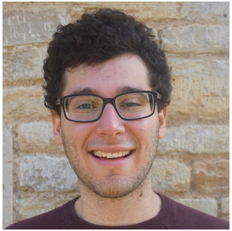
The latent variable proximal point algorithm for variational problems with inequality constraints

Patrick E. Farrell[1,2] Brendan Keith[3] Thomas Surowiec[4,3] Jørgen S. Dokken[4]
Ioannis P. A. Papadopoulos[5]



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13 April 2026

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over the constrained set

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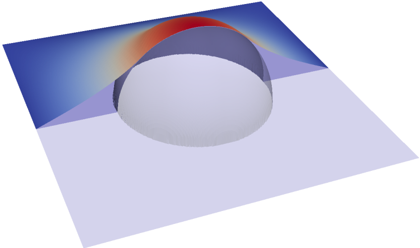

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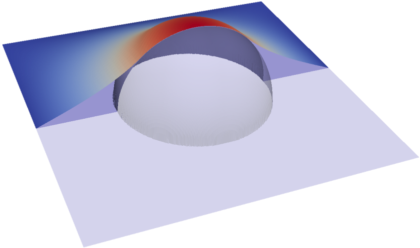

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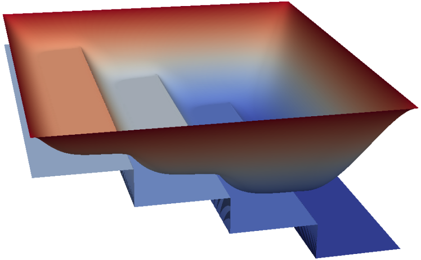

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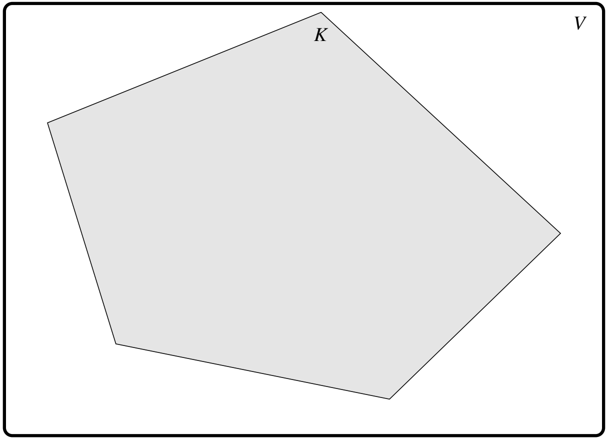

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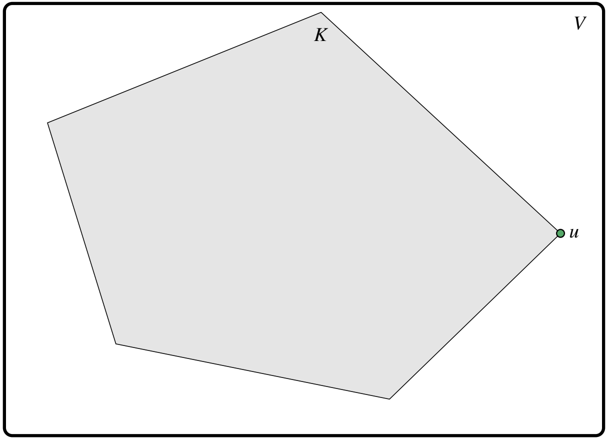

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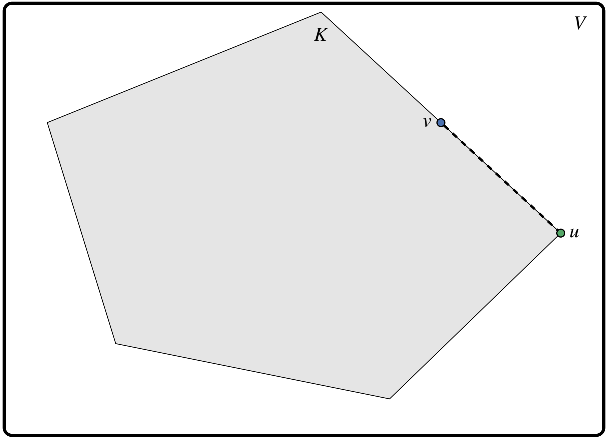

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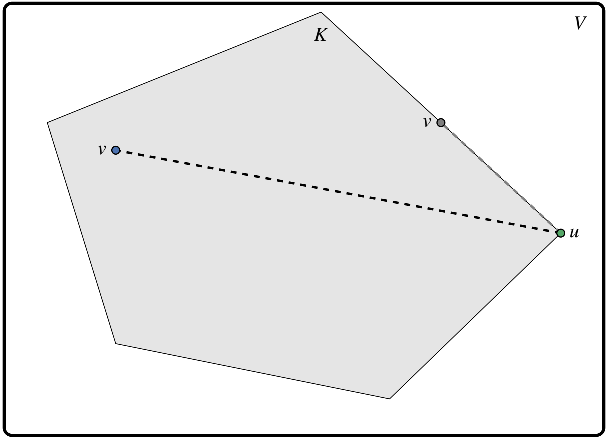

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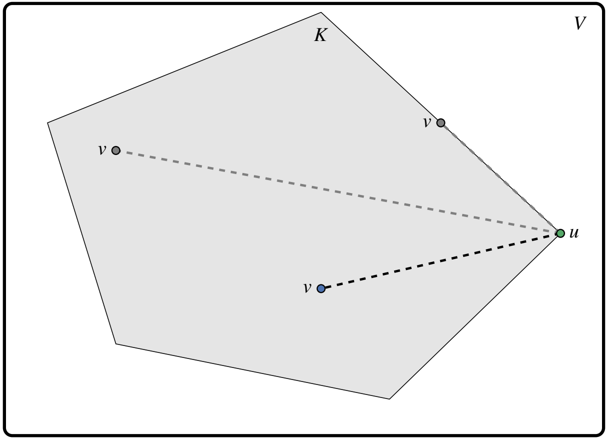

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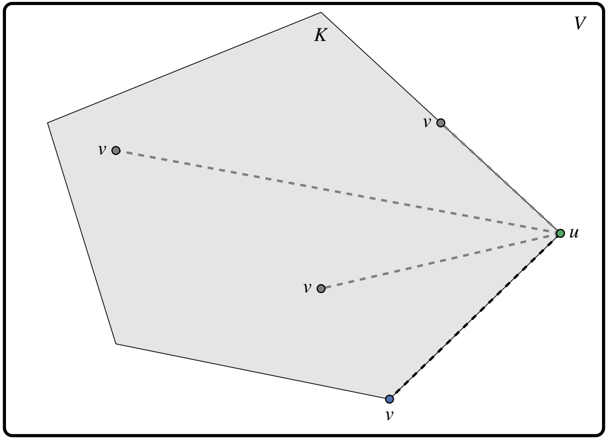

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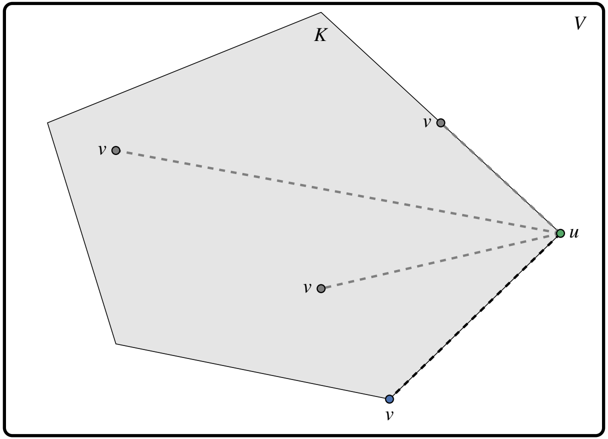

At a solution u, the energy must not decrease *along any feasible direction* $v - u$.

