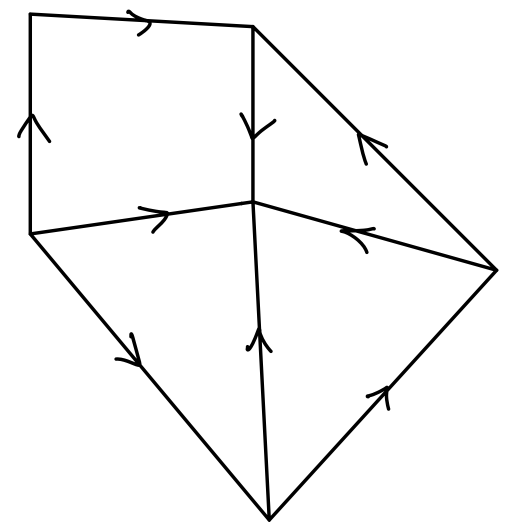
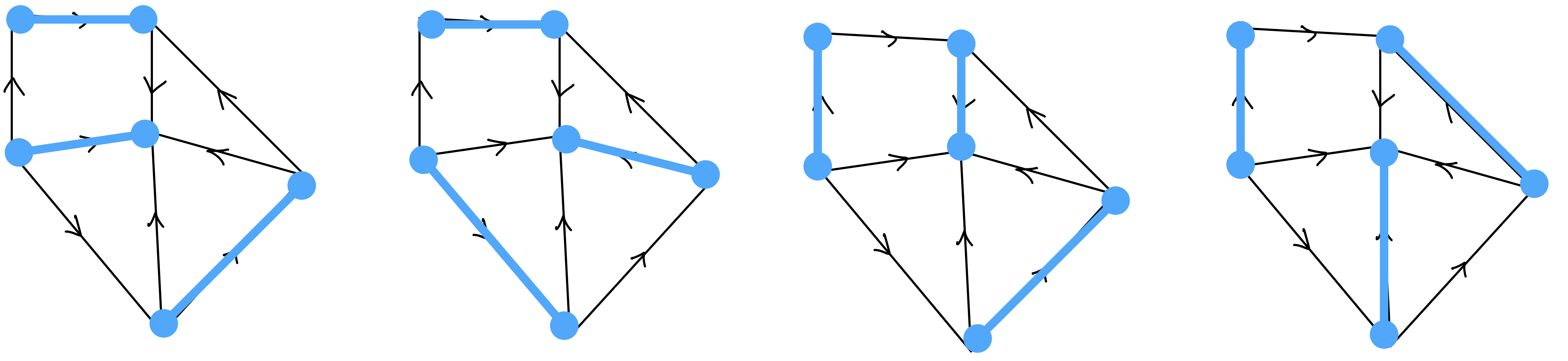
Richard Kenyon (Yale)

based on joint work with

Daniel Douglas, Nicholas Ovenhouse, Haolin Shi, David Wilson, Haihan Wu

Dimer covers

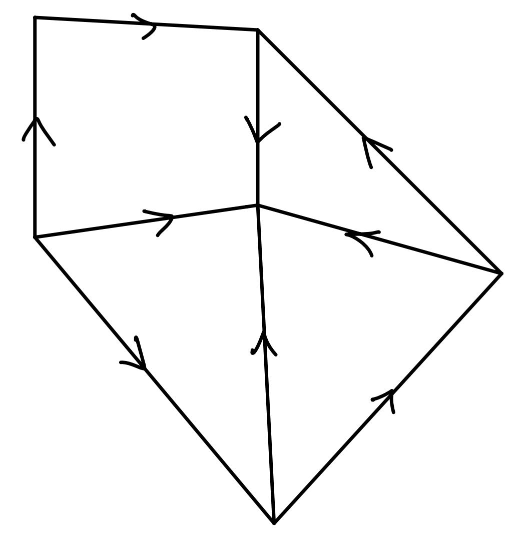




Dimers and Kasteleyn theory

Let G be a planar graph.

Choose a "clockwise odd" orientation: each face has an odd number of arrows oriented in the clw direction.



Let K be the Kasteleyn matrix: $K : \mathbb{R}^V \to \mathbb{R}^V$

$$K_{uv} = \begin{cases} 1 & u \to v \\ -1 & v \to u \\ 0 & \text{else.} \end{cases}$$

Thm [Kasteleyn, 1965]: $|\mathrm{Pf}(K)| = \#\{\text{dimer covers}\}$

Recall that for an antisymmetric matrix K,

$$\operatorname{Pf} K = \sum_{\sigma} (-1)^{\sigma} K_{\sigma(1)\sigma(2)} \dots K_{\sigma(2n-1)\sigma(2n)}$$

where the sum is over pairings

$$\{\sigma(1), \sigma(2)\}, \dots, \{\sigma(2n-1), \sigma(2n)\},$$

with $\sigma(2i-1) < \sigma(2i)$.

$$\operatorname{Pf} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Bipartite case

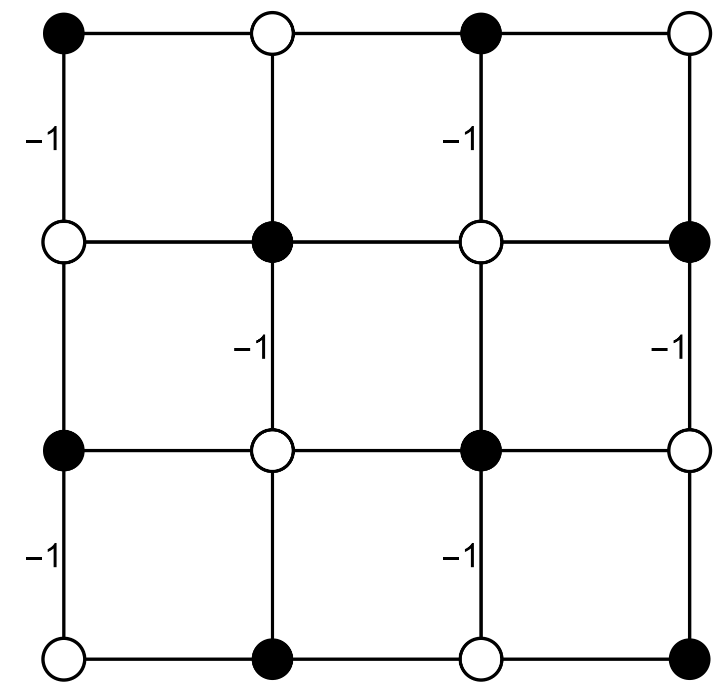
Let G be a planar, bipartite graph.

Let K be the Kasteleyn matrix: $K : \mathbb{C}^B \to \mathbb{C}^W$

"Adjacency matrix with Kasteleyn connection"

