_h^n) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$ and $(\psi_h, \phi_h) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$,

$$(\varepsilon\, \delta_t E_h^{n+1}, \psi_h) + (\sigma\, E_h^{n+\frac{1}{2}}, \psi_h) - (\mu^{-1} B_h^{n+\frac{1}{2}}, \operatorname{curl}\, \psi_h) = (f^{\,n+\frac{1}{2}}, \psi_h),$$
$$(\delta_t B_h^{n+1}, \phi_h) + (\operatorname{curl}\, E_h^{n+\frac{1}{2}}, \phi_h) = 0,$$

with the continuous weak form it converges to (as we will prove),

$$\begin{aligned} \langle \varepsilon\, \partial_t E,\, \psi\rangle_{H_0(\mathrm{curl};\Omega)} + (\sigma E, \psi) - (\mu^{-1} B, \operatorname{curl} \psi) &= (f, \psi), && \forall\, \psi \in H_0(\mathrm{curl}; \Omega), \\ \langle \partial_t B,\, \phi\rangle_{H(\mathrm{curl};\Omega)} + (E, \operatorname{curl} \phi) &= 0, && \forall\, \phi \in H(\mathrm{curl}; \Omega). \end{aligned}$$

Time-discrete quantities used for the analysis

We use a blend of piecewise constant and piecewise affine/linear reconstructions or lifts.

- For midpoints define the (right–continuous) midpoint lift

$$(g^{1/2})(t) := g^{n+\frac{1}{2}} \quad \text{for } t \in (t^n, t^{n+1}], \;\; n = 0, \ldots, N_t - 1.$$

- Lift the discrete time derivative as a piecewise constant function

$$(\partial_t^{\Delta t} g)(t) := \delta_t g^{n+1} \quad \text{for } t \in (t^n, t^{n+1}], \;\; n = 0, \ldots, N_t - 1.$$

- We also use the continuous, piecewise affine reconstruction $g^{\mathrm{pl}} : (0, T] \to X$ defined by

$$g^{\mathrm{pl}}(t) := g^n + (t - t^n)\, \delta_t g^{n+1} \qquad \text{for } t \in (t^n, t^{n+1}], \;\; n = 0, \ldots, N_t - 1.$$

  Then $g^{\mathrm{pl}} \in C([0, T]; X)$, $g^{\mathrm{pl}}(t^n) = g^n$, and

$$\partial_t g^{\mathrm{pl}}(t) = \delta_t g^{n+1} \quad \text{for } t \in (t^n, t^{n+1}], \qquad g^{\mathrm{pl}}(t^{n+\frac{1}{2}}) = g^{n+\frac{1}{2}}.$$

Rewrite the discrete CN system:

$$(\varepsilon\, \partial_t E_h^{\mathrm{pl}}(t), \psi_h) + (\sigma\, E_h^{1/2}(t), \psi_h) - (\mu^{-1} B_h^{1/2}(t), \operatorname{curl} \psi_h) = (f^{1/2}(t), \psi_h), \quad \forall\, \psi_h \in \mathcal{N}_h^0,$$

$$(\partial_t B_h^{\mathrm{pl}}(t), \phi_h) + (\operatorname{curl} E_h^{1/2}(t), \phi_h) = 0, \qquad \forall\, \phi_h \in \mathcal{RT}_h.$$

Crank-Nicolson well-posedness, energy identity, and stability

Lemma 1 (Crank-Nicolson well-posedness, energy identity, and stability)

Let the discrete energy be

$$\mathcal{E}^n := \tfrac{1}{2}\Big(\|\varepsilon^{1/2} E_h^n\|^2_{L^2(\Omega)} + \|\mu^{-1/2} B_h^n\|^2_{L^2(\Omega)}\Big).$$

Given $(E_h^n, B_h^n) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$, *problem* (5) *admits a unique solution* $(E_h^{n+1}, B_h^{n+1}) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$, *and the solution depends continuously on the data* $(E_h^n, B_h^n, f^{n+\frac{1}{2}})$. *Moreover, the exact balance*