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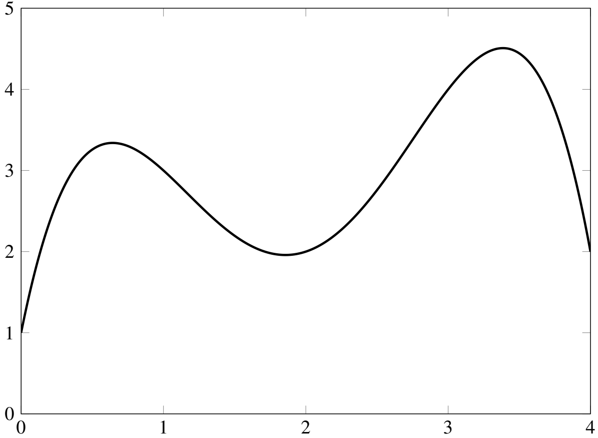

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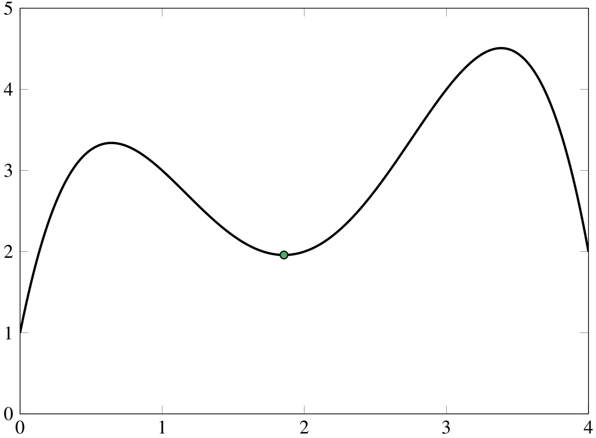

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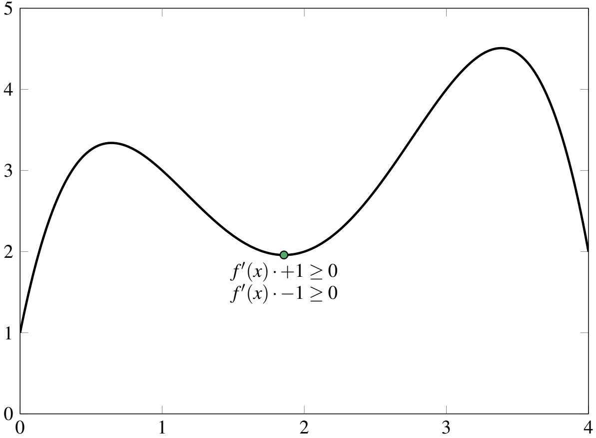

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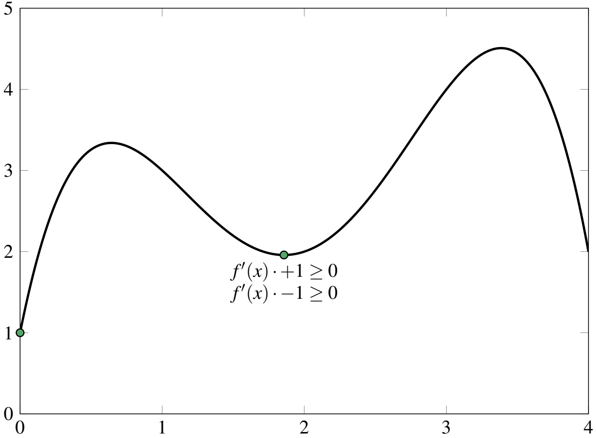

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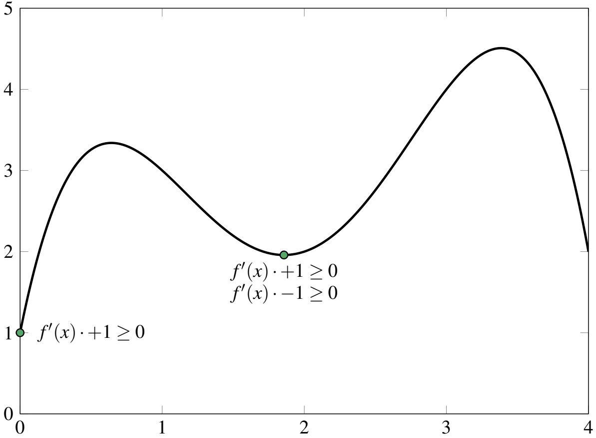

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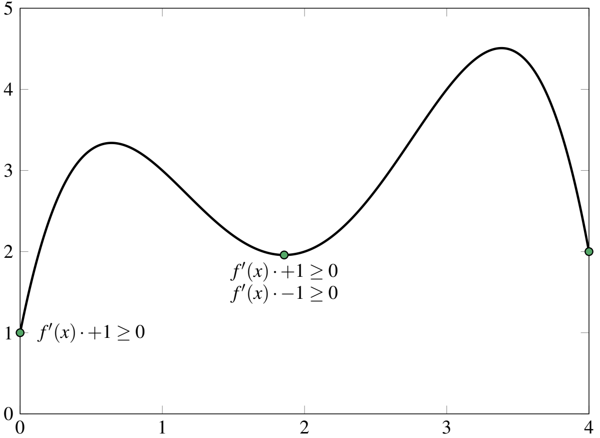

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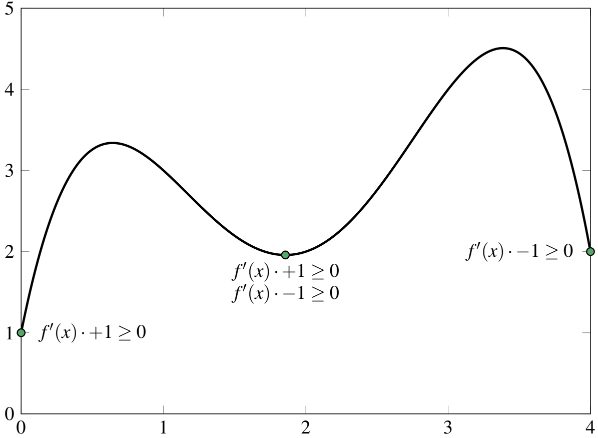

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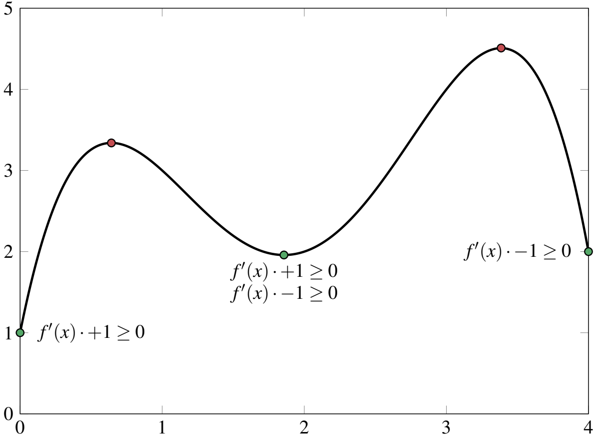

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This talk

A new framework for solving infinite-dimensional variational inequalities ...

...with major advantages over existing algorithms.

Section 2

Latent variable proximal point

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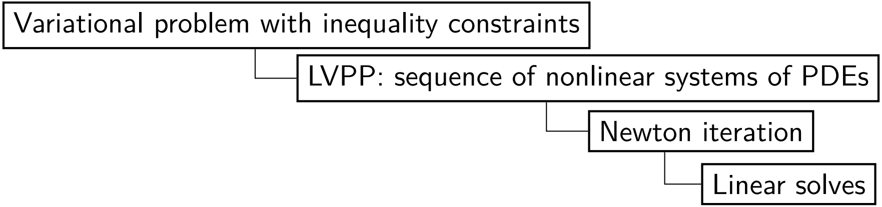
LVPP breaks down a VI into a sequence of nonlinear PDE solves.

We then use Newton's method to break down nonlinear PDE solves into linear PDE solves.



- Variational problem with inequality constraints
  - LVPP: sequence of nonlinear systems of PDEs
    - Newton iteration
      - Linear solves

Schematic solver diagram.

Subsection 1

Legendre functions

The obstacle problem has feasible set

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Here $C(x) \subset \mathbb{R}^m$ is the nonempty, closed, convex *feasible image* at x.