$$K_{wb} = \begin{cases} \pm 1 & w \sim b \\ 0 & \text{else.} \end{cases}$$

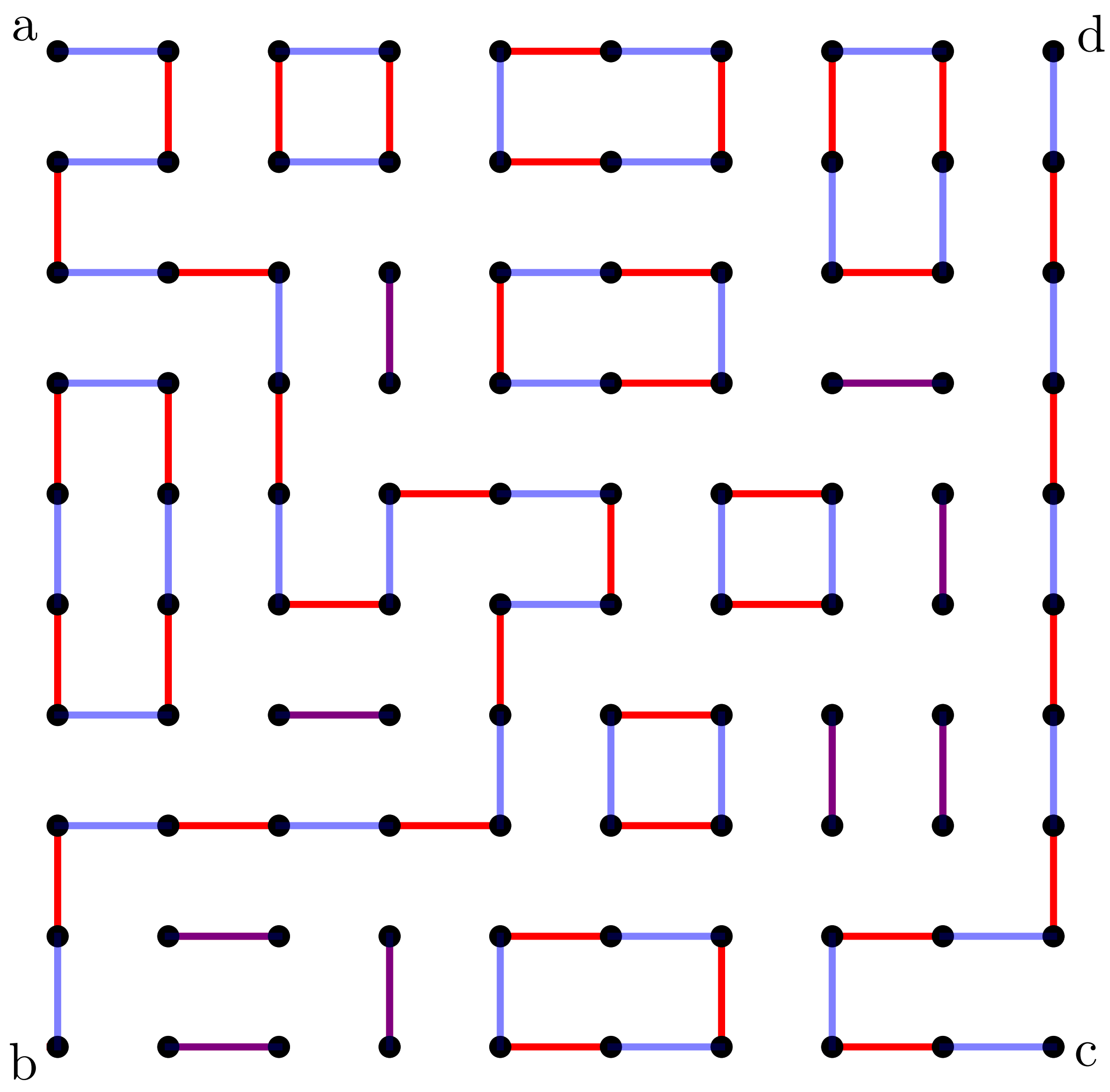
where a face of length l has monodromy $(-1)^{l/2+1}$.



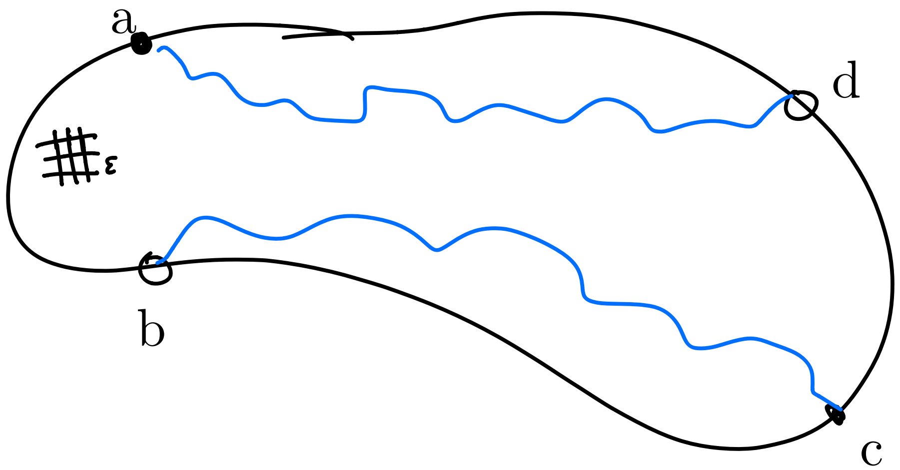
Kasteleyn, Temperley/Fisher (1963) proved

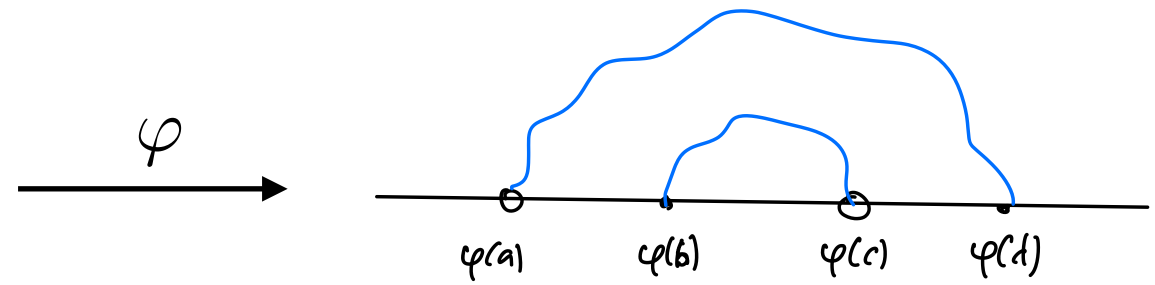
Thm: $|\det K| = \#\{\text{dimer covers}\}$

Connection probabilities in double-dimers (**2**-multiwebs)



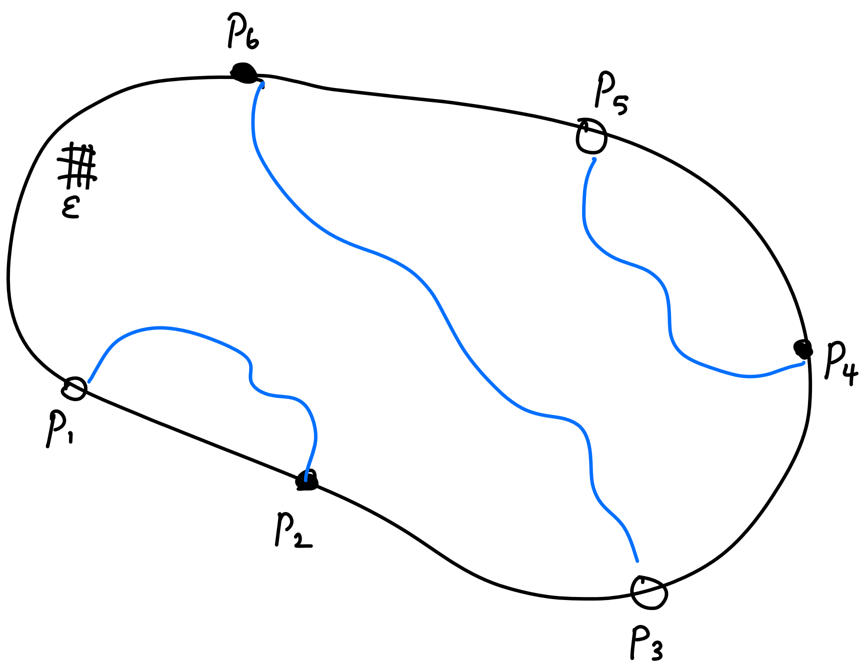
Take two dimer covers of a rectangle, one of which misses the four corners. What is the probability that, in the union, the corner connection goes top-to-bottom?

