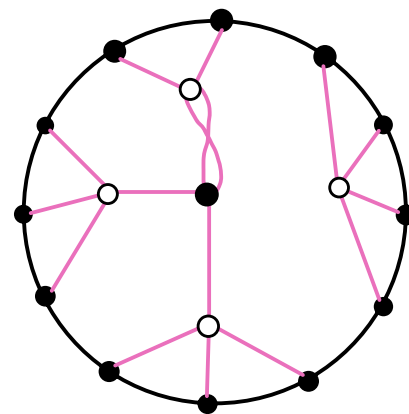
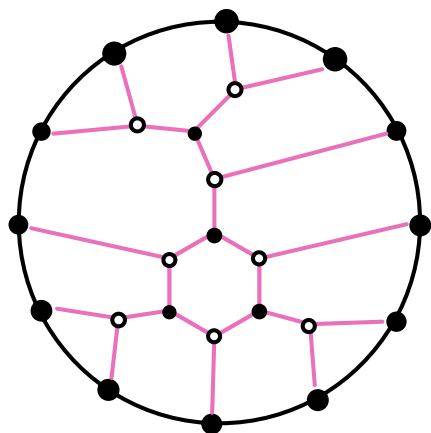


Webs & Grassmannian Cluster Algebras

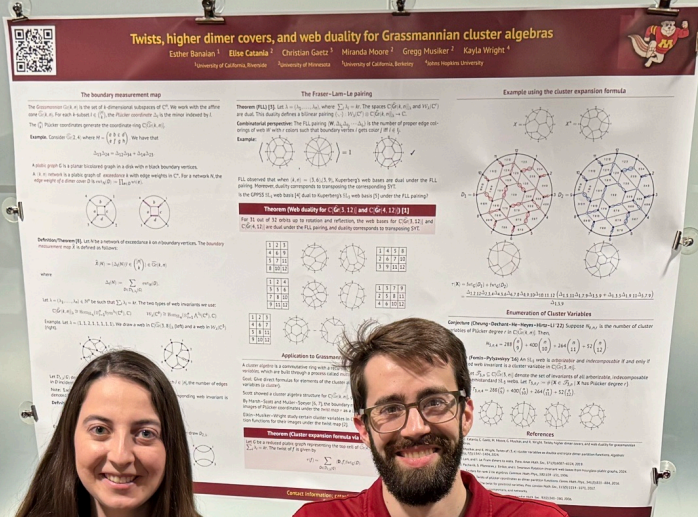
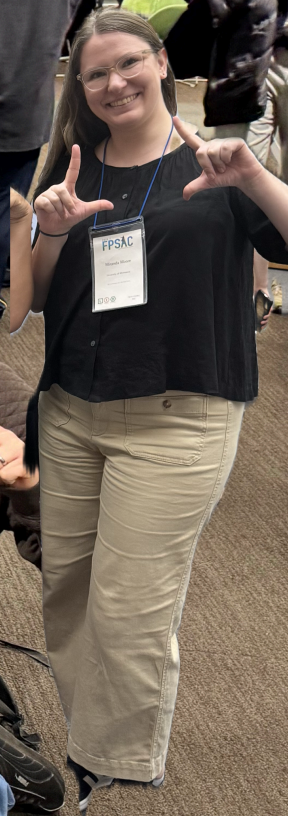
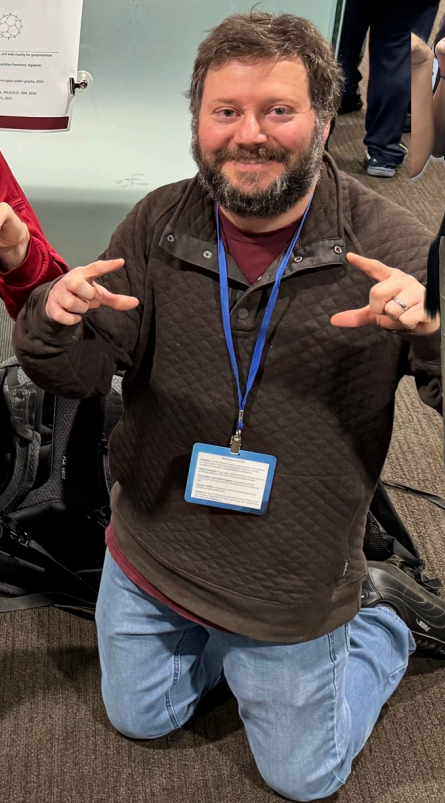