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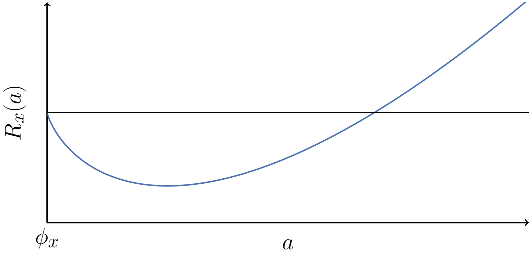
Example

For the obstacle problem, $C_x = [\phi_x, \infty)$, and we choose a modified Shannon entropy:

$$R_x(a) = (a - \phi_x)\log(a - \phi_x) - (a - \phi_x), \quad \nabla R_x(a) = \log(a - \phi_x).$$



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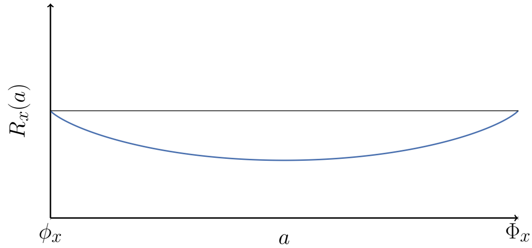
For a *double obstacle* problem, $C_x = [\phi_x, \Phi_x]$, and we choose the Fermi–Dirac entropy:

$$R_x(a) = (a - \phi_x) \log (a - \phi_x) + (\Phi_x - a) \log (\Phi_x - a), \quad \nabla R_x(a) = \log (a - \phi_x) + \log (\Phi_x - a).$$



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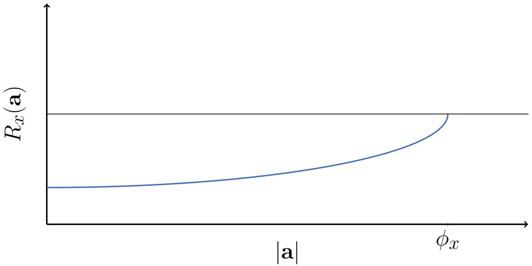
Example

For gradient constraints, $B = \nabla$, $C_x = \mathcal{B}(0, \phi_x)$, and we choose a modified Hellinger entropy:

$$R_x(\mathbf{a}) = -\sqrt{\phi_x^2 - |\mathbf{a}|^2}, \quad \nabla R_x(\mathbf{a}) = \mathbf{a}/\sqrt{\phi_x^2 - |\mathbf{a}|^2}.$$



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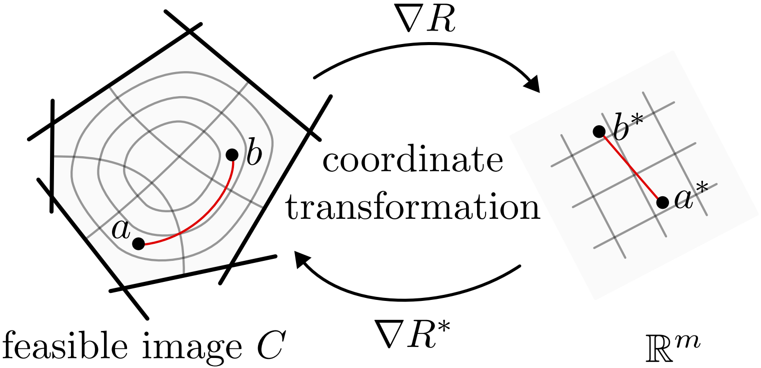
Theorem (Rockafellar (1967))

A proper convex function R is a Legendre function if and only if its convex conjugate R^* is also a Legendre function. Moreover,

$$\nabla R\colon \operatorname{int}(\operatorname{dom} R) \to \operatorname{int}(\operatorname{dom} R^*)$$

is a topological isomorphism with $(\nabla R)^{-1} = \nabla R^*$.

R. Tyrell Rockafellar


∇R
coordinate transformation
∇R^*
a
b
b^*
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feasible image C
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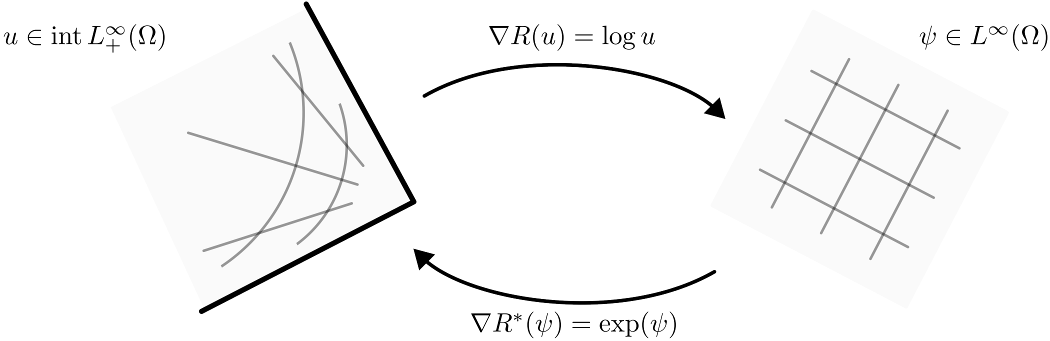

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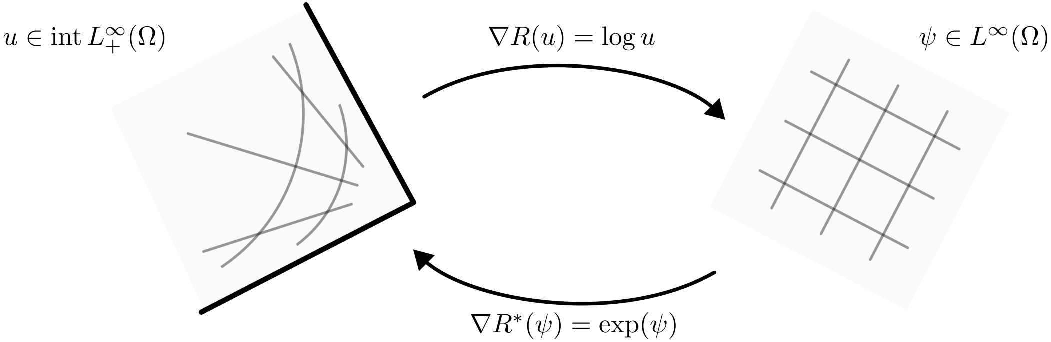

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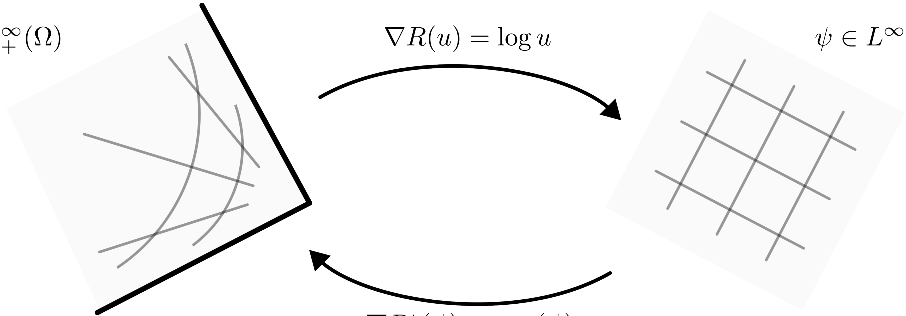

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...or more precisely any *strictly* feasible function.

Subsection 2

Proximal point

Proximal point is a fundamental algorithm in nonsmooth, convex optimisation.

To solve

$$u \in \operatorname*{argmin}_{v \in K} \; J(v)$$

Bernard Martinet

Osman Güler

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$$u^k \in \operatorname*{argmin}_{v \in K} \left\{ J(v) + \frac{1}{\alpha^k} \|v - u_{k-1}\|_V^2 \right\} \quad \text{for} \quad \{\alpha^k\},\ \alpha^k > 0.$$

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Amazingly, for strictly convex J, this converges arbitrarily quickly:

$$J(u^k) - J(u) \leq \frac{\|u^0 - u\|_V^2}{\sum_{i=1}^k \alpha_i}.$$

Osman Güler

Bad news

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The regularisation

$$\frac{1}{\alpha^k}\|v - u_{k-1}\|_V^2$$

does *not respect the geometry of the constraints*. The subproblems remain VIs.

Bad news

Applying proximal point to the obstacle problem gives *subproblems that are as hard to solve.*

The regularisation

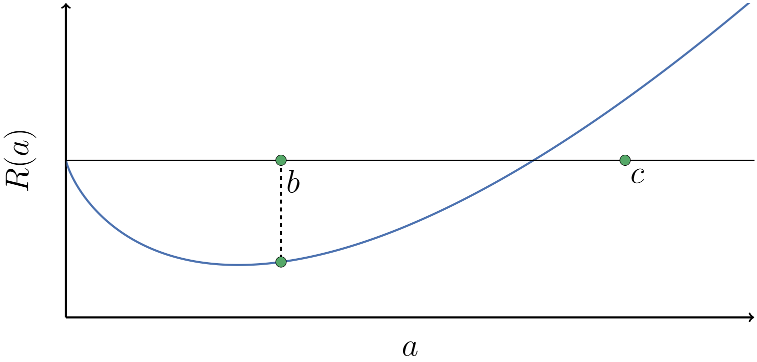
$$\frac{1}{\alpha^k}\|v - u_{k-1}\|_V^2$$

does *not respect the geometry of the constraints*. The subproblems remain VIs.