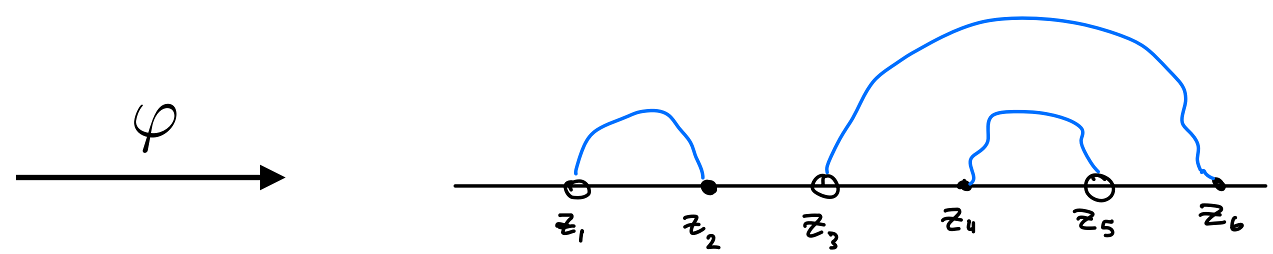




$$Pr = \frac{(\varphi(b) - \varphi(a))(\varphi(d) - \varphi(c)))}{(\varphi(c) - \varphi(a))(\varphi(d) - \varphi(b))}$$

Thm[K-Wilson '06] In the scaling limit for a domain with $2n$ marked boundary points $p_1, \ldots, p_{2n}$ (and appropriate boundary conditions) each connection probability is an explicit rational function of the $\phi(p_i)$'s.

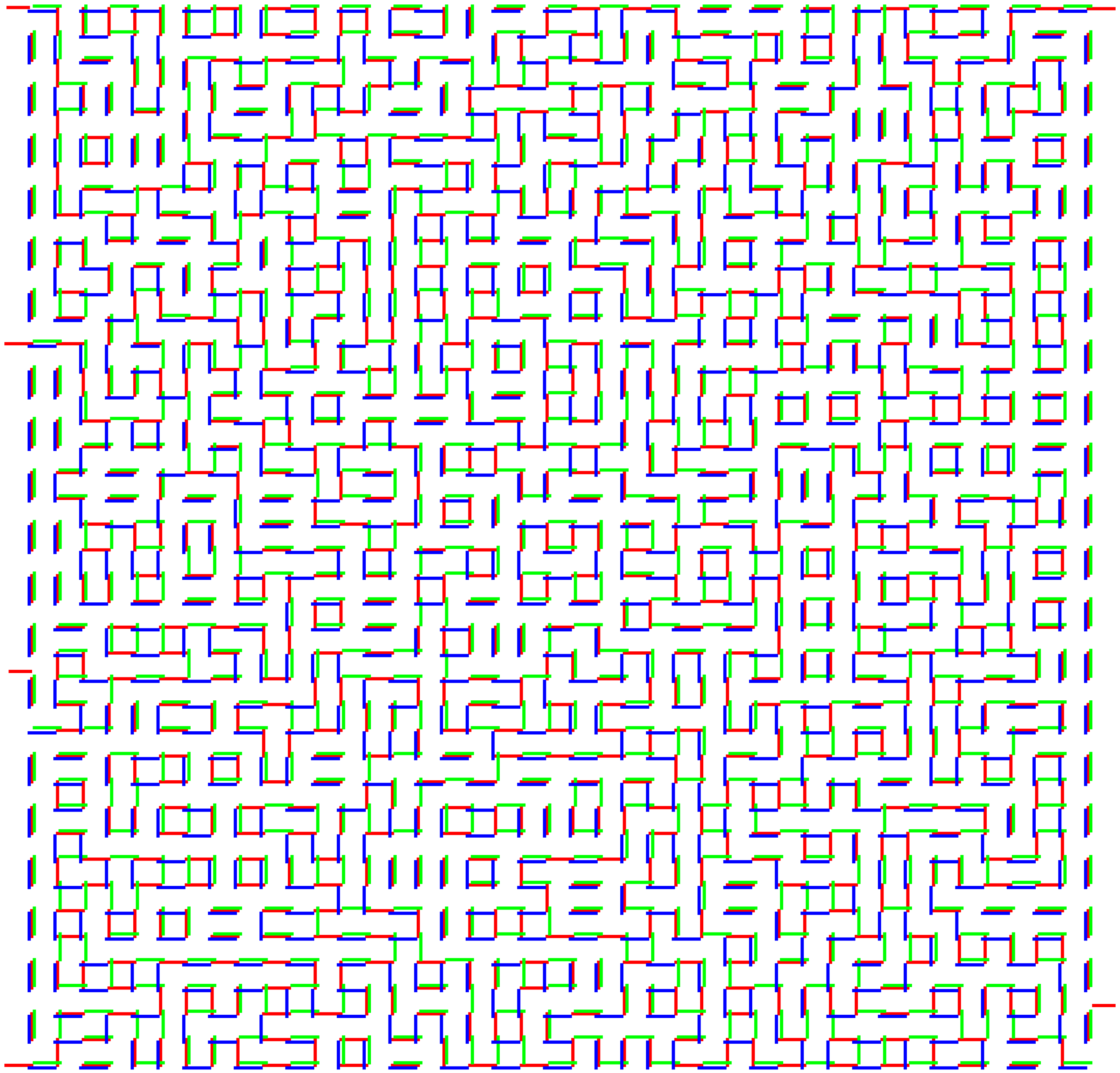




$$z_i = \varphi(p_i)$$

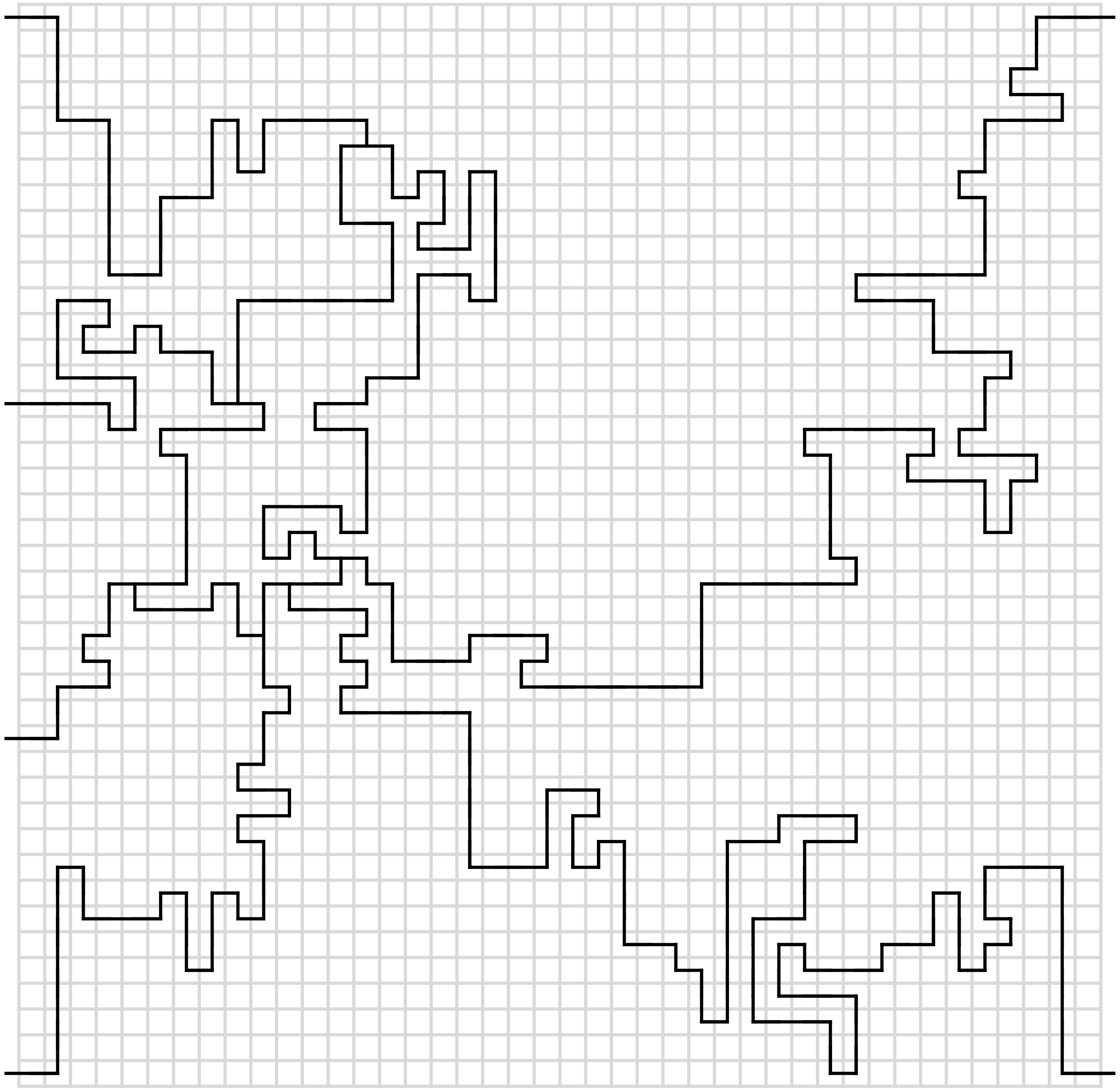
$$Pr = \frac{(z_3 - z_2)(z_4 - z_3)(z_6 - z_1)(z_6 - z_5)}{(z_3 - z_1)(z_5 - z_3)(z_6 - z_2)(z_6 - z_4)}$$

