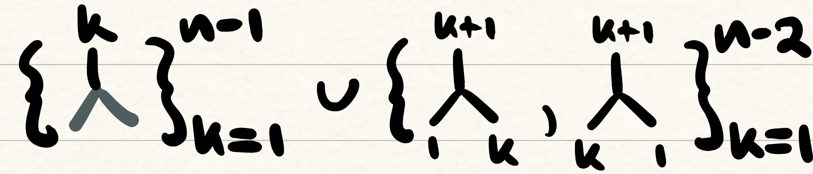
Type B Webs

+ Spin Link Homology

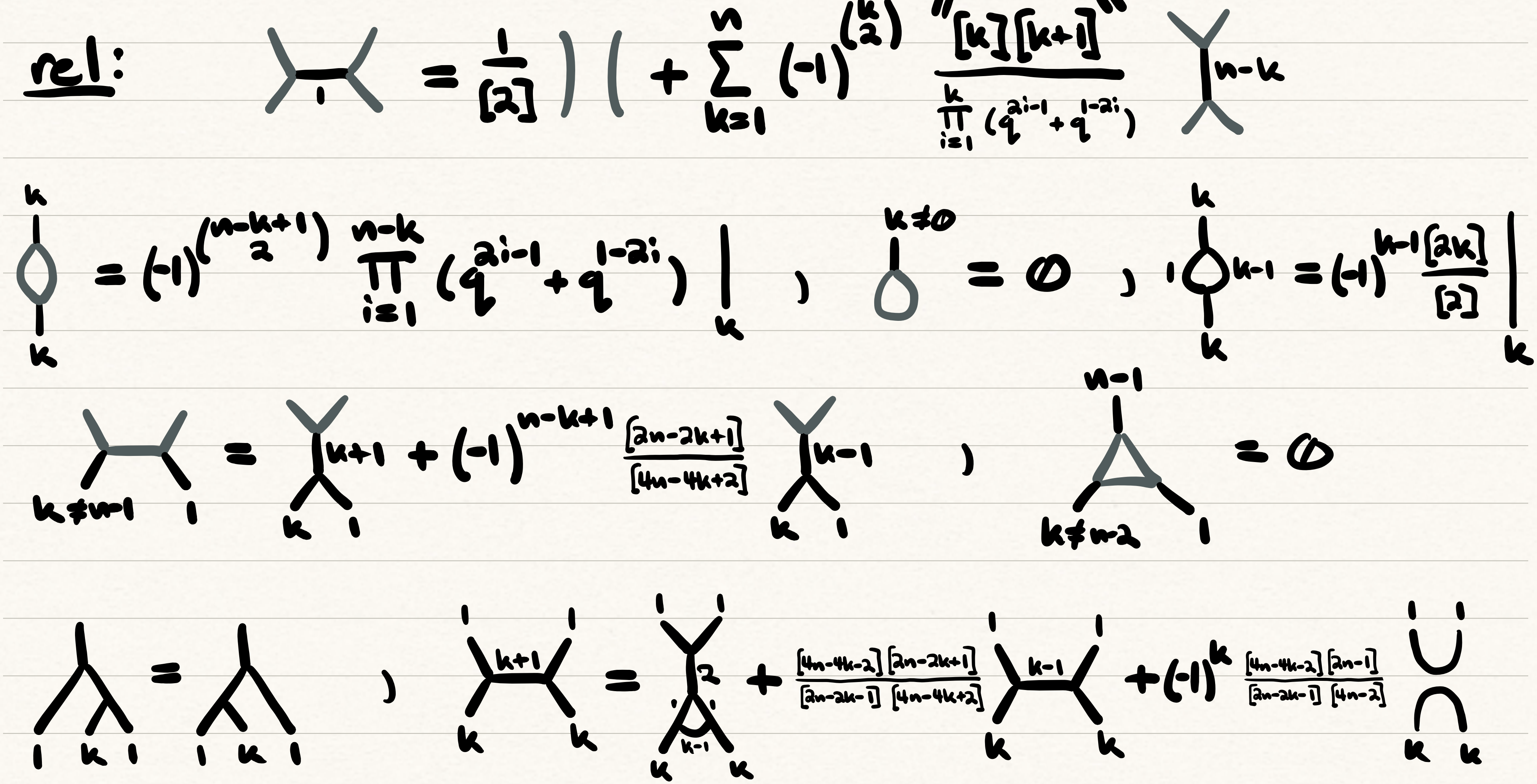
(joint w/ Bodish + Elias)