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} + \|\sqrt{\sigma}\, E_h^{n+\frac{1}{2}}\|^2_{L^2(\Omega)} = \big(f^{n+\frac{1}{2}}, E_h^{n+\frac{1}{2}}\big)$$

holds for every n. *In particular, if* $f \equiv 0$ *then the scheme is energy-dissipative for* $\sigma \geq 0$ *and strictly conservative when* $\sigma \equiv 0$. *Finally, there exists a constant* $C_T > 0$ *depending only on* T, *the ellipticity bounds of* ε, μ, *and* $\|\varepsilon\|_{L^\infty}, \|\mu^{-1}\|_{L^\infty}$ *(but not on* $h, \Delta t$*), such that for all* $m = 1, \ldots, N_t$,

$$\mathcal{E}^m + \Delta t \sum_{n=0}^{m-1} \|\sqrt{\sigma}\, E_h^{n+\frac{1}{2}}\|^2_{L^2(\Omega)} \leq C_T \Bigg(\mathcal{E}^0 + \sum_{n=0}^{m-1} \Delta t\, \|\varepsilon^{-1/2} f^{n+\frac{1}{2}}\|^2_{L^2(\Omega)}\Bigg).$$

Discrete Gauss's law for magnetism

Lemma 2 (Discrete Gauss's law for magnetism is preserved)

Let $\{(E_h^n, B_h^n)\}$ *solve the fully discrete system* (5a) *and* (5b). *Let* $\mathrm{div}_h := \mathrm{div}\big|_{\mathcal{RT}_h} : \mathcal{RT}_h \to \mathcal{Q}_h$. *Then the discrete divergence evolves in* $\mathcal{Q}_h$ *according to*

$$\delta_t\left(\mathrm{div}_h\, B_h^{n+1}\right) + \mathrm{div}_h\left(\mathrm{curl}\, E_h^{n+\frac{1}{2}}\right) = 0 \quad \text{in } \mathcal{Q}_h, \tag{6}$$

which implies that $\delta_t(\mathrm{div}_h\, B_h^{n+1}) = 0$ *in* $\mathcal{Q}_h$. *In particular, if the initial data satisfy* $\mathrm{div}_h\, B_h^0 = 0$ *in* $\mathcal{Q}_h$*, then* $\mathrm{div}_h\, B_h^n = 0$ *in* $\mathcal{Q}_h$ *for all* n.

Proof.

Let $r_h := \delta_t B_h^{n+1} + \mathrm{curl}\, E_h^{n+\frac{1}{2}}$. From (5b), we have that $(r_h, \phi_h) = 0, \quad \forall \phi_h \in \mathcal{RT}_h$, i.e., $r_h \in \mathcal{RT}_h^{\perp}$ (with respect to the L^2 inner product). Since $B_h^n, B_h^{n+1} \in \mathcal{RT}_h$, by definition $\delta_t B_h^{n+1} \in \mathcal{RT}_h$.

From the sequence $\mathcal{N}_h^0 \xrightarrow{\mathrm{curl}} \mathcal{RT}_h \xrightarrow{\mathrm{div}} \mathcal{Q}_h$, we have $\mathrm{curl}\, \mathcal{N}_h^0 \subset \mathcal{RT}_h$. Therefore $\mathrm{curl}\, E_h^{n+\frac{1}{2}} \in \mathcal{RT}_h$ implying $r_h \in \mathcal{RT}_h$. Thus, $r_h \in \mathcal{RT}_h \cap \mathcal{RT}_h^{\perp}$, whence $r_h = 0$ in $\mathcal{RT}_h$. Applying the divergence operator $\mathrm{div}_h : \mathcal{RT}_h \to Q_h$ to r_h, we get (6) in Q_h. So, $\delta_t(\mathrm{div}_h\, B_h^{n+1}) = 0$ in Q_h. Clearly, if $\mathrm{div}_h\, B_h^0 = 0$, then $\mathrm{div}_h\, B_h^n = 0$ in $\mathcal{Q}_h$ for all n. □

Discrete Gauss's law for electricity

Lemma 3 (Discrete Gauss's law for electricity is preserved in a weak sense)

Let $\{(E_h^n, B_h^n)\}$ *solve the fully discrete system* (5a) *and* (5b)*. Define the discrete divergence functional*

$$\langle \mathrm{div}_h(\varepsilon E_h^{n+1}), q_h\rangle_{\mathcal{P}_h^*,\mathcal{P}_h} := -(\varepsilon E_h^{n+1}, \nabla q_h) \qquad \forall q_h \in \mathcal{P}_h.$$