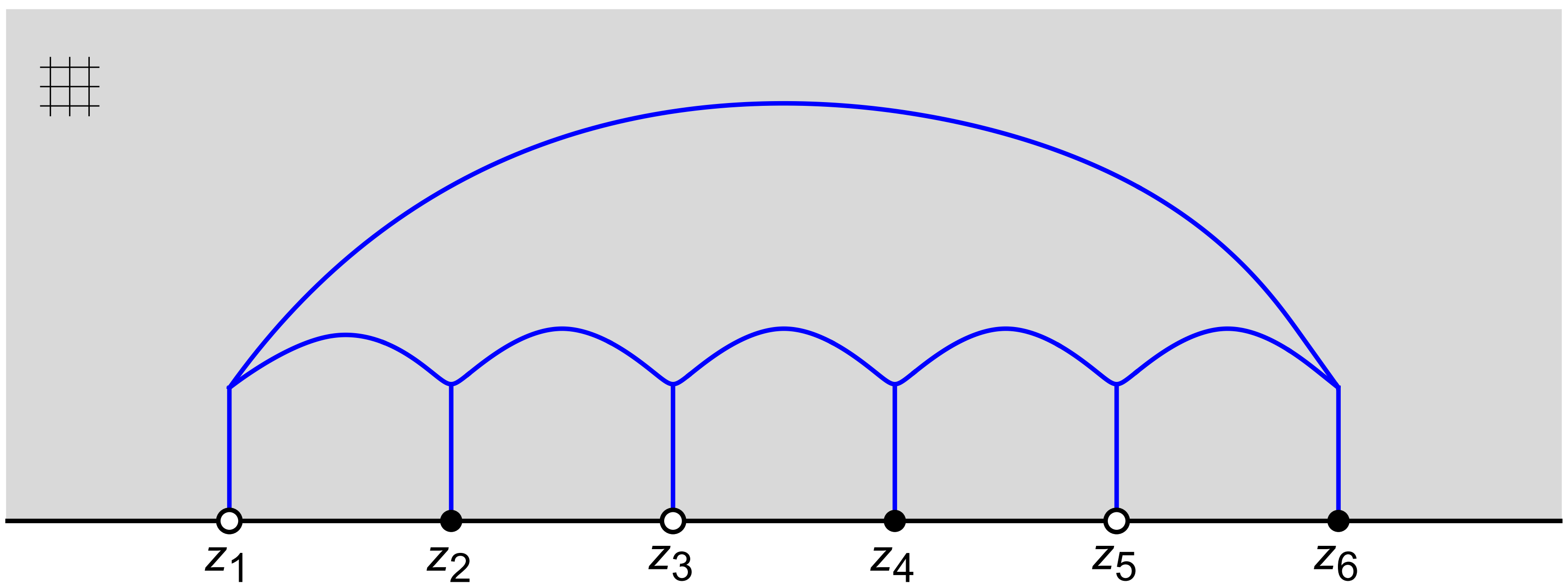
Q. What happens when we superpose multiple dimer covers?

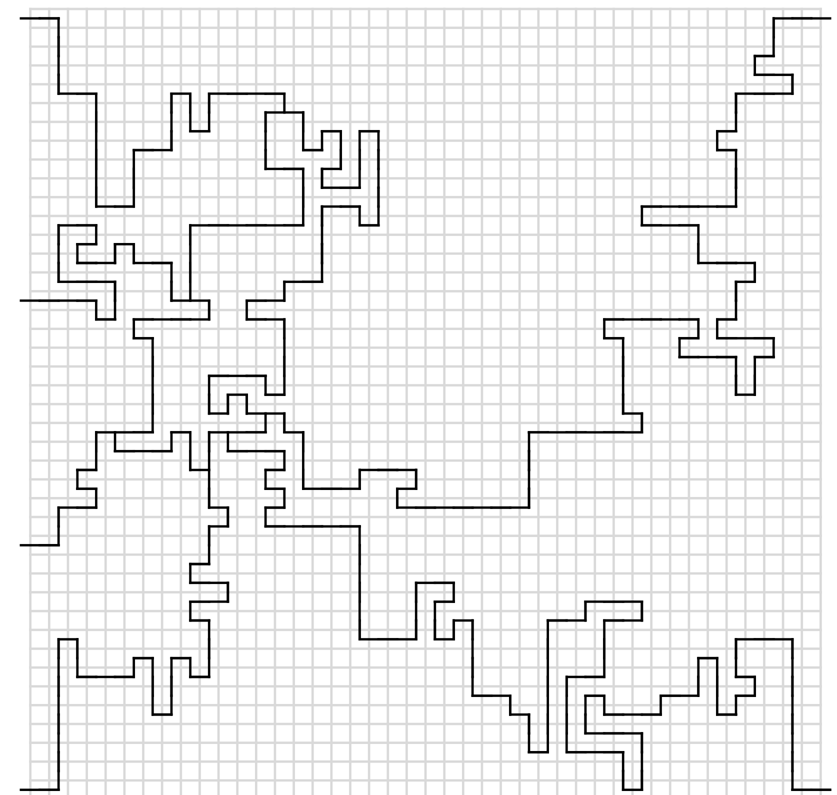


internal structure?

reduce...







In scaling limit,

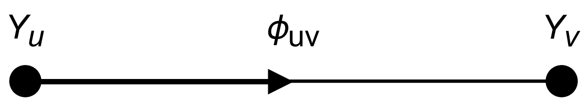
$$\mathrm{Pr} = \frac{2(z_2 - z_1)(z_3 - z_2)(z_4 - z_3)(z_5 - z_4)(z_6 - z_5)(z_6 - z_1)}{(z_3 - z_1)(z_4 - z_2)(z_5 - z_3)(z_6 - z_4)(z_5 - z_1)(z_6 - z_2)}$$

Graph connections

Let $G = (V, E)$ be a planar graph.