arXiv:2601.???? ± ε + 2407.00189

def: $\mathrm{Web}(so_{2n+1})$ is the $\mathbb{C}(q)$-linear pivotal category w/

gen: ob. $\{1, 2, \dots, n-1, S\}$, mor. 

rel:



thm (Bodish-Elias-R.): $\mathrm{Web}(\mathfrak{so}_{2n+1}) \cong \mathrm{FRep}(U_q(\mathfrak{so}_{2n+1}))$

history: analogous thm for $\mathfrak{g} = \mathfrak{sl}_2$ Folklore ($\approx$ '32)

inform our presentation $\leftarrow$
- $\mathfrak{sl}_3$, $\mathfrak{so}_5 \cong \mathfrak{sp}_4$, $\mathfrak{g}_2$ Kuperberg '96
- $\mathfrak{sl}_n$ Cautis-Kamnitzer-Morrison '12
- $\mathfrak{sp}_{2n}$ Bodish-Elias-R.-Tatham '21

↑ today ↓

proof (sketch): ① find presentation
② construct functor
③ prove fully faithful ▨

uses a) diagrammatic / topological arguments
b) duality $U'_{-q^2}(\mathfrak{so}_m) \twoheadrightarrow \mathrm{End}_{U_q(\mathfrak{so}_{2n+1})}(S^{\otimes m})$

[Wenzl '20]

an observation: $\mathrm{Web}(\mathfrak{sl}_4)$ vs. $\mathrm{Web}(\mathfrak{so}_5)$

$$\bigcirc_2 = q^4+q^2+2+q^{-2}+q^{-4} = q^{-4}\sum_{\lambda\subseteq\boxplus} q^{2|\lambda|}$$

$$\bigcirc_S = -(q^4+q^2+q^{-2}+q^{-4}) = -q^{-4}\sum_{\substack{\lambda\subseteq\boxplus\\ \lambda=\lambda^T}} q^{2|\lambda|}$$

mitigated by categorification:

$$KhR_{\mathfrak{sl}_{2n}}(\bigcirc_n) \stackrel{\wedge}{=} q^{-n^2}H^*(Gr_n(\mathbb{C}^{2n})) \cong \Bbbk\{s_\lambda \mid \lambda\subseteq \boxplus_{n\times n}\} \circlearrowleft \tau:\lambda\mapsto\lambda^T$$

$\dim_q$:

$$\bigcirc_n = \begin{bmatrix}2n\\n\end{bmatrix} \quad \text{in } \mathrm{Web}(\mathfrak{sl}_{2n})$$

tr_q:

$$\text{“}\begin{bmatrix}2n\\n\end{bmatrix}\text{”} := \prod_{i=1}^{n}(q^{2i-1}+q^{1-2i}) = \pm\,\bigcirc_S \quad \text{in } \mathrm{Web}(\mathfrak{so}_{2n+1})$$

topological generalization

can be defined via $\mathrm{Foam}^+(\mathfrak{sl}_{2n})$ [Queffelec-R.]
which categorifies $\mathrm{Web}^+(\mathfrak{sl}_{2n})$

thm (Bodish-Elias-R.): let $\mathcal{L} \subseteq S^3$ be a link

① $\forall n \geq 1$, $\exists$ involution $\tau_n \circlearrowright \mathrm{KhR}_{\mathfrak{sl}_{2n}}(\mathcal{L}^n)$