Good news

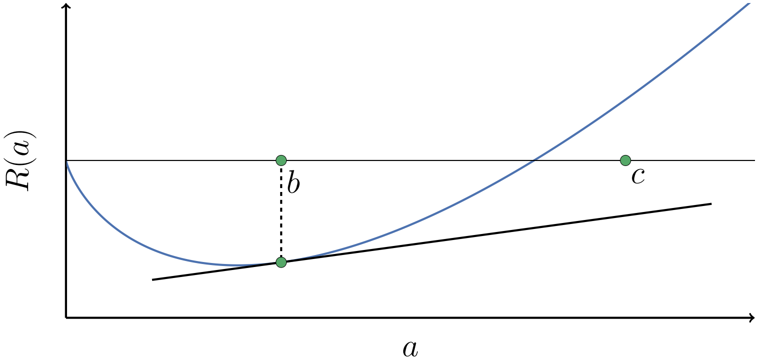
The Legendre function gives a notion of distance where the subproblems *do* simplify.

To define the *Bregman distance* $D_R(c, b)$ between base b and c, proceed as follows.



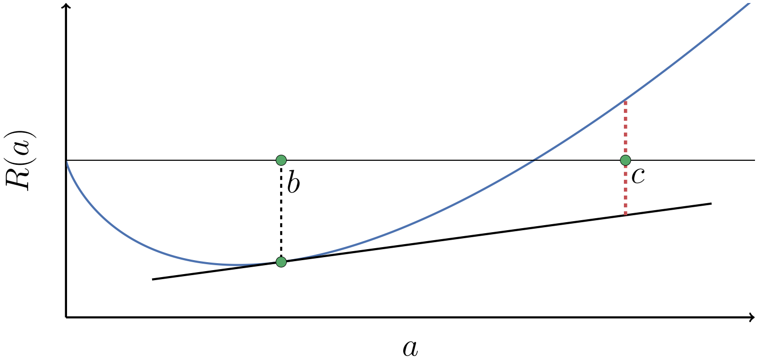
Start with the Legendre function R.

To define the *Bregman distance* $D_R(c, b)$ between base b and c, proceed as follows.



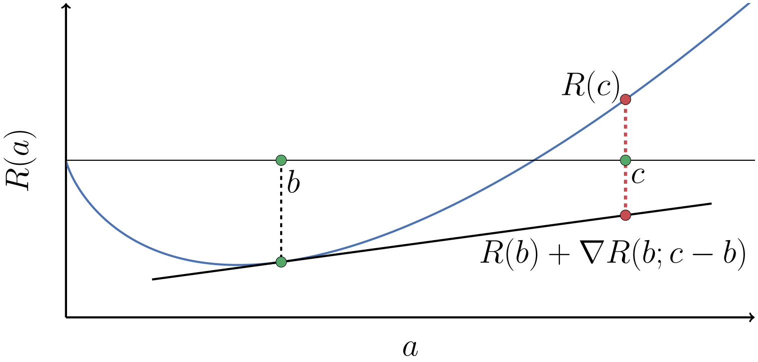
Build the tangent at b.

To define the *Bregman distance* $D_R(c,b)$ between base b and c, proceed as follows.



Measure distance between tangent and R at c.

To define the *Bregman distance* $D_R(c, b)$ between base b and c, proceed as follows.



$$D_R(c, b) = R(c) - R(b) - \nabla R(b; c - b).$$

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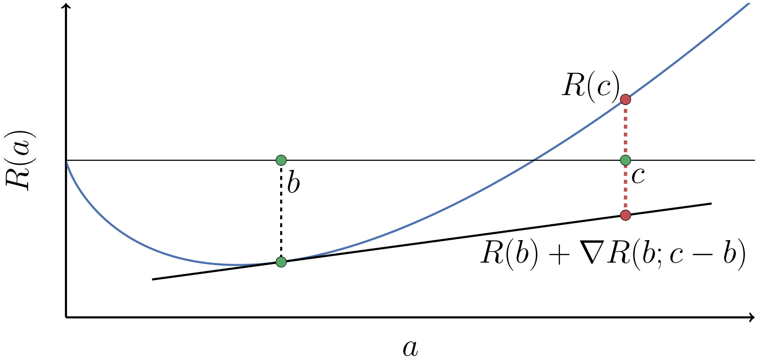


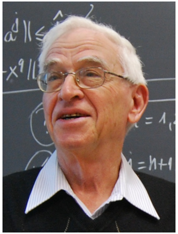
$$D_R(c, b) = R(c) - R(b) - \nabla R(b; c - b).$$

Example

If R is the Shannon entropy, then D_R is the Kullback–Leibler divergence.

To solve

$$u \in \operatorname*{argmin}_{v\in K} \; J(v)$$



Yair Censor

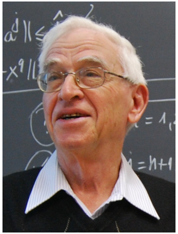
Stavros Zenios

To solve

$$u \in \operatorname*{argmin}_{v \in K} \; J(v)$$

with Bregman proximal point, we iterate

$$u^k \in \operatorname*{argmin}_{v \in K} \left\{ J(v) + \frac{1}{\alpha^k} \int_{\Omega_d} D_R(Bv, Bu^{k-1}) \, \mathrm{d}x \right\} \quad \text{for} \quad \{\alpha^k\}, \; \alpha^k > 0.$$



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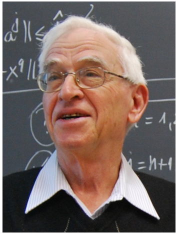

Yair Censor

Good news

For many problems, this forces u^k to be strictly feasible, and the subproblem optimality condition *becomes a PDE*:

$$u^k \in K : \alpha^k J'(u^k) + B^* \nabla R(Bu^k) - B^* \nabla R(Bu^{k-1}) = 0.$$

Stavros Zenios

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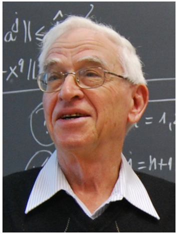

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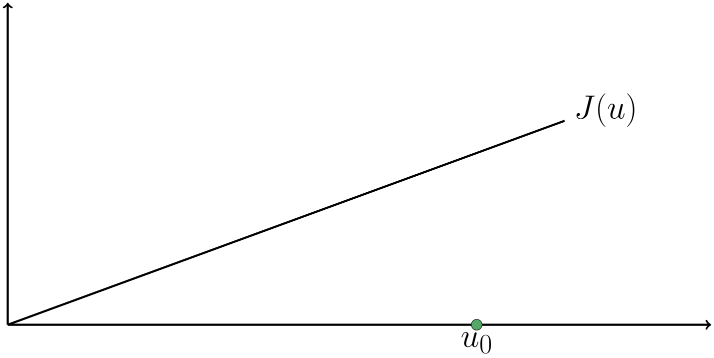
Stavros Zenios

This breaks down the VI into a sequence of nonlinear PDEs!

Consider the toy problem

$$u \in \operatorname*{argmin}_{v \in [0,\infty)} J(v) = v$$

with solution $u = 0$.