Assign to a vertex v a vector space $Y_v \cong \mathbb{R}^n$.

A *connection* is a collection of linear maps $\Phi = \{\phi_{uv}\}_{uv \in E}$ with $\phi_{vu} = \phi_{uv}^{-1}$.



Two connections Φ, Φ' are *gauge equivalent* if

$$\phi'_{uv} = g_v \phi_{uv} g_u^{-1}$$

for some maps $g_v : Y_v \to Y_v$.

For a matrix group H, Φ is an *H-connection* if Φ is gauge equivalent to a connection with values in H.

Ex. $H = \mathrm{SL}_n \quad H = \mathrm{SO}(n), \quad H = \mathrm{Sp}(2n)$

These three families of groups have invariant bilinear pairings

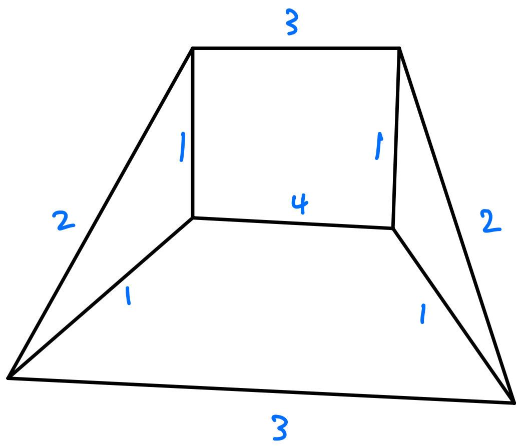
$$\mathrm{SL}_n \quad Y \otimes Y^* \to \mathbb{R} : \quad u \otimes v^* \to v^*(u)$$

$$\mathrm{SO}_n \quad Y \otimes Y \to \mathbb{R} : \quad u \otimes v \to u \cdot v$$

$$\mathrm{Sp}_{2n} \quad Y \otimes Y \to \mathbb{R} : \quad u \otimes v \to \omega(u, v)$$

Multiwebs

An *n-multiweb* in G is a function $m : E \to \mathbb{Z}_{\geq 0}$ summing to n at each vertex v:



$\Omega_{\mathbf{n}}$ is the set of $\mathbf{n}$-multiwebs.

Ex: For $\mathbf{n} \equiv 1$, $\Omega_1 = \{\text{dimer covers}\}$

Ex: Superposing n dimer covers gives an n-multiweb.

Prop: If G is bipartite, every n-multiweb is a superposition of n dimer covers.

Generalized Kasteleyn Theorem

We define a trace function $\mathrm{Tr} = \mathrm{Tr}_\Phi : \Omega_n \to \mathbb{R}$ and a matrix $\tilde{K}(\Phi)$ (later) so that

Thm [Douglas-K-Shi, K-Ovenhouse-Wu]: For an H-connection Φ on a postively ciliated planar graph,

$$\mathrm{Pf}\tilde{K}(\Phi) = \pm \sum_{m \in \Omega_n} \mathrm{Tr}(m).$$

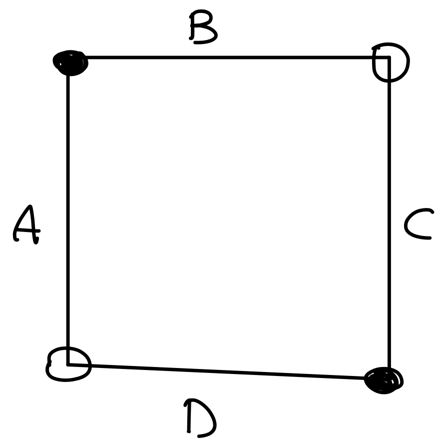
SL_n case

Let G be a bipartite planar graph with SL_n connection Φ. Define a Kasteleyn matrix $K = K(\Phi)$:

$$K(w,b) = \begin{cases} \pm\phi_{bw} & b \sim w \\ 0 & \text{else.} \end{cases}$$

"tensor Φ with the Kasteleyn connection."

Ex.

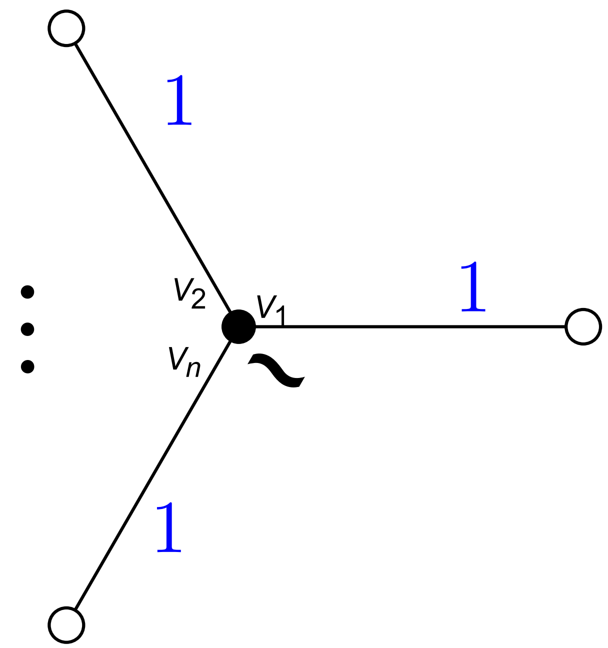


$$K(\Phi) = \begin{pmatrix} A & -D \\ B & C \end{pmatrix}$$

$$\tilde{K}(\Phi) = \begin{pmatrix} a_{11} & a_{21} & -d_{11} & -d_{21} \\ a_{12} & a_{22} & -d_{12} & -d_{22} \\ b_{11} & b_{21} & c_{11} & c_{21} \\ b_{12} & b_{22} & c_{12} & c_{22} \end{pmatrix}$$

Trace of an n-multiweb

First assume $m_e = 0$ or 1 for all edges



$V_i \cong Y$ with basis $e_1, \ldots, e_n$

Define $v_b \in V_1 \otimes \cdots \otimes V_n$ by $$v_b = \sum_{\sigma \in S_n} (-1)^\sigma e^1_{\sigma(1)} \otimes \cdots \otimes e^n_{\sigma(n)}$$

the "codeterminant"

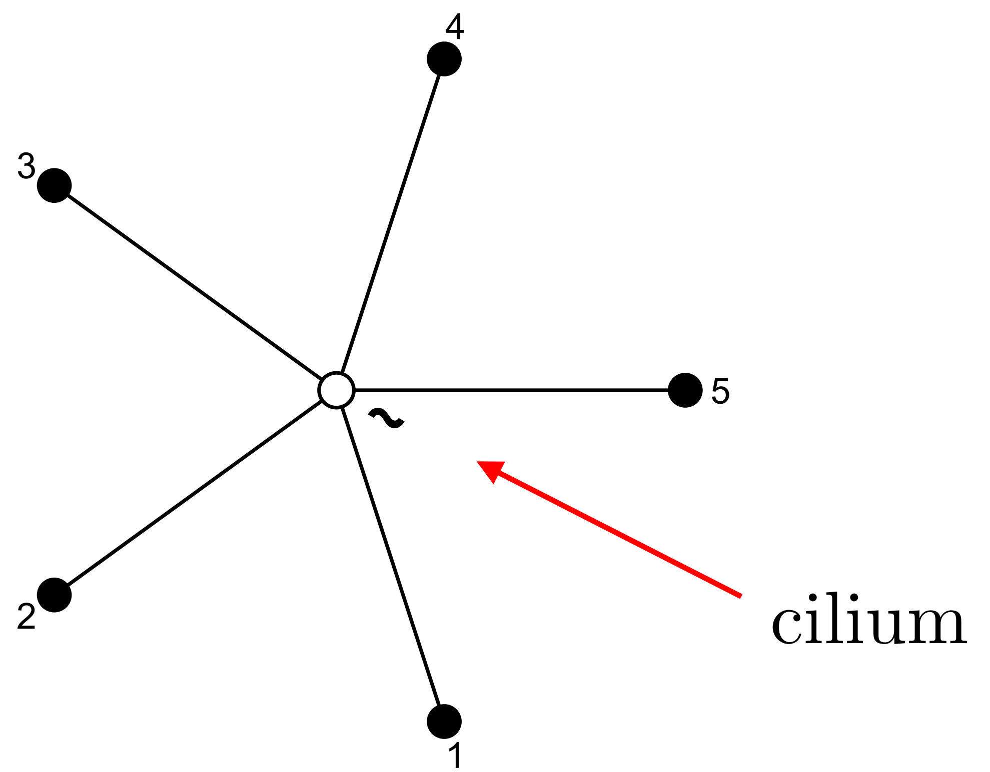
invariant under SL_n-base change

Similarly define v_w using Y^*.