$\rightsquigarrow$ eigenspaces are link invts $SH_n^+(\mathcal{L})$, $SH_n^-(\mathcal{L})$

② for $n = 1,2,3$ (and $n \geq 4$, assuming technical conj.)

$$P_{\mathfrak{so}_{2n+1}}(\mathcal{L}^S) = \mathrm{tr}_q(\tau_n)$$

$$= \dim_q(SH_n^+(\mathcal{L})) - \dim_q(SH_n^-(\mathcal{L}))$$

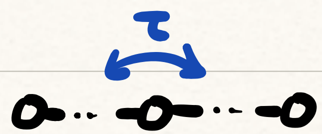
defined via braiding on $\mathrm{Rep}(U_q(\mathfrak{so}_{2n+1}))$

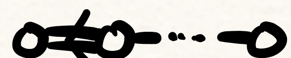
meta-conj: $\mathfrak{g}$ simply laced, τ Dynkin automorphism $\Longrightarrow$ "equivariant (bi)category" $\mathcal{C}^\tau$

$$\tau \circlearrowright \mathcal{C} \subseteq \mathrm{Foam}(\mathfrak{g}) \xrightarrow{K_0} \mathrm{Web}(\mathfrak{g}) \quad \rightsquigarrow \quad \mathcal{C}^\tau \xrightarrow{K_0^\tau} \mathrm{Web}(\mathfrak{g}_\tau)$$

Foam/Web analogue of "folding":
[Jantzen, Lusztig]

$\mathfrak{g} \xrightarrow{\text{fold root system}} \mathfrak{g}_\tau$
- assoc. to root system w/ simple roots $= (2\cdot)\sum_{\text{orbits}} \alpha_i$
- weights $= \tau$-invt weights

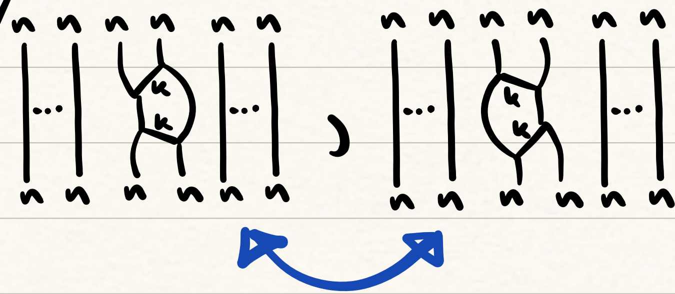
A_{2n-1}: 

B_n: 

$\mathfrak{sl}_{2n} \longrightarrow \mathfrak{so}_{2n+1}$

$\Lambda^n(\mathbb{C}^{2n}) \qquad S$

rigorous instantiation

$$\mathcal{C}_n^m := \Big\langle$$  $$\Big\rangle_{0 \le k \le n} \xrightarrow[\text{sub-bicategory}]{\text{full}} \mathbf{Foam}^+(\mathfrak{sl}_{2n})$$

[Queffelec-R.]

↑ monoidal category

thm (Bodish-Elias-R.): let $m, n \geq 1$

① $\exists$ non-trivial involution $\tau_n \circlearrowright \mathcal{C}_n^m$

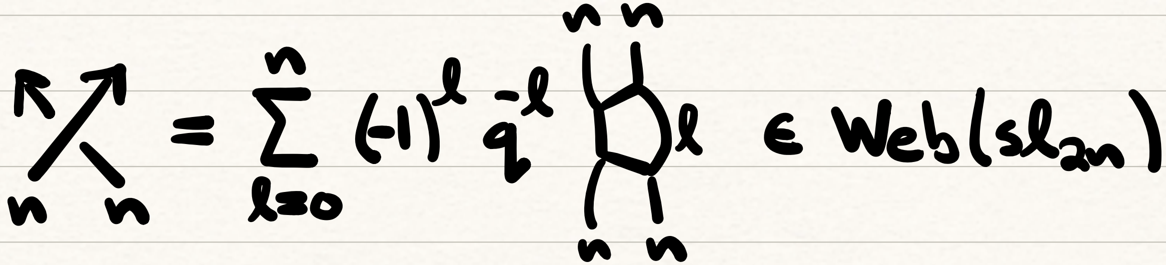
② let $(\mathcal{C}_n^m)^\tau$ = equivariant category

then $\exists$ alg. hom.