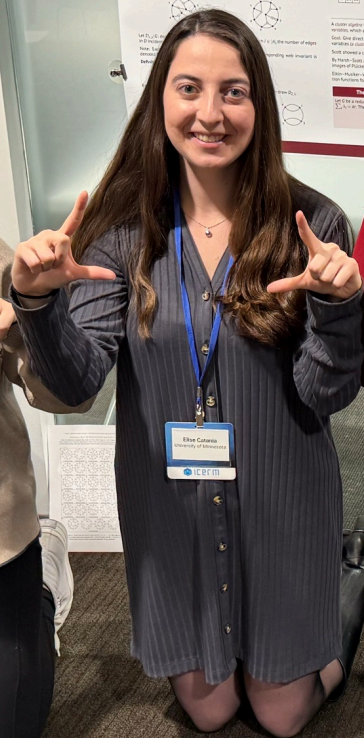
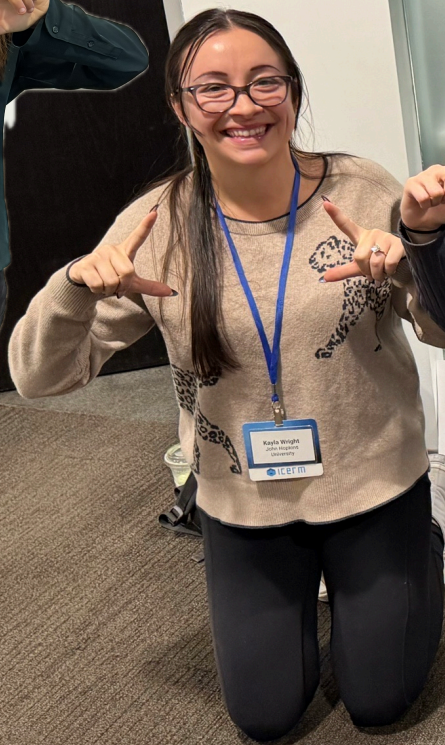
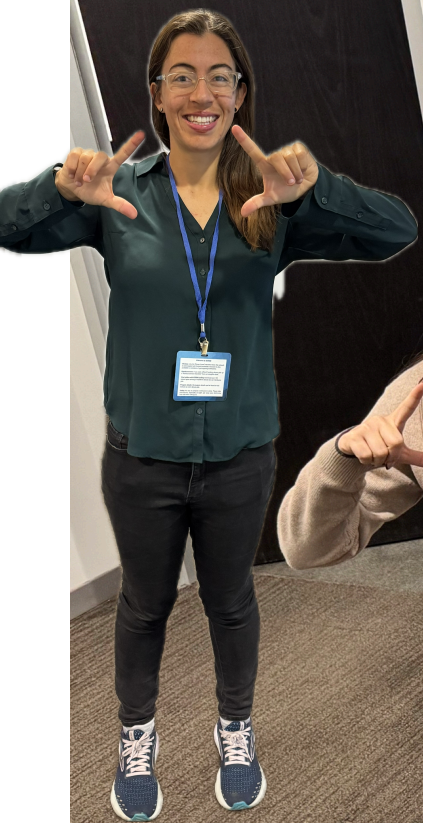
Kayla Wright

Johns Hopkins University

ICERM Workshop on Webs



joint with Esther Banaian, Elise Catania,
Christian Gaetz, Miranda Moore & Gregg Musiker