Similarly, define

$$\langle \mathrm{div}_h(\sigma E_h^{n+\frac{1}{2}} - f^{n+\frac{1}{2}}), q_h\rangle_{\mathcal{P}_h^*,\mathcal{P}_h} := -(\sigma E_h^{n+\frac{1}{2}} - f^{n+\frac{1}{2}}, \nabla q_h) \qquad \forall q_h \in \mathcal{P}_h.$$

Then, for all n,

$$\partial_t\, \mathrm{div}_h(\varepsilon E_h^{n+1}) + \mathrm{div}_h(\sigma E_h^{n+\frac{1}{2}} - f^{n+\frac{1}{2}}) = 0 \qquad \text{in } \mathcal{P}_h^*.$$

If, for all n,

$$\mathrm{div}_h(\sigma E_h^{n+\frac{1}{2}} - f^{n+\frac{1}{2}}) = 0 \qquad \text{in } \mathcal{P}_h^*,$$

and the initial data satisfies

$$\mathrm{div}_h(\sigma E_h^{0}) = 0 \qquad \text{in } \mathcal{P}_h^*,$$

then, for all n,

$$\mathrm{div}_h(\varepsilon E_h^{n}) = 0 \qquad \text{in } \mathcal{P}_h^*.$$

Convergence of the fully discrete Crank-Nicolson scheme

Lemma 4 (Midpoint and piecewise–linear reconstructions have the same limit)

Let (E_h^n, B_h^n) solve the CN scheme (5)*, and let $E_h^{pl}, B_h^{pl}, E_h^{1/2}, B_h^{1/2}$ be the lifts. Let $f \in L^2(0,T;L^2(\Omega)^3)$. Then there exist functions $E, B \in L^\infty(0,T;L^2(\Omega)^3)$ and a subsequence (not relabeled) such that*

$$E_h^{pl} \overset{*}{\rightharpoonup} E, \qquad B_h^{pl} \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty(0,T;L^2(\Omega)^3).$$

Moreover, the midpoint lifts converge to the same limits in the following sense: for every $\Phi \in L^1(0,T;L^2(\Omega)^3)$,

$$\int_0^T \big(E_h^{1/2}(t), \Phi(t)\big)_{L^2}\, dt \longrightarrow \int_0^T \big(E(t), \Phi(t)\big)_{L^2}\, dt, \quad \int_0^T \big(B_h^{1/2}(t), \Phi(t)\big)_{L^2}\, dt \longrightarrow \int_0^T \big(B(t), \Phi(t)\big)_{L^2}\, dt.$$

In other words, $E_h^{1/2} \to E$ and $B_h^{1/2} \to B$ in the duality with $L^1(0,T;L^2(\Omega)^3)$.

Theorem 2 (Convergence of the fully discrete Crank-Nicolson scheme)

The limit from Lemma 4 is the unique solution to (3) *in the sense of Definition 1.*

Optimal control problem: continuous theory

Lemma 5 (Well-posedness and boundedness of the control-to-state map)

Without loss of generality, assume $\Omega_{src} = \Omega_{obs} = \Omega_{ctrl} = \Omega$. For every $z \in Z$ there exists a unique weak solution (E, B) of the optimal control problem (1) *in the sense of Definition 1, with $f := I_{src} + \operatorname{curl} z$. Moreover,*

$$\|\partial_t E\|_{L^2(0,T;H_0(\mathrm{curl};\Omega)^*)} + \|\partial_t B\|_{L^2(0,T;H(\mathrm{curl};\Omega)^*)} + \|E\|_{C([0,T];L^2(\Omega)^3)} + \|B\|_{C([0,T];L^2(\Omega)^3)}$$
$$\leq C\left(\|I_{src}\|_{L^2(0,T;L^2(\Omega)^3)} + \|z\|_Z + \|E_0\|_{L^2(\Omega)^3} + \|B_0\|_{L^2(\Omega)^3}\right),$$

with a constant $C > 0$ independent of z. Consequently, the control-to-state map $S : Z \to Y$, $S(z) = (E, B)$, is affine and bounded. Writing $S(z) = S_0 z + (E^{src}, B^{src})$, the operator $S_0 : Z \to Y$ is linear and bounded.