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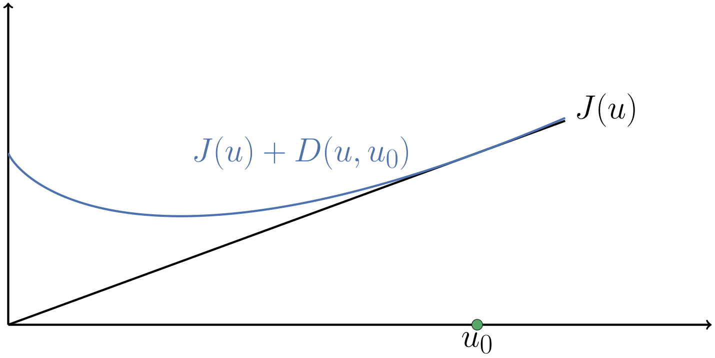

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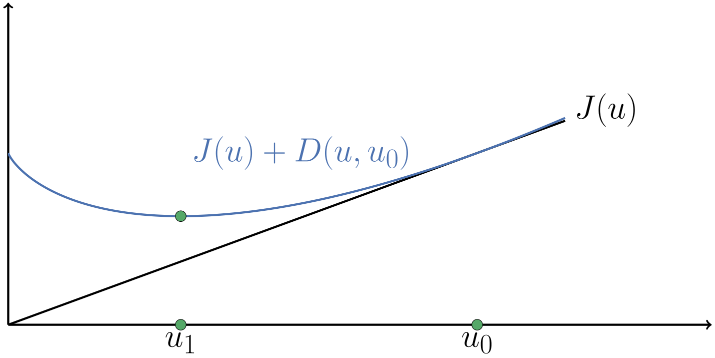

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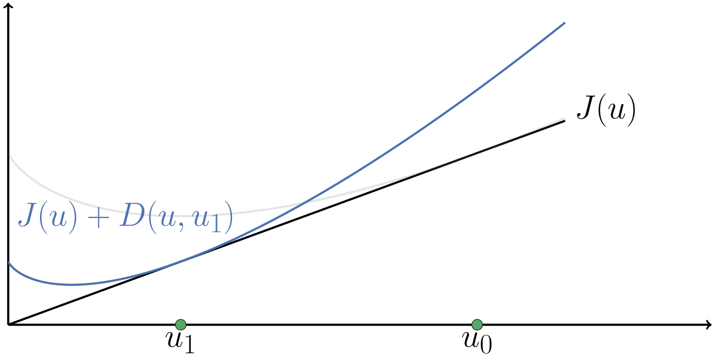

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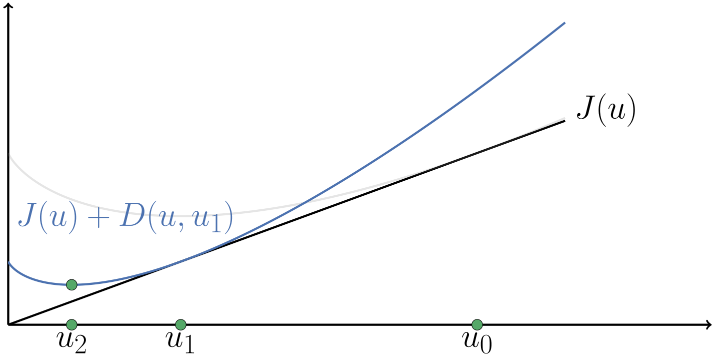

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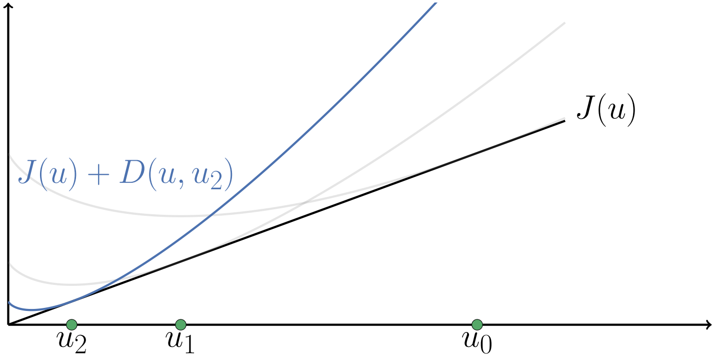

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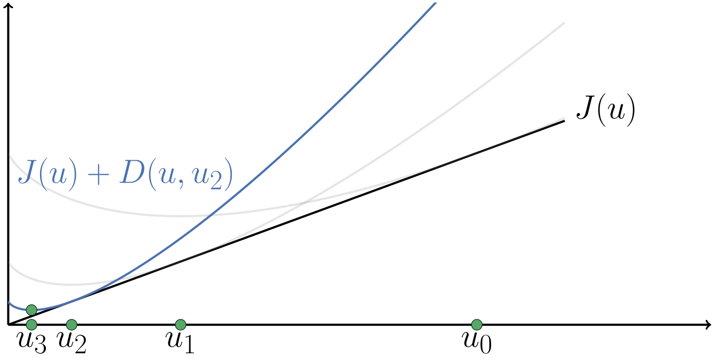

Observation

The choice of Legendre function is important!

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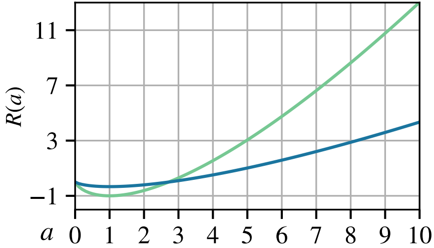

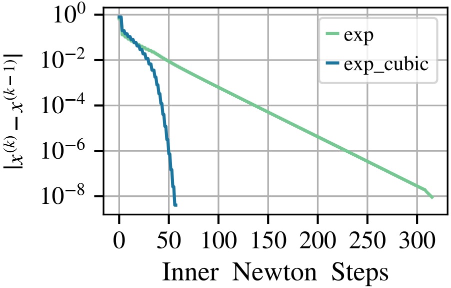

Jongho Park

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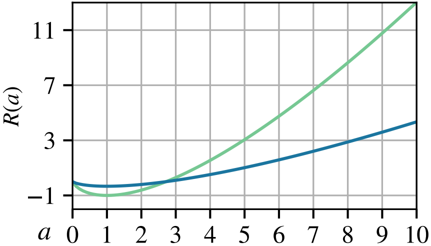

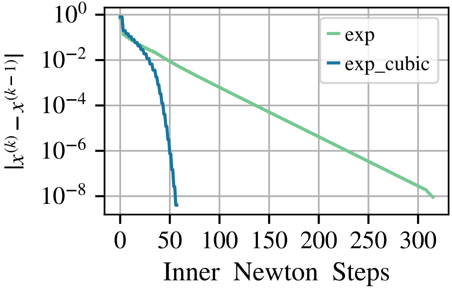

Jongho Park

Analysis

Less steep gradient at the boundary $\partial C(x)$ means faster convergence.

Subsection 3

Mixed formulation

Good news

The iterates are strictly feasible, so the optimality condition becomes a PDE:

$$u^k \in K : \alpha^k J'(u^k) + B^* \nabla R(B u^k) - B^* \nabla R(B u^{k-1}) = 0.$$

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We still need to discretise $u \in K$, because R is only defined for feasible Bu.

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Bad news

We still need to discretise $u \in K$, because R is only defined for feasible Bu.

Idea

Introduce a latent variable $\psi \in W$ and express $Bu = \nabla R^*(\psi)$!

Latent variable proximal point

For some $\psi^0 \in W$, find $(u^k, \psi^k) \in V \times W$ s. t.

$$\alpha_k J'(u^k) + B^*\psi^k = B^*\psi^{k-1},$$
$$Bu^k - \nabla R^*(\psi^k) = 0.$$

Latent variable proximal point

For some $\psi^0 \in W$, find $(u^k, \psi^k) \in V \times W$ s. t.

$$\alpha_k J'(u^k) + B^* \psi^k = B^* \psi^{k-1},$$
$$B u^k - \nabla R^*(\psi^k) = 0.$$

Important observation

This new formulation only requires discretising $u \in V$, not $u \in K$!

This is much, much simpler, because V is a Banach space but K is not.

Section 3

Bound constraints

Consider again the obstacle problem:

$$u \in \underset{v \in K}{\operatorname{argmin}}\ J(v) = \frac{1}{2} \int_\Omega \nabla v \cdot \nabla v \ \mathrm{d}x - \int_\Omega f v \ \mathrm{d}x,$$

for feasible set

$$K = \left\{ v \in H_0^1(\Omega) \mid v \geq \phi \text{ a.e. in } \Omega \right\}.$$

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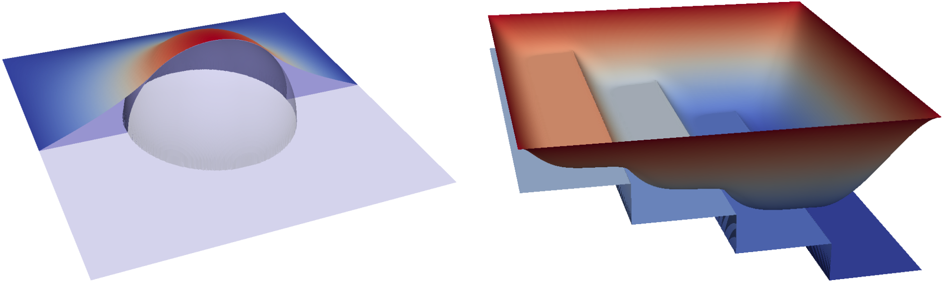
$$K = \left\{ v \in H_0^1(\Omega) \mid v \geq \phi \text{ a.e. in } \Omega \right\}.$$

The LVPP formulation becomes: for $\psi^0 = 0$, find $(u^k, \psi^k) \in H_0^1(\Omega) \times L^\infty(\Omega)$ s. t.

$$\begin{aligned} \alpha_k(\nabla u^k, \nabla v) + (\psi^k, v) &= \alpha_k(f, v) + (\psi^{k-1}, v), \\ (u^k, w) - (\exp(\psi^k) + \phi, w) &= 0, \end{aligned}$$

for all $(v, w) \in H_0^1(\Omega) \times L^\infty(\Omega)$.