1014

Chenchen Zhao
Jianping Pan
Trevor Karn
George Seelinger
Aram Wingham

rooms 1001-1014
←
rooms 1015-1028,
accessible restroom
→

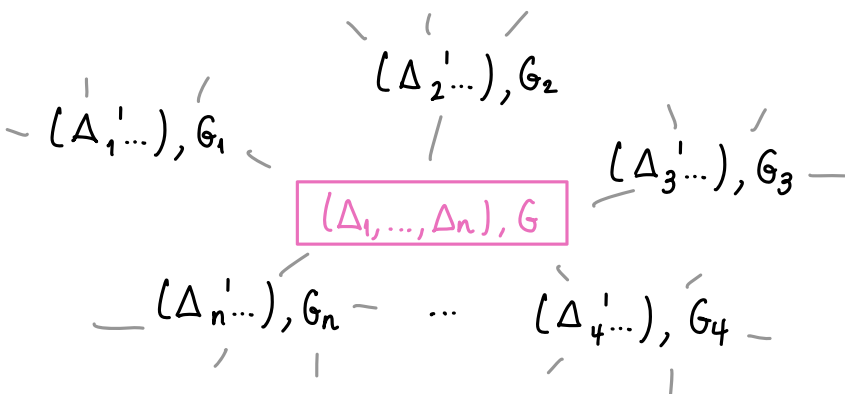
Outline

- I. Grassmannian Cluster Algebras
- II. Dimers & Webs
- III. Web Duality & Cluster Variables

Cluster Algebras

Cluster algebras are combinatorially defined subrings of the field of rational functions in variables $\Delta_1, \Delta_2, \dots, \Delta_n$.

$$(\underbrace{\Delta_1, \Delta_2, \dots, \Delta_n}_{\text{variables}}, \underbrace{\Delta_1, \dots, \Delta_n}_{\text{coefficients}}, \underbrace{G}_{\text{combinatorial data}}) \longleftarrow \text{initial seed}$$



$$\Delta_i' = \frac{\Delta_a \Delta_b + A_c \Delta_d}{\Delta_i}$$

recursively mutate
to obtain new
generators

$$\mathbb{C}[\Delta]$$

cluster algebra
generated by all cluster
variables in all seeds

Grassmannian $\text{Gr}(k, n)$

Fix $0 \leq k \leq n$.

$\text{Gr}(k, n) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}$

$= \{k \left\{ \underbrace{\quad}_n \right\} : \text{full rank}\} / \text{row operations}$

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$\text{Gr}(k, n)$ is coordinatized by determinants of submatrices. Plücker coordinates Δ_J :

Ex $\text{Gr}(2, 4)$: $M = \begin{pmatrix} \overset{1}{a} & \overset{2}{b} & \overset{3}{c} & \overset{4}{d} \\ e & f & g & h \end{pmatrix}$

$$\Delta_{\{1, 2\}} = af - be$$

$$\Delta_{\{2, 4\}} = bh - df$$

Plücker Relations

Plücker coordinates satisfy a set of algebraic relations called Plücker relations

Ex] $\text{Gr}(2, 4) : M = \begin{pmatrix} \overset{1}{1} & \overset{2}{0} & \overset{3}{a} & \overset{4}{b} \\ 0 & 1 & c & d \end{pmatrix}$

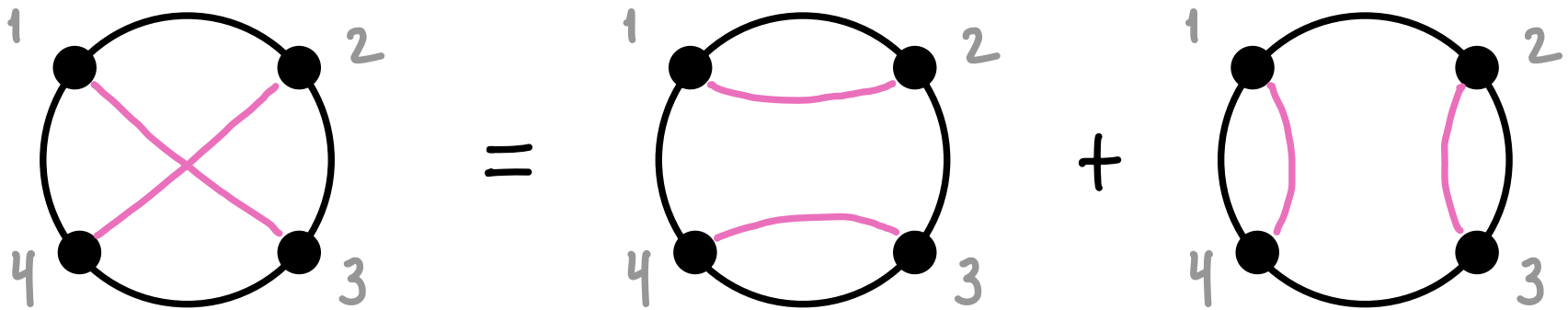
$$\Delta_{12} = 1; \Delta_{13} = c; \Delta_{14} = d; \Delta_{23} = -a; \Delta_{24} = -b; \\ \Delta_{34} = ad - bc$$