Reduced objective functional: Given the target fields $(E_d, B_d) \in Y$ and $\alpha_1, \alpha_2 > 0$, consider

$$\min_{z\in Z} \ \mathcal{J}(z) := \frac{1}{2}\|S(z) - (E_d, B_d)\|_Y^2 + \frac{\alpha_1}{2}\|z\|^2_{L^2(0,T;L^2(\Omega)^3)} + \frac{\alpha_2}{2}\|\operatorname{curl} z\|^2_{L^2(0,T;L^2(\Omega)^3)}. \qquad (7)$$

Theorem 3 (Existence and uniqueness of an optimal control)

The reduced optimization problem (7) *admits a unique minimizer $z^* \in Z$.*

Optimal control problem: continuous theory

Definition 3 (Weak solution to adjoint)

A pair (p_E, p_B) *is called an adjoint for* $(q_E, q_B) \in Y$ *if*

$$p_E \in H^1(0, T; H_0(\mathrm{curl};\Omega)^*) \cap C([0, T]; L^2(\Omega)^3),$$
$$p_B \in H^1(0, T; H(\mathrm{curl};\Omega)^*) \cap C([0, T]; L^2(\Omega)^3),$$

$p_E(T) = 0$, $p_B(T) = 0$ *in* $L^2(\Omega)^3$ *such that*

$$\langle -\varepsilon\partial_t\, p_E,\, \psi\rangle_{H_0(\mathrm{curl};\Omega)} + (p_E, \sigma\psi) + (\mathrm{curl}\,\psi,\, p_B) = (q_E, \psi) \quad \forall\psi \in H_0(\mathrm{curl};\Omega))$$
$$\langle -\partial_t p_B,\, \phi\rangle_{H(\mathrm{curl};\Omega)} - \langle \mu^{-1}\phi,\, \mathrm{curl}\, p_E\rangle_{H(\mathrm{curl};\Omega)} = (q_B, \phi) \quad \forall\phi \in H(\mathrm{curl};\Omega).$$

Optimal control problem: continuous theory

Theorem 4 (Optimality system)

Let $\bar{z} \in Z$ be the unique minimizer from Theorem 3, and let $(\bar{E}, \bar{B}) = S\bar{z}$. Then there exists a unique adjoint $(\bar{p}_E, \bar{p}_B)$ in the sense of Definition 3 with data $(q_E, q_B) = (\bar{E} - E_d,\ \bar{B} - B_d)$ such that the variational optimality condition

$$\alpha_1 \int_0^T (\bar{z}, w)\, dt + \alpha_2 \int_0^T (\operatorname{curl} \bar{z}, \operatorname{curl} w)\, dt + \int_0^T (\bar{p}_E, \operatorname{curl} w)\, dt = 0 \qquad \forall\, w \in Z$$

holds. Equivalently, in Z^*,

$$\alpha_1 \bar{z} + \mathcal{C}^*\big(\alpha_2 \mathcal{C}\bar{z} + \bar{p}_E\big) = 0 \quad \text{in } Z^* \qquad \text{with } \mathcal{C}z := \operatorname{curl} z,\ \langle \mathcal{C}^* v, w\rangle_{Z^*,Z} = \int_0^T (v, \operatorname{curl} w).$$

Optimal control problem: discrete theory

Discrete controls: Nédélec space, piecewise constant in time. Let

$$\mathcal{Z}_h := \mathcal{N}_h, \qquad \mathcal{Z}_{h,\Delta t} := \left\{ z_h^{1/2} : (0,T] \to \mathcal{Z}_h \text{ right–continuous}, z_h^{1/2}(t) = z_h^{n+\frac{1}{2}} \in \mathcal{Z}_h \text{ on } (t^n, t^{n+1}] \right\}.$$

For $z_h^{1/2} \in \mathcal{Z}_{h,\Delta t}$ we use the midpoint (Bochner) norms

$$\|z_h^{1/2}\|^2_{L^2(0,T;L^2)} := \sum_{n=0}^{N_t-1} \Delta t \, \|z_h^{n+\frac{1}{2}}\|^2_{L^2(\Omega)}, \quad \|\operatorname{curl} z_h^{1/2}\|^2_{L^2(0,T;L^2)} := \sum_{n=0}^{N_t-1} \Delta t \, \|\operatorname{curl} z_h^{n+\frac{1}{2}}\|^2_{L^2(\Omega)}.$$

We write $\|z_h^{1/2}\|^2_Z := \alpha_1 \|z_h^{1/2}\|^2_{L^2(0,T;L^2)} + \alpha_2 \|\operatorname{curl} z_h^{1/2}\|^2_{L^2(0,T;L^2)}$.