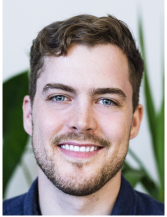


Good news

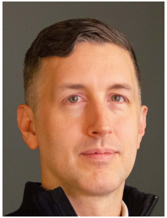
Complete mesh independence *and* strict feasibility for *any* approximation order.

For the proof that the Bregman proximal point iterations for the obstacle problem are PDEs, not VIs, see

B. Keith and T. M. Surowiec. “Proximal Galerkin: a structure-preserving finite element method for pointwise bound constraints”. In: *Foundations of Computational Mathematics* (2024). DOI: 10.1007/s10208-024-09681-8.



Brendan Keith



Thomas Surowiec

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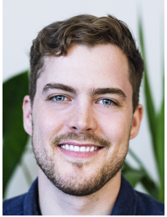

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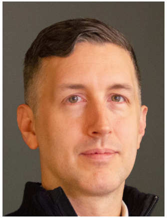

Thomas Surowiec

For the proof that the convergence of proximal Galerkin is mesh-independent for the obstacle and Signorini problems, see

B. Keith, R. Masri, and M. Zeinhofer. *A priori error analysis of the proximal Galerkin method.* arXiv:2507.13516. 2025.

Rami Masri

Marius Zeinhofer

There are several algorithms for obstacle-type VIs. How does LVPP compare?

	feasible?	inf-dim?	mesh-indep?	no param to $0/\infty$?
active set/semismooth Newton	✓	✓	✗	✓
penalty/augmented Lagrangian	✗	✓	✓	✗
monotone multigrid	✓	✗	✓	✓
interior point	✓	✓	✓	✗
latent variable proximal point	✓	✓	✓	✓

Latent variable proximal point

Only method that combines all desirable properties!

Section 4

Gradient constraints

Take again the Dirichlet energy

$$u \in \operatorname*{argmin}_{v \in K} J(v) = \frac{1}{2} \int_\Omega \nabla v \cdot \nabla v \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x,$$

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but now impose *both obstacle and gradient* constraints:

$$K = \left\{ v \in H_0^1(\Omega) \mid v \geq \phi \text{ and } |\nabla v| \leq \Phi \text{ a.e. in } \Omega \right\}.$$

Take again the Dirichlet energy

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Now $B = (\mathrm{id}, \nabla)^\top$ and $C(x) = [\phi(x), \infty) \times \mathcal{B}(0, \Phi(x))$, $\mathcal{B}$ the Euclidean ball.

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Legendre functions for intersection

Legendre functions for intersections of sets are additive:

$$R(a, \mathbf{a}) = (a - \phi) \log (a - \phi) - (a - \phi) - \sqrt{\Phi^2 - |\mathbf{a}|^2}.$$

The induced isomorphism has two components:

$$\nabla R^* \left((a^*, \mathbf{a}^*) \right) = \begin{pmatrix} \phi + \exp a^* \\ \dfrac{\Phi \, \mathbf{a}^*}{\sqrt{1 + |\mathbf{a}^*|^2}} \end{pmatrix} .$$

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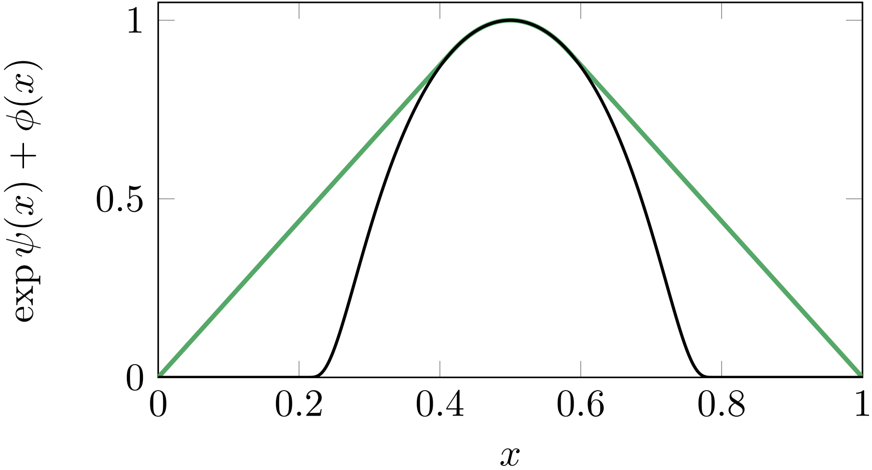
$$\nabla R^*\left((a^*, \mathbf{a}^*)\right) = \begin{pmatrix} \phi + \exp a^* \\ \dfrac{\Phi\,\mathbf{a}^*}{\sqrt{1 + |\mathbf{a}^*|^2}} \end{pmatrix}.$$

The LVPP iteration becomes: find $(u^k, \psi^k, \Psi^k) \in H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega, \mathbb{R}^n)$ s. t.

$$\begin{aligned} \alpha_k(\nabla u^k, \nabla v) + (\psi^k, v) + (\Psi^k, \nabla v) &= (\psi^{k-1}, v) + (\Psi^{k-1}, \nabla v) \\ (u^k, w) - (\exp\psi^k + \phi, w) &= 0 \\ (\nabla u^k, W) - \left(\frac{\Phi\,\Psi^k}{\sqrt{1 + |\Psi^k|^2}}, W\right) &= 0 \end{aligned}$$

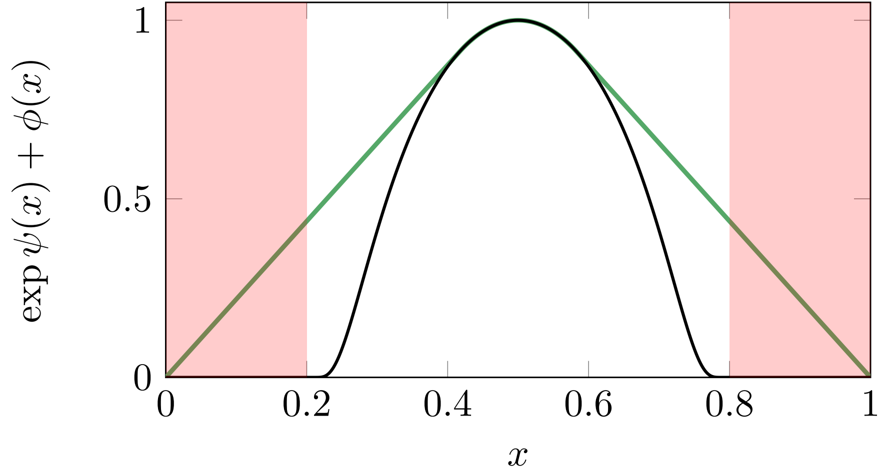
for all $(v, w, W) \in H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega, \mathbb{R}^n)$.

We set $f = 0$ and vary the gradient constraints.



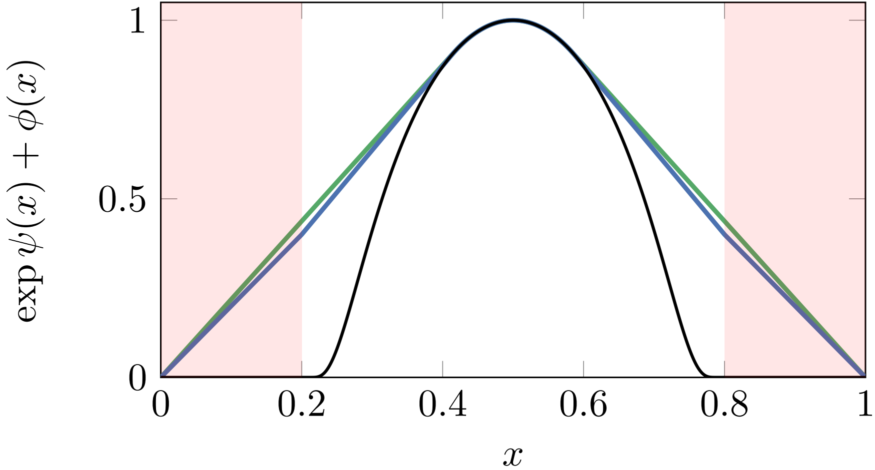
No gradient constraints.