$$\Delta_{13} \cdot \Delta_{24} = \Delta_{12} \cdot \Delta_{34} + \Delta_{14} \cdot \Delta_{23}$$

$$c \cdot (-b) = 1 \cdot (ad - bc) + d \cdot (-a)$$

Plücker Relations via Tensor Diagrams

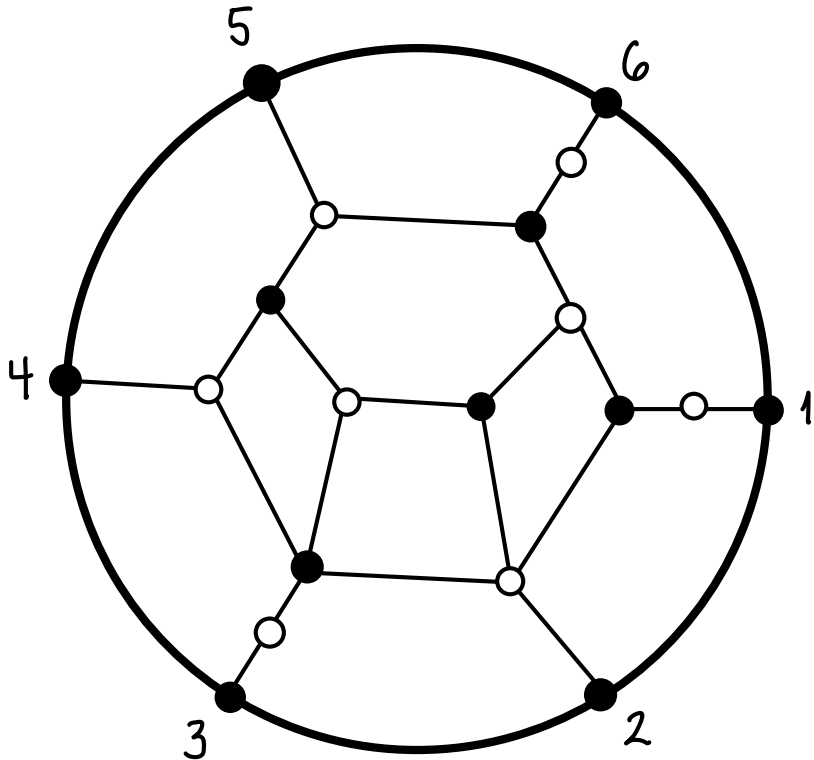
One way we can visualize the Plücker relation is using tensor diagrams.



$$\Delta_{13} \cdot \Delta_{24} = \Delta_{12} \cdot \Delta_{34} + \Delta_{14} \cdot \Delta_{23}$$