Discrete state equation: Given $z_h^{1/2} \in \mathcal{Z}_{h,\Delta t}$, solve, for $n = 0, \ldots, N_t - 1$, the CN step

$$(\varepsilon\, \delta_t E_h^{n+1}, \psi_h) + (\sigma\, E_h^{n+\frac{1}{2}}, \psi_h) - (\mu^{-1} B_h^{n+\frac{1}{2}}, \operatorname{curl}\, \psi_h) = (I_{\text{src}}^{n+\frac{1}{2}} + \operatorname{curl} z_h^{n+\frac{1}{2}}, \psi_h), \quad \forall\, \psi_h \in \mathcal{N}_h^0, \tag{8a}$$

$$(\delta_t B_h^{n+1}, \phi_h) + (\operatorname{curl}\, E_h^{n+\frac{1}{2}}, \phi_h) = 0, \qquad \forall\, \phi_h \in \mathcal{RT}_h. \tag{8b}$$

Existence/uniqueness and the energy identity of Lemma 1 hold with $f^{n+\frac{1}{2}} = I_{\text{src}}^{n+\frac{1}{2}} + \operatorname{curl} z_h^{n+\frac{1}{2}}$.

Optimal control problem: discrete theory

Discrete objective functional: With the desired fields given as midpoint lifts $(E_d^{1/2}, B_d^{1/2})$,

$$\mathcal{J}_{h,\Delta t}(z_h^{1/2}) := \frac{1}{2} \sum_{n=0}^{N_t-1} \Delta t \left(\|E_h^{n+\frac{1}{2}} - E_d^{n+\frac{1}{2}}\|^2_{L^2(\Omega)} + \|B_h^{n+\frac{1}{2}} - B_d^{n+\frac{1}{2}}\|^2_{L^2(\Omega)} \right) + \frac{1}{2} \|z_h^{1/2}\|^2_Z,$$

where $(E_h^{n+\frac{1}{2}}, B_h^{n+\frac{1}{2}}) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$ is the CN–state (8) driven by $z_h^{1/2} \in \mathcal{Z}_{h,\Delta t}$.

Theorem 5 (Existence and uniqueness of a fully discrete optimal control)

Assuming $\alpha_1, \alpha_2 > 0$, the problem

$$\min_{z_h^{1/2} \in \mathcal{Z}_{h,\Delta t}} \mathcal{J}_{h,\Delta t}(z_h^{1/2})$$

admits a unique minimizer $\bar{z}_h^{1/2} \in \mathcal{Z}_{h,\Delta t}$.

Optimal control problem: discrete optimality system

Backward-in-time discrete adjoint: Define backward difference and midpoints

$$\delta_t^- p^n := \frac{p^n - p^{n+1}}{\Delta t}, \qquad p^{n+\frac{1}{2}} := \tfrac{1}{2}(p^n + p^{n+1}).$$

Theorem 6 (Fully discrete first–order optimality system)

Let $\bar{z}_h^{1/2} \in \mathcal{Z}_{h,\Delta t}$ be the unique minimizer of $\mathcal{J}_{h,\Delta t}$, and let $(\bar{E}_h^n, \bar{B}_h^n)$ be the associated CN state (8). Then there exist unique adjoint sequences $(\bar{p}_{E,h}^n, \bar{p}_{B,h}^n) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$, $n = 0, \ldots, N_t$, with terminal data

$$\bar{p}_{E,h}^{N_t} = 0 \in \mathcal{N}_h^0, \qquad \bar{p}_{B,h}^{N_t} = 0 \in \mathcal{RT}_h,$$

solving, backward for $n = N_t - 1, \ldots, 0$,

$$(\varepsilon\, \delta_t^-\, \bar{p}_{E,h}^n, \psi_h) + (\sigma\, \bar{p}_{E,h}^{n+\frac{1}{2}}, \psi_h) + (\mathrm{curl}\ \psi_h, \bar{p}_{B,h}^{n+\frac{1}{2}}) = (\bar{E}_h^{n+\frac{1}{2}} - E_d^{n+\frac{1}{2}}, \psi_h), \qquad \forall\, \psi_h \in \mathcal{N}_h^0, \tag{9a}$$