Then define $$Tr(m) = \left\langle \bigotimes_{w \in W} v_w \middle| \bigotimes_{e=wb} \phi_{wb} \middle| \bigotimes_{b \in B} v_b \right\rangle$$

Ciliations

We need a linear order of the edges out of each vertex: use the circular order, plus a starting edge, at black vertices, and the anticircular order, plus starting edge, at white vertices.

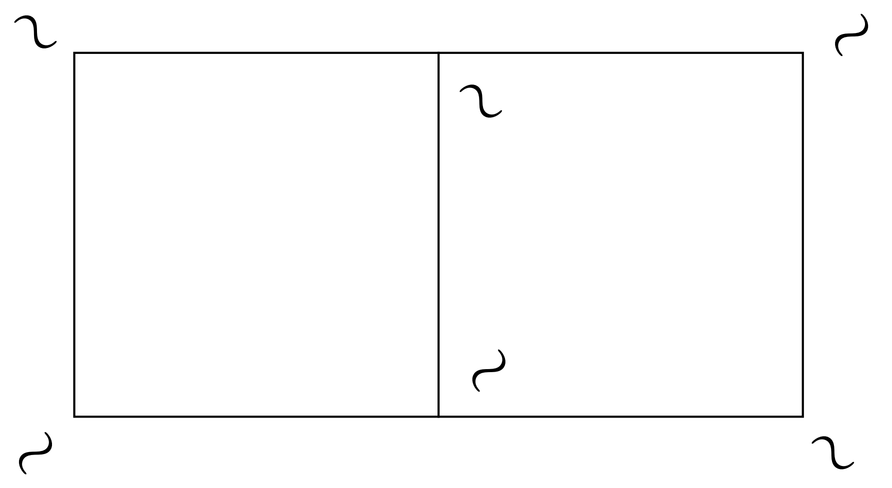


(In non-bipartite cases, use circular order at all vertices.)

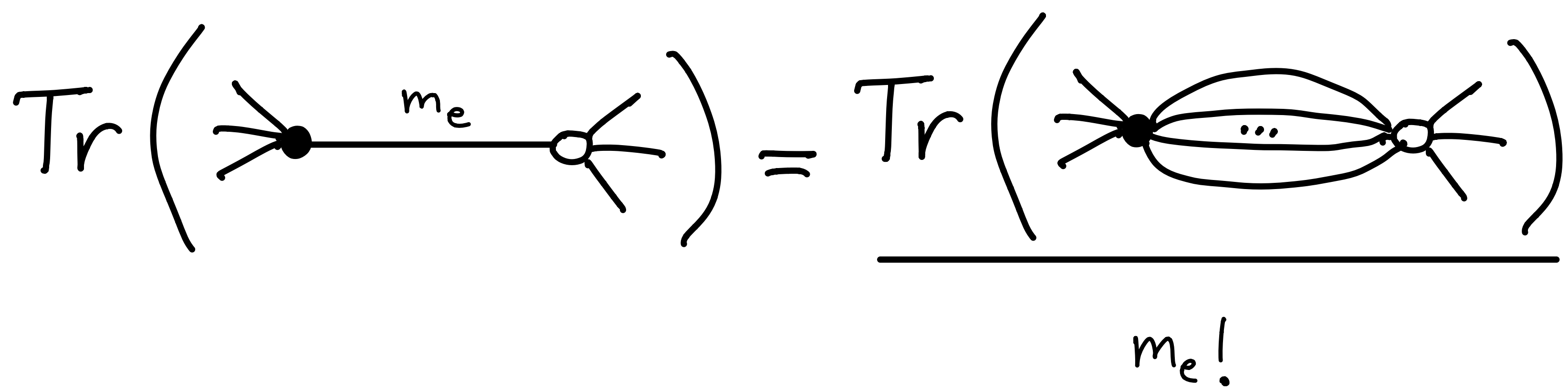
If n is even, the sign of the trace depends on this linear order.

If n is odd, trace is independent of cilia.

A ciliation is *positive* if each face contains an even number of cilia.



If edges have multiplicity > 1:



Thm[Douglas-K-Shi]: For an SL_n-connection Φ on a (positively ciliated) bipartite planar graph,

$$\det \tilde{K}(\Phi) = \pm \sum_{m \in \Omega_n} \mathrm{Tr}(m).$$

Thm[K-Ovenhouse-Wu]: For an SO_n-connection Φ on a (positively ciliated) planar graph,

$$\mathrm{Pf}\tilde{K}(\Phi) = \pm \sum_{m \in \Omega_n} \mathrm{Tr}(m).$$

Thm[K-Wu]: For an Sp_{2n}-connection Φ on a planar graph with standard orientation and cilia,

$$\mathrm{Pf}\tilde{K}(\Phi) = \pm \sum_{m \in \Omega_n} \mathrm{Tr}(m).$$

SO(n) case

The planar graph G is not necessarily bipartite.

The trace of an n-multiweb is defined as for SL_n but the tensor contraction uses the *inner product* rather than the duality.

Choose a (clockwise odd) Kasteleyn orientation of edges of G. Define

$$K(u,v) = \begin{cases} \phi_{uv} & u \to v \\ -\phi_{uv} & v \to u \\ 0 & \text{else} \end{cases}.$$

Note if $u \to v$,

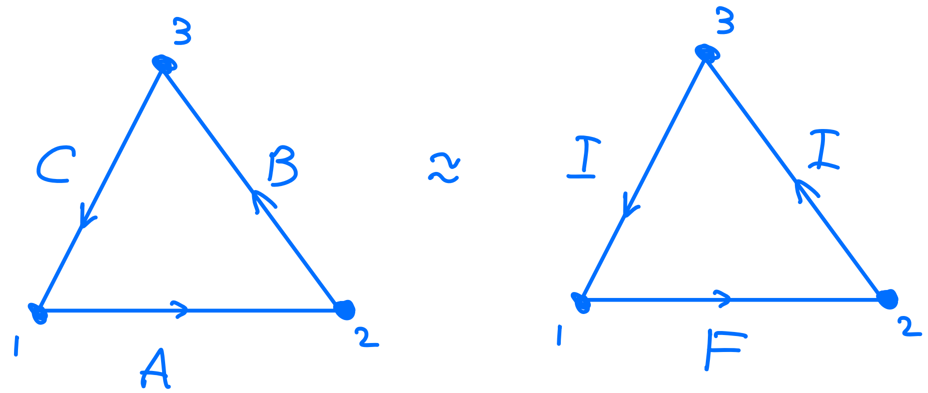
$$K_{uv} = \phi_{uv} = \phi_{vu}^{-1} = \phi_{vu}^{t} = -(-\phi_{vu})^t = -(K_{vu})^t.$$

So $\tilde{K}$ is antisymmetric.