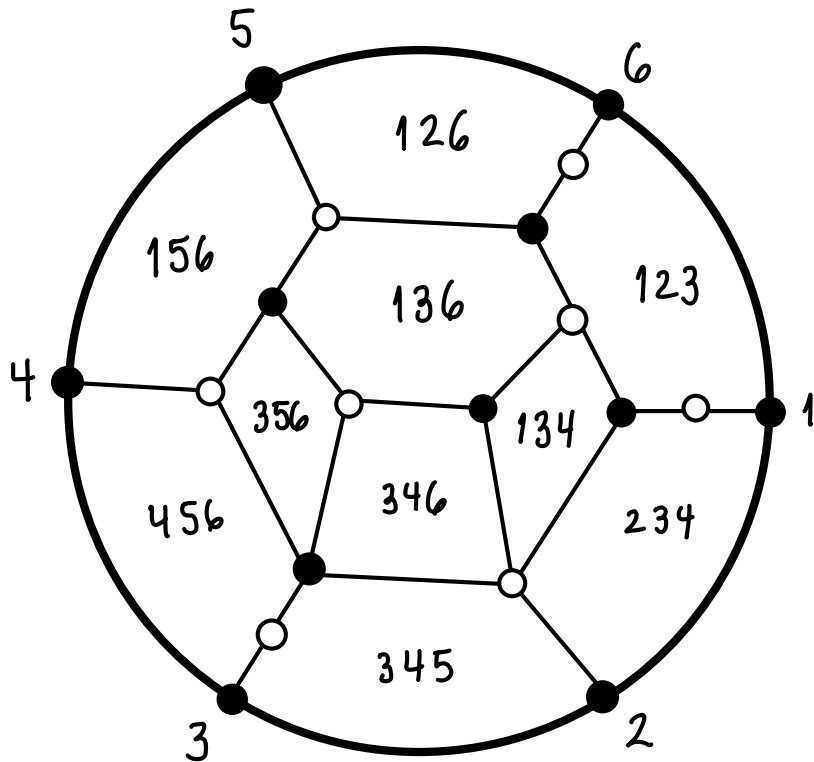
Plabic Graphs

A **plabic graph** (planar bi-colored) is a combinatorial tool used to define the cluster structure on $Gr(k,n)$.



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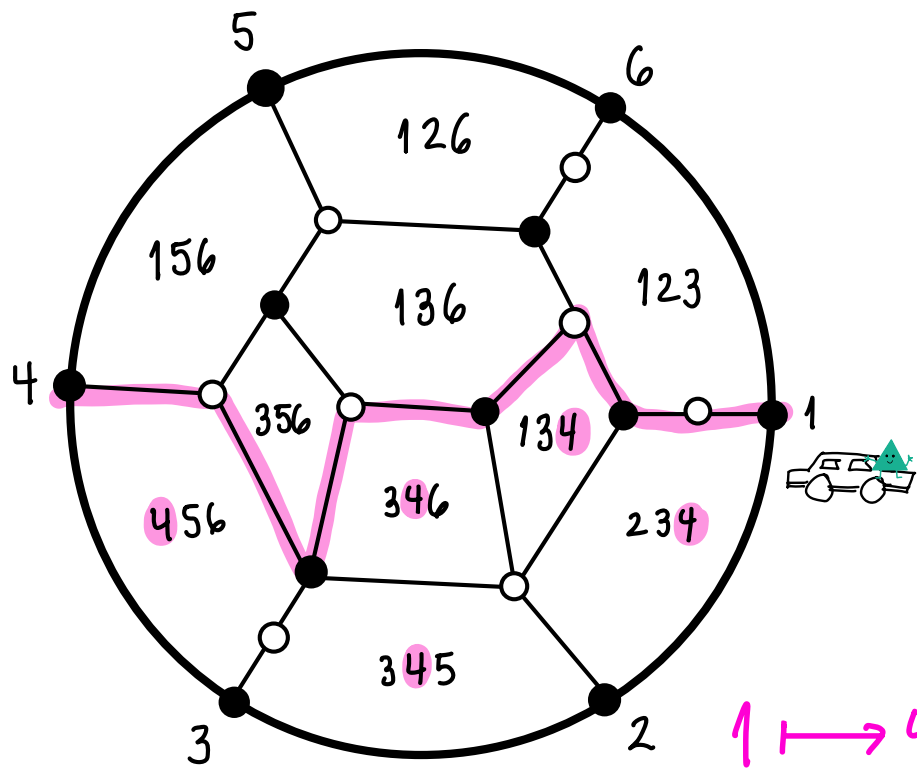
Face labels $J \leftrightarrow \Delta_J$