$$(\delta_t^-\, \bar{p}_{B,h}^n, \phi_h) - (\mu^{-1}\phi_h, \mathrm{curl}\ \bar{p}_{E,h}^{n+\frac{1}{2}}) = (\bar{B}_h^{n+\frac{1}{2}} - B_d^{n+\frac{1}{2}}, \phi_h), \qquad \forall\, \phi_h \in \mathcal{RT}_h, \tag{9b}$$

and the discrete stationarity condition …

Optimal control problem: discrete optimality system

Continued (Discrete stationarity condition)

$$\alpha_1 \sum_{n=0}^{N_t-1} \Delta t \, (\bar{z}_h^{n+\frac{1}{2}}, w_h^{n+\frac{1}{2}}) + \alpha_2 \sum_{n=0}^{N_t-1} \Delta t \, (\operatorname{curl} \bar{z}_h^{n+\frac{1}{2}}, \operatorname{curl} w_h^{n+\frac{1}{2}}) + \sum_{n=0}^{N_t-1} \Delta t \, (\bar{p}_{E,h}^{n+\frac{1}{2}}, \operatorname{curl} w_h^{n+\frac{1}{2}}) = 0 \qquad (10)$$

for all $w_h^{1/2} \in \mathcal{Z}_{h,\Delta t}$. *Equivalently, time-cellwise,*

$$\alpha_1 \, \bar{z}_h^{n+\frac{1}{2}} \; + \; \mathcal{C}_h^*\big(\alpha_2 \, \operatorname{curl} \bar{z}_h^{n+\frac{1}{2}} + P_h \, \bar{p}_{E,h}^{n+\frac{1}{2}}\big) = 0 \quad \text{in } \mathcal{N}_h, \qquad n = 0, \ldots, N_t - 1,$$

where $\mathcal{C}_h^* : \mathcal{RT}_h \to \mathcal{N}_h$ *is the (discrete) adjoint of curl defined by* $(\mathcal{C}_h^* q_h, w_h) = (q_h, \operatorname{curl} w_h)$ *for all* $q_h \in \mathcal{RT}_h$, $w_h \in \mathcal{N}_h^0$, *and* P_h *is the global* L^2 *orthogonal projection onto* $\mathcal{RT}_h$. *Here we assume that* $\mathcal{C}_h : \mathcal{N}_h \to \mathcal{RT}_h$.

Optimal control problem: discrete control convergence

Theorem 7 (Convergence of the discrete control, with structure-preserving state)

Let $\bar{z} \in Z$ *be the unique minimizer of the continuous reduced problem* (7) *with associated state* $(\bar{E}, \bar{B}) = S(\bar{z})$ *and adjoint* $(\bar{p}_E, \bar{p}_B)$ *per Definition 3.*

Let $\bar{z}_{h,\Delta t}^{1/2} \in \mathcal{Z}_{h,\Delta t} := L^2(0,T;\mathcal{N}_h^0)$ *be the unique minimizer of the fully discrete reduced functional* $\mathcal{J}_{h,\Delta t}$*, defined using the CN state solver* (8)*, midpoint lifts, and discrete tracking terms.*

Then, as $h, \Delta t \to 0$ *(up to a subsequence),*

$$\bar{z}_{h,\Delta t}^{1/2} \rightharpoonup \bar{z} \quad \text{in } L^2(0,T;L^2(\Omega)^3), \qquad \operatorname{curl} \bar{z}_{h,\Delta t}^{1/2} \rightharpoonup \operatorname{curl} \bar{z} \quad \text{in } L^2(0,T;L^2(\Omega)^3).$$

Moreover, if $\alpha_1 > 0$ *and* $\alpha_2 > 0$*, then the convergence is strong in* Z*:*

$$\bar{z}_{h,\Delta t}^{1/2} \to \bar{z} \ \text{ in } L^2(0,T;L^2(\Omega)^3), \qquad \operatorname{curl} \bar{z}_{h,\Delta t}^{1/2} \to \operatorname{curl} \bar{z} \ \text{ in } L^2(0,T;L^2(\Omega)^3).$$

Numerical results

Approach and computational tools

- Solve equation (5) forward in time for the states (E_h^n, B_h^n), then solve equation (9) backward in time for the adjoint states $(\bar{p}_{E,h}^n, \bar{p}_{B,h}^n)$, for all n. Compute the discrete gradient in (10).