$$\mathrm{End}_{U_q(\mathfrak{so}_{2n+1})}(S^{\otimes m}) \longrightarrow K_0^\tau\big((\mathcal{C}_n^m)^\tau\big)$$

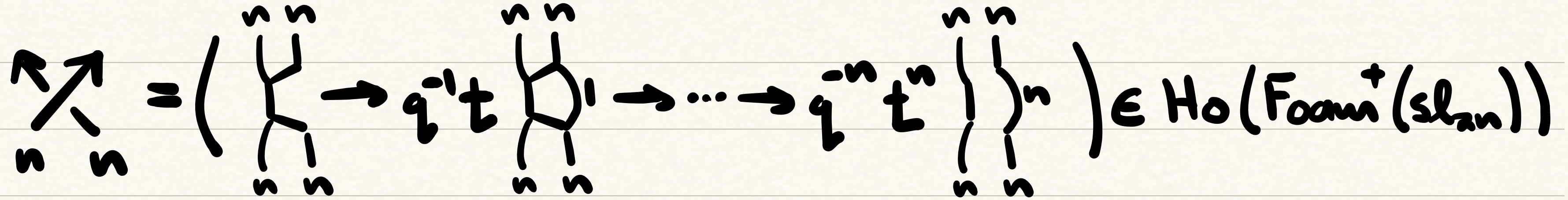
③ when $m=2$, this is an iso. (conj: always)

implication for webs



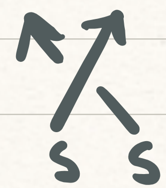
$\updownarrow K_0$

"Rickard complex"



thm (Bodish-Elias-R.): $\exists$ equivariant lift to $\mathrm{Ho}((\mathcal{C}_n^2)^{\tau})$

$\Rightarrow \exists\, X^{(k)} \in \mathrm{End}_{U_q(so_{2n+1})}(S \otimes S)$ s.t.



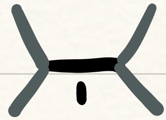
$= \sum_{l=0}^{n} \pm q^{-l} X^{(n-l)}$

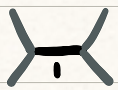
also, must have $X^{(k)} \cdot X^{(1)} = (-1)^k \left(\text{"}[k+1]^2\text{"}\, X^{(k+1)} + \text{"}[k][k+1]\text{"}\, X^{(k)} \right)$

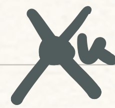
and $X^{(n+1)} = 0$

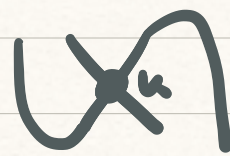
implication for webs

$$\mathrm{Web}(\mathfrak{so}_{2n+1}) \cong \mathrm{FRep}(U_q(\mathfrak{so}_{2n+1}))$$

 $- \frac{1}{[2]} \,)(\, \longleftrightarrow X^{(1)}$

 $= \frac{1}{[2]} \,)(\, + \sum_{k=1}^{n} (-1)^{\binom{k}{2}} \frac{\text{"}[k][k+1]\text{"}}{\prod_{i=1}^{k} (q^{2i-1} + q^{1-2i})}$  $\longleftrightarrow X^{(n+1)} = 0$

 $\longleftrightarrow X^{(k)}$

conj (Bodish-Elias-R.):  = 

holds when $k=0,n$ and when $n=1,2$

$\Rightarrow$ $\{$  $\}_{k=0}^{n}$ is rot. invt. basis for $\mathrm{End}_{\mathrm{Web}(\mathfrak{so}_{2n+1})}(S \otimes S)$