Theorem:

$$\mathrm{Pf}\tilde{K} = \pm \sum_{m \in \Omega_n} \mathrm{Tr}(m).$$

Example



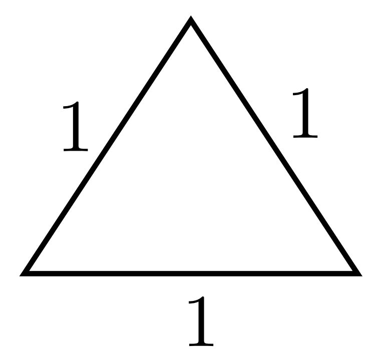
$$A, B, C \in \mathrm{SO}(2)$$

$$F = CBA = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & A & C^t \\ -A^t & 0 & B \\ -C & -B^t & 0 \end{pmatrix}$$

$$\operatorname{Pf} \tilde{K} = 2\sin\theta$$

Note there is only one 2-multiweb.



Sp(2n)

Sp($2n$) is the set of matrices in SL_n preserving the (standard) symplectic form.

$$\mathrm{Sp}(2n) = \{M \in \mathrm{SL}(2n, \mathbb{R}) \mid M^t J M = J\},$$

where J is the matrix

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

The trace of a $2n$-multiweb is defined as for $O(2n)$ but the contraction uses the symplectic form rather than the inner product.

Define

$$K(u,v) = \begin{cases} J\phi_{uv} & u \sim v \\ 0 & \text{else} \end{cases}.$$

Note

$$K_{uv} = J\phi_{uv} = J\phi_{vu}^{-1} = \phi_{vu}^t J = -(J\phi_{vu})^t = -(K_{vu})^t$$

so $\tilde{K}$ is antisymmetric.

Theorem:

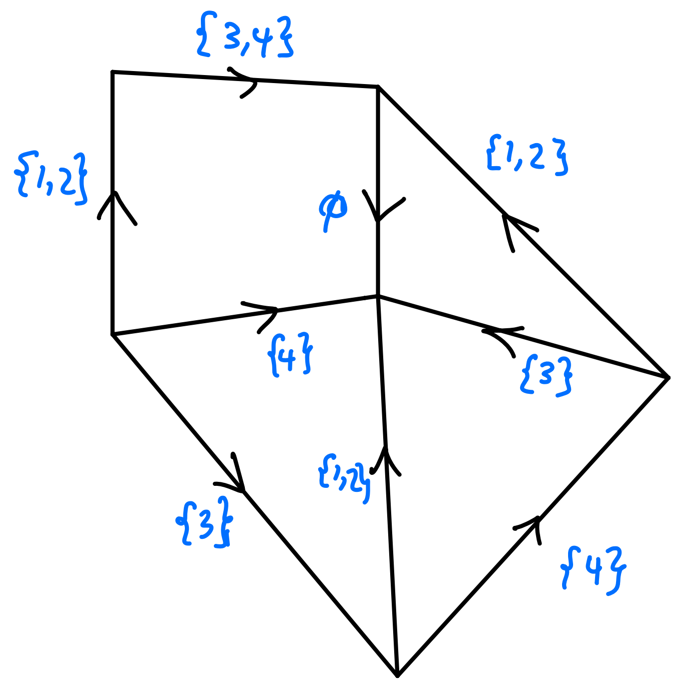
$$\mathrm{Pf}\tilde{K} = \pm \sum_{m \in \Omega_n} \mathrm{Tr}(m).$$

Different types of connections

	General	Identity/Flat	Positive
SL_n	[DKS]	[DKS]	[KO]
$SO(n)$	[KOW]	[KOW]	?
$Sp(2n)$	[KW]	[KW]	?

Edge colorings

An *edge-n-coloring* of an n-multiweb m is a map $c : E \to 2^{[n]}$ with $m_e = |c|$ and so that the union of the color sets at each vertex is $[n]$.



colors $= \{1, 2, 3, 4\}$

Prop. (SO(n) or SL$_n$)

For the identity connection and positive cilia, $\mathrm{Tr}_I(m)$ is $(-1)^{Vn(n-1)/4}$ times the number of edge-n-colorings.

SO_3 and the 4-color theorem

Thm: For a triangulation T of S^2, let m be the dual 3-web. Then

$$4Tr_I(m) = (-1)^{V/2} N_c,$$

where N_c is the number of proper 4-colorings of T.

Cor: For a planar 3-web m, $\mathrm{Tr}_I(m) \neq 0$.

Proof: The 4 color theorem. □

Thm: For each edge e of a triangulation, pick a random unit vector $u_e \in \mathbb{R}^3$. Then

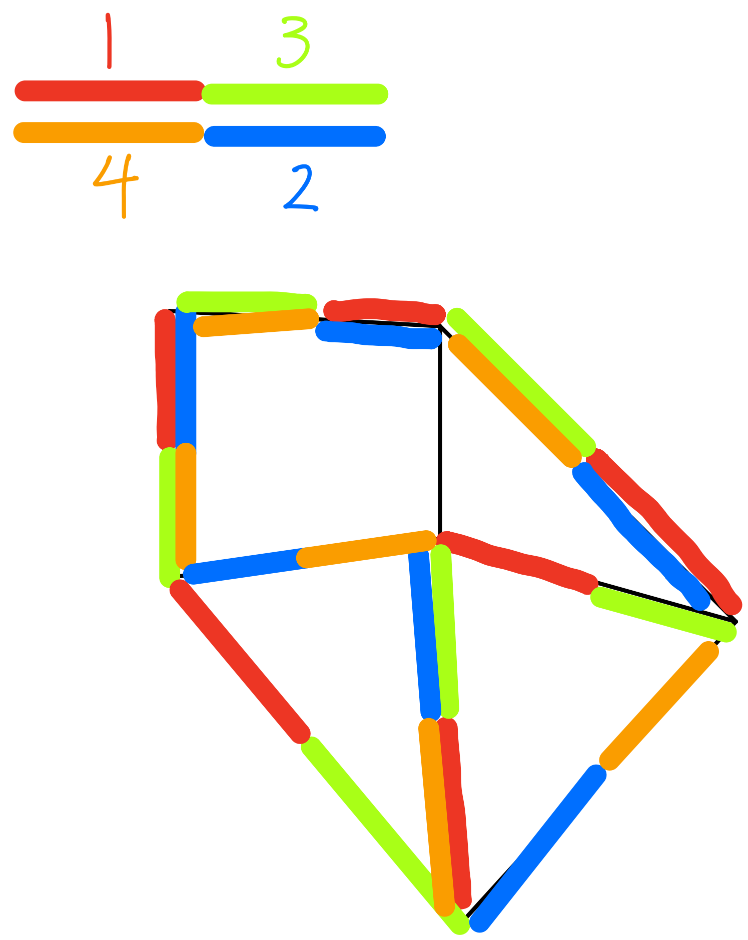
$$N_c = 4(-3)^{V/2} \mathbb{E}[\prod_F \det(u_1, u_2, u_3)]$$

Symplectic edge colorings

Complementation is the map $[2n] \to [2n]$ taking i to $i \pm n$.