- Line-search quasi-Newton method based on the limited-memory BFGS secant approximation to update the control.

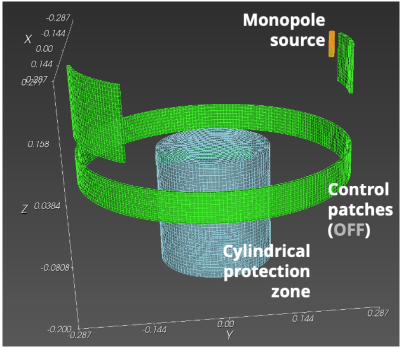
- Alternative: Trust regions with a truncated conjugate gradient method and analytic Hessian-vector products.

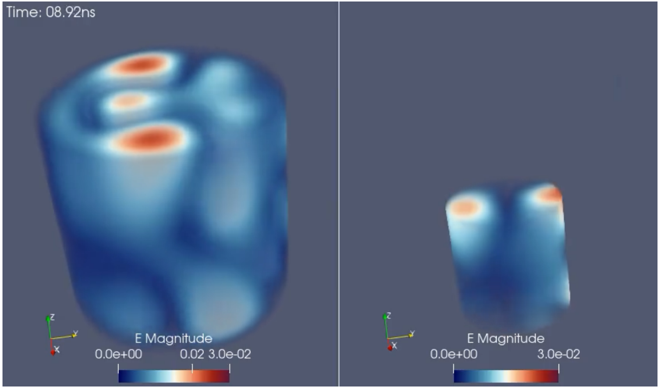
- Two high-performance computing frameworks:
  - MrHyDE: A framework for solving Multi-resolution Hybridized Differential Equations to solve Maxwell's equations and compute adjoint sensitivities and gradients; and
  - Rapid Optimization Library (ROL) to solve the optimization problem.

Numerical results

Solution of the state equation using CN+Nédélec/RT

- Source: Sinusoid in time, windowed by a Gaussian.
- Center frequency is 1 GHz, with a 500 kHz bandwidth.
- Controls are off, i.e., simple wave propagation in a cavity with reflecting walls.

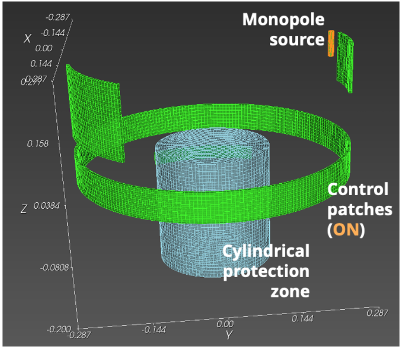


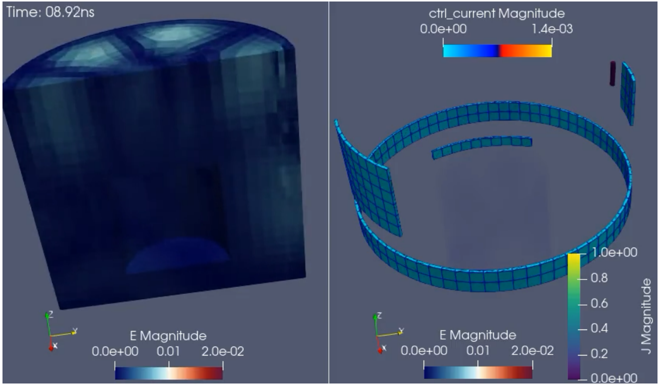


Numerical results

Structure-preserving solution of the optimal control problem

- Source: Sinusoid in time, windowed by a Gaussian.
- Center frequency is 1 GHz, with a 500 kHz bandwidth.
- Controls are on: the EM energy is rerouted around the protection zone.





Open research questions and challenges

- Leapfrog time stepping: Explicit methods are computationally more efficient in certain settings (ongoing work by Yaw Owusu-Agyemang).
- Consistent discrete inner products and reduced Hessian preconditioning: The number of optimization iterations does not scale with mesh and time step refinement in our current implementation (ongoing work by Yaw Owusu-Agyemang and Alexey Voronin).
- Optimal (sparse) placement of controls.
- Packed cavity: How much will additional interior geometry affect the solution of the optimal control problem?
- Robustness of controls to (unknown) source perturbations.
- Combining sensing and control: Patch antennas are used as sources and receivers.