Obtain labeling with **trips**

↳ black = right
white = left

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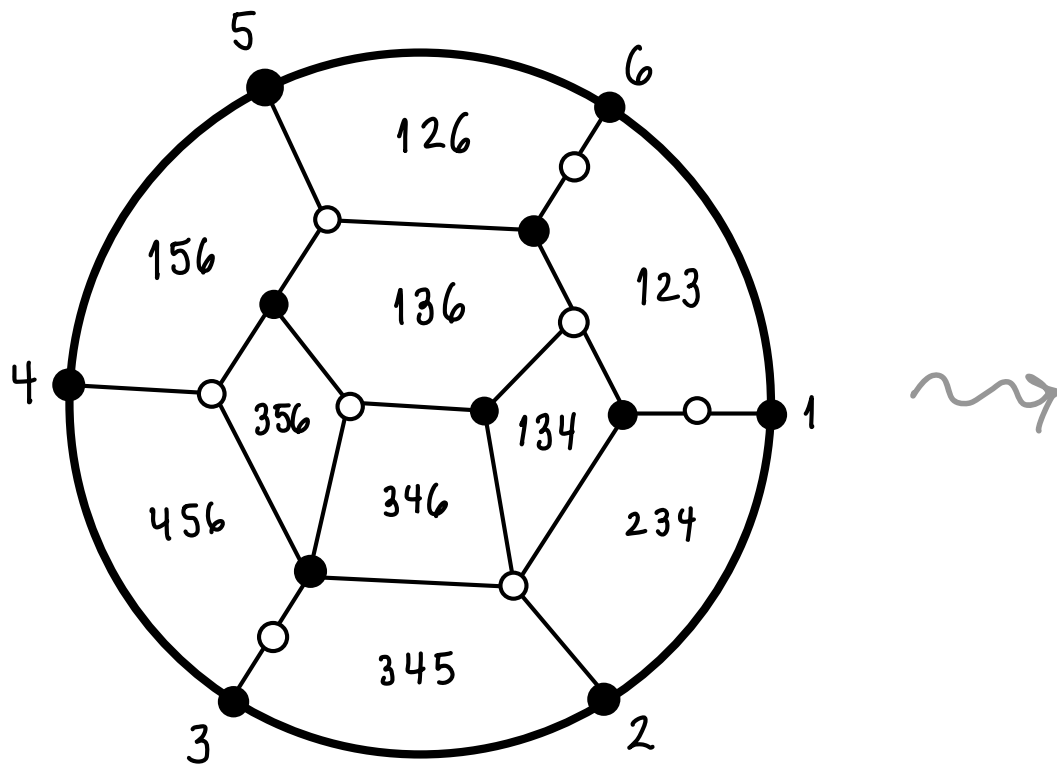
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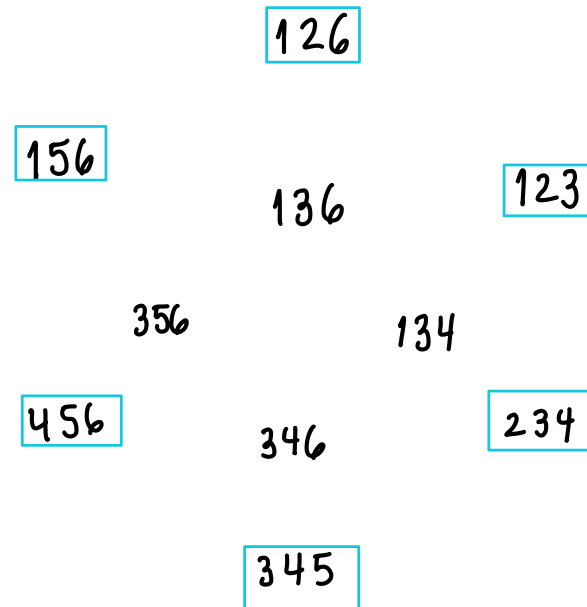