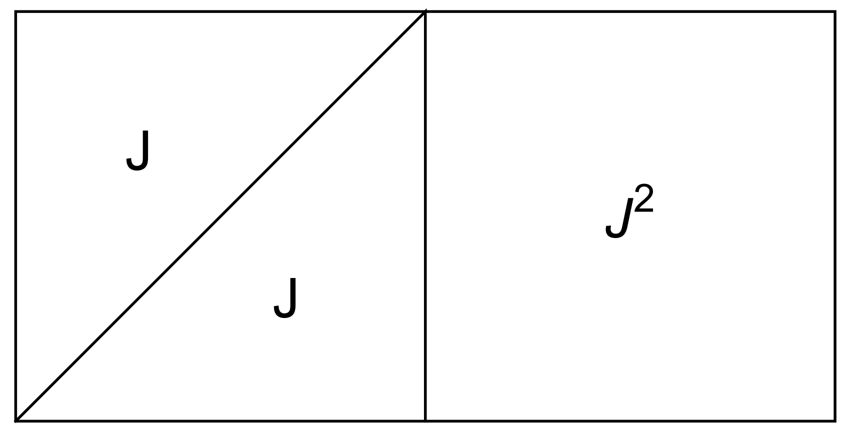
e.g. when $2n = 4$, $\quad 1 \leftrightarrow 3, \quad 2 \leftrightarrow 4$

A *symplectic edge-n-coloring* of a $2n$-multiweb m is a map $c : E_{\pm} \to 2^{[2n]}$ with $m_e = |c|$ so that the two half-edges e_+, e_- of an edge have complementary color sets and so that the union of the color sets at each vertex is $[2n]$.



Sp($2n$)-Kasteleyn connection

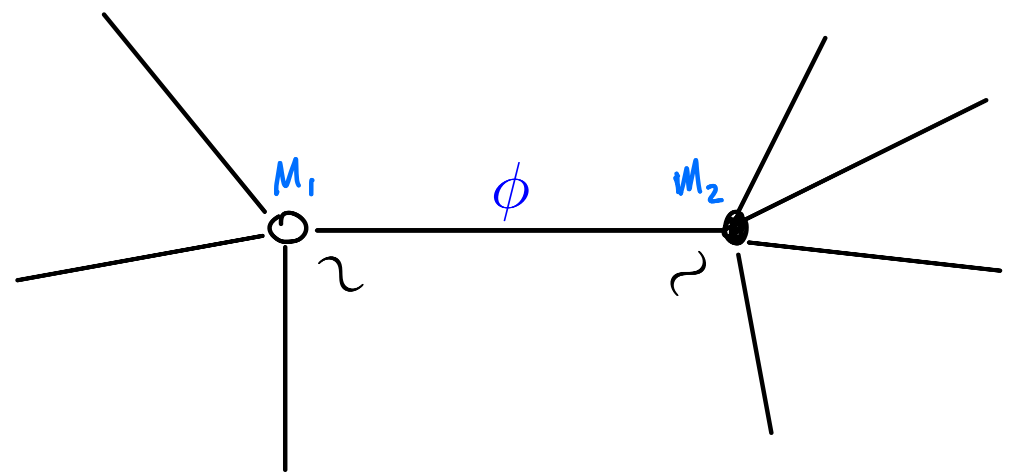
The *symplectic Kastelyn connection* gives each face of length l monodromy J^{l-2}.



Prop.

For the Kasteleyn connection Φ_K and standard orientation and cilia, $\mathrm{Tr}_\Phi(m)$ is $(-1)^{Vn(n-1)/4}$ times the number of symplectic edge colorings.

Construction of positive connections (SL_n)



$$M_1 \in \mathrm{Gr}^+_{n,4n} \qquad M_2 \in \mathrm{Gr}^+_{n,5n}$$

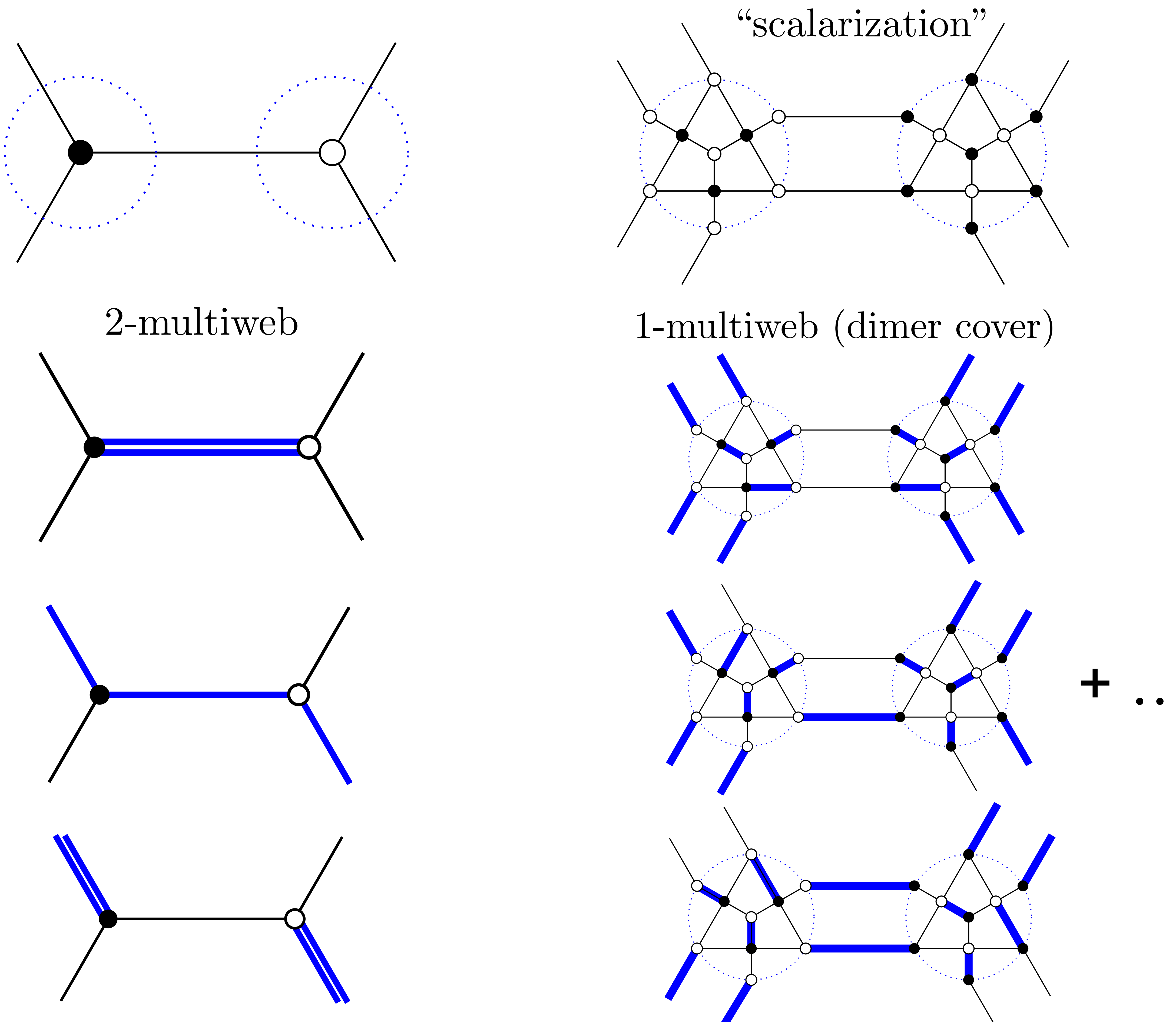
$$M_1 = (A_1\ A_2\ A_3\ A_4) \qquad M_2 = (B_1\ B_2\ B_3\ B_4\ B_5)$$

$$\phi = A_1 B_1^t$$

scale columns so that each $\phi \in \mathrm{SL}_n$.

Postnikov '96 showed how to associate to an element of $Gr^+_{m,n}$ a planar bipartite network with positive edge weights...

SL_2 example



Q. Is there an analogous procedure for $\mathrm{SO}(n)$ and $\mathrm{Sp}(2n)$ connections?

JOURNAL OF OPEN MATHEMATICAL PROBLEMS

* open access: no charges to authors or readers

* surveys about "famous" open problems in all fields of math