We set $f = 0$ and vary the gradient constraints.



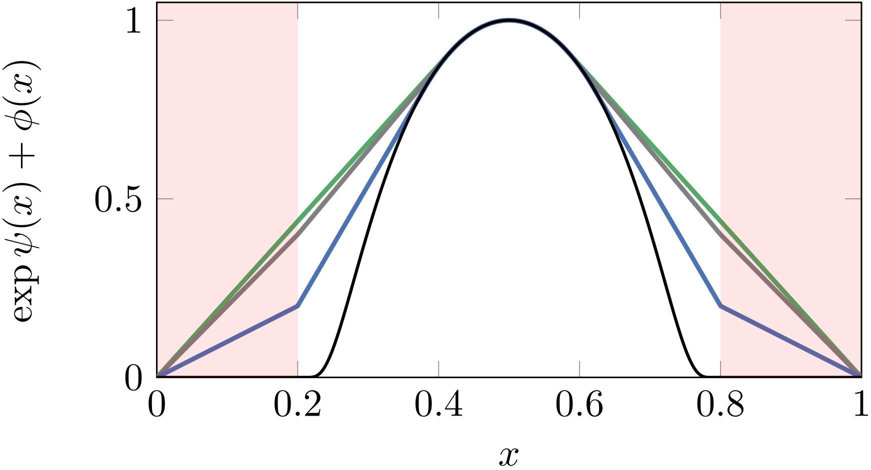
Apply gradient constraints on $[0, 0.2] \cup [0.8, 1]$.

We set $f = 0$ and vary the gradient constraints.



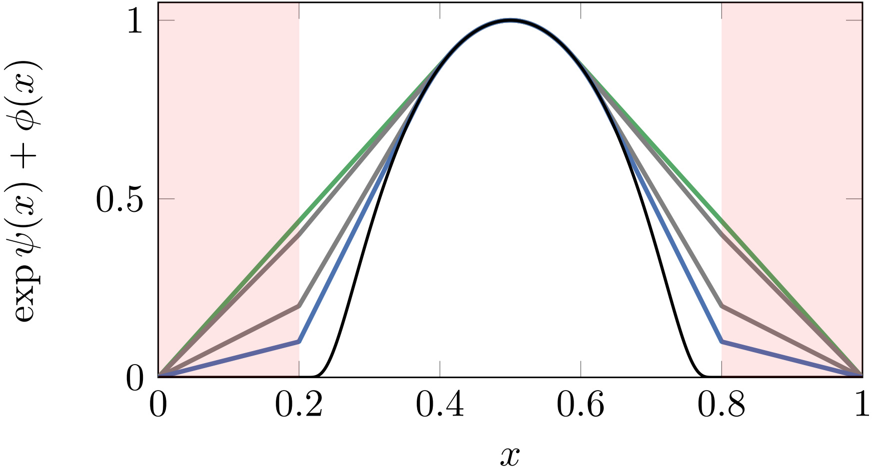
Light gradient constraints.

We set $f = 0$ and vary the gradient constraints.



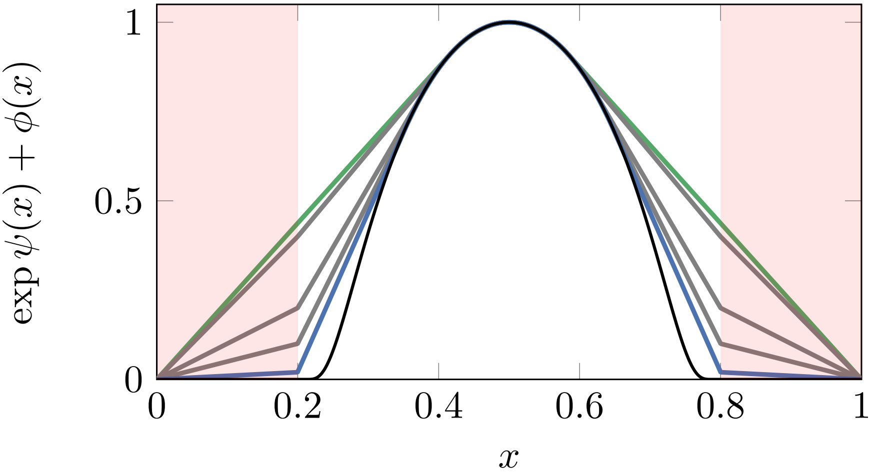
Medium gradient constraints.

We set $f = 0$ and vary the gradient constraints.



Heavy gradient constraints.

We set $f = 0$ and vary the gradient constraints.



Extreme gradient constraints.

Section 5

Eigenvalue constraints

Eigenvalue constraints are very important, but very difficult to enforce numerically.

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The Landau–de Gennes model of nematic liquid crystals minimises

$$J(Q) = \frac{1}{2}\int_\Omega \nabla Q \,\vdots\, \nabla Q \ \mathrm{d}x + \frac{1}{2}\int_\Omega A\,\mathrm{tr}(Q^2) \ \mathrm{d}x + \frac{1}{4}\int_\Omega C(\mathrm{tr}(Q^2)^2) \ \mathrm{d}x$$

for a symmetric traceless matrix field Q.

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for a symmetric traceless matrix field Q.

To be physical, Q must satisfy eigenvalue constraints ($n =$ spatial dimension)

$$\lambda_i(Q) \in [-1/n, (n-1)/n], \quad i = 1, \ldots, n,$$

but this is usually ignored as too difficult.

Fix $n = 2$ for simplicity. We employ as Legendre function

$$R(A) = \operatorname{tr}\big((A + I/2)\log(A + I/2) + (I/2 - A)\log(I/2 - A)\big),$$

$$\text{with } \nabla R^*(A^*) = \operatorname{tanhm}(A^*/2)/2,$$

where $\log$ and tanhm are the matrix logarithm and hyperbolic tangent functions.

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where $\log$ and tanhm are the matrix logarithm and hyperbolic tangent functions.

The LVPP iteration becomes: find $(Q^k, \psi^k) \in H^1_D(\Omega, \mathbb{R}^{2\times 2}_{\text{sym,tr}}) \times L^\infty(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym,tr}})$ s. t.

$$\alpha_k J'(Q; V) + (\psi^k, V) = (\psi^{k-1}, V)$$
$$(Q, w) - (\frac{1}{2}\operatorname{tanhm}(\psi/2), w) = 0$$

for all $(V, w) \in H^1_0(\Omega, \mathbb{R}^{2\times 2}_{\text{sym,tr}}) \times L^\infty(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym,tr}})$.

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where $\log$ and tanhm are the matrix logarithm and hyperbolic tangent functions.

The LVPP iteration becomes: find $(Q^k, \psi^k) \in H^1_D(\Omega, \mathbb{R}^{2\times 2}_{\text{sym,tr}}) \times L^\infty(\Omega, \mathbb{R}^{2\times 2}_{\text{sym,tr}})$ s. t.

$$\alpha_k J'(Q; V) + (\psi^k, V) = (\psi^{k-1}, V)$$

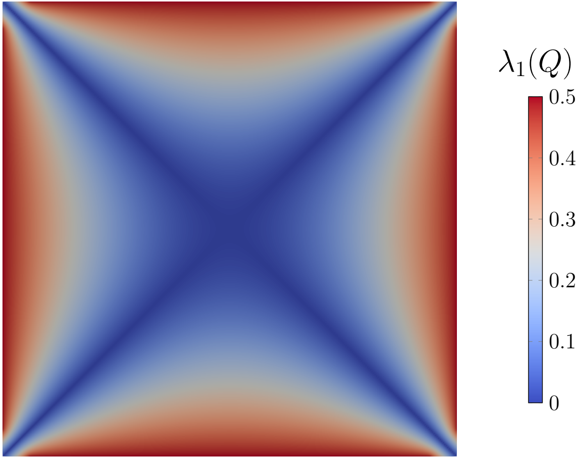
$$(Q, w) - (\frac{1}{2}\operatorname{tanhm}(\psi/2), w) = 0$$

for all $(V, w) \in H^1_0(\Omega, \mathbb{R}^{2\times 2}_{\text{sym,tr}}) \times L^\infty(\Omega, \mathbb{R}^{2\times 2}_{\text{sym,tr}})$.

All of this extends straightforwardly to $n > 2$.

Good news

Mesh-independent convergence, $\sim$ 6 proximal steps, $\sim$ 11 Newton iterations.



The larger eigenvalue $\lambda_1(Q)$. Both eigenvalues satisfy the inequality constraints.

Section 6

Multiple solutions

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Given a *set* S of solutions to the previous iteration, solve

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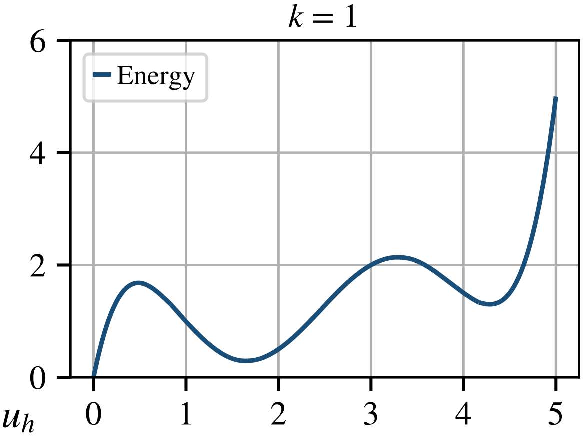
We can search for multiple solutions of this nonsmooth PDE at every LVPP iteration.

Use $u = 1$ as initial guess for the toy problem

$$u \in \underset{v \in [0,\infty)}{\text{argmin}} \; J(v)$$



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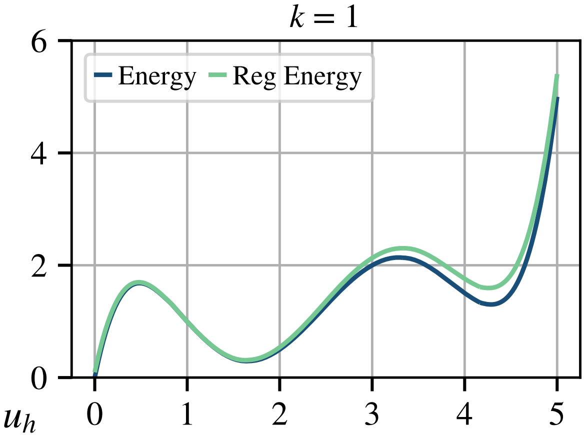

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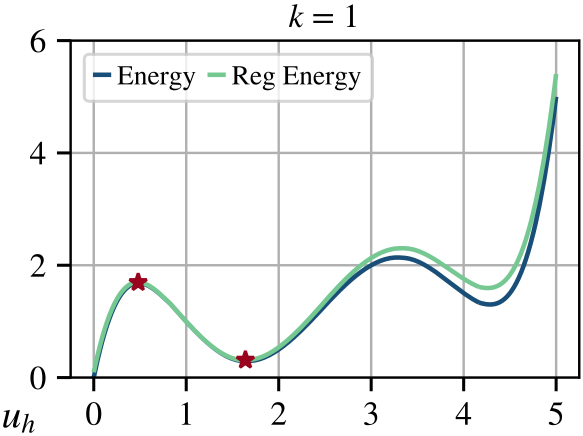

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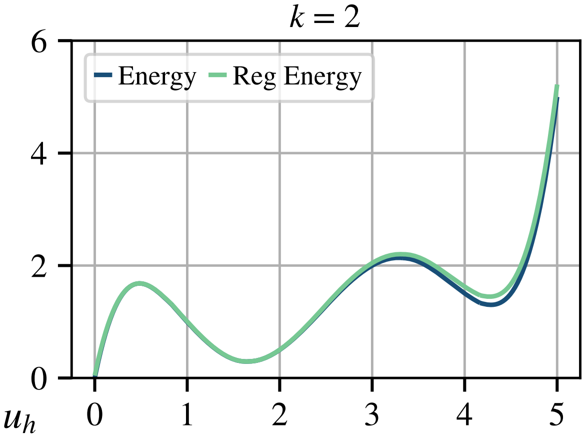

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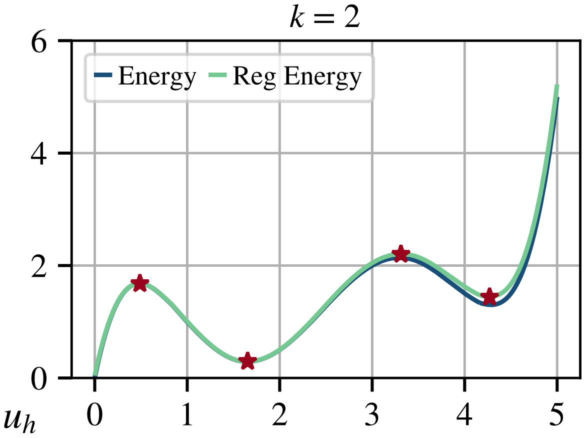

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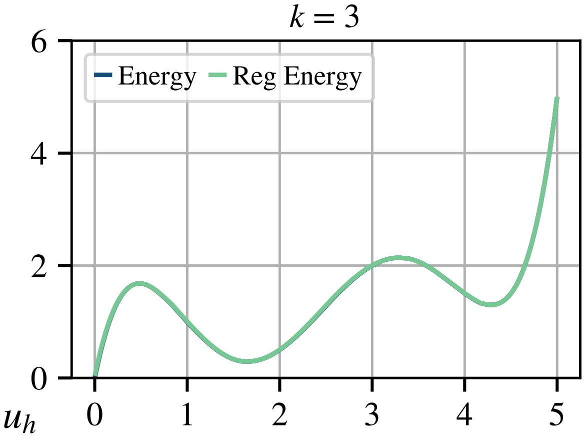

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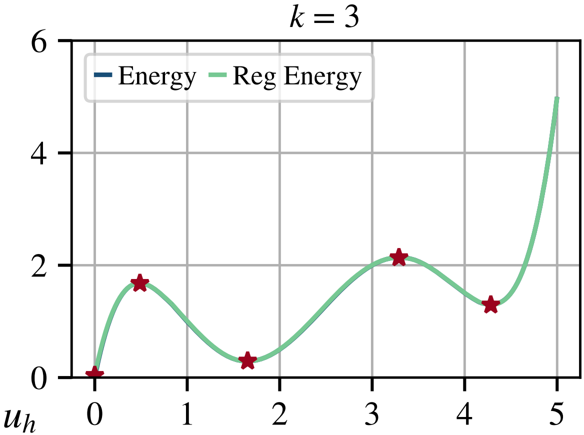
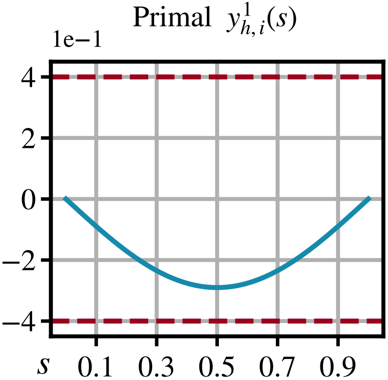


Consider the Zeidler beam problem

$$u \in \underset{v \in H^2(0,L) \cap H^1_0(0,L)}{\text{argmin}} \quad J(v) = \int_0^L B(v'')^2 - P(v')^2 - \rho g v \, \mathrm{d}x$$

subject to $|y| \leq \alpha = 0.4$.



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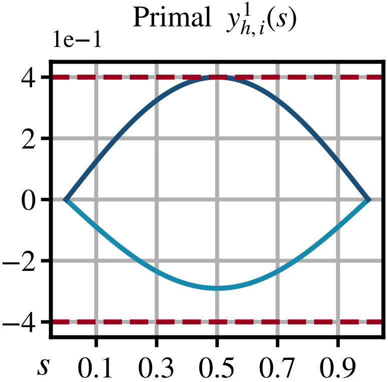

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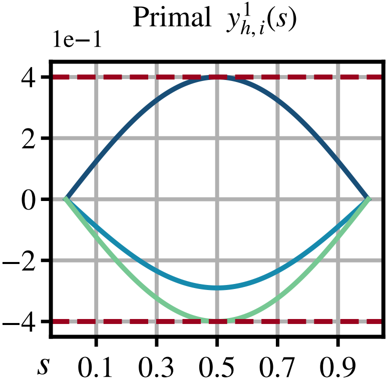

Section 7

Conclusions

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Good news

Many open questions remain! Proofs, discretisations, solvers, nonconvex constraints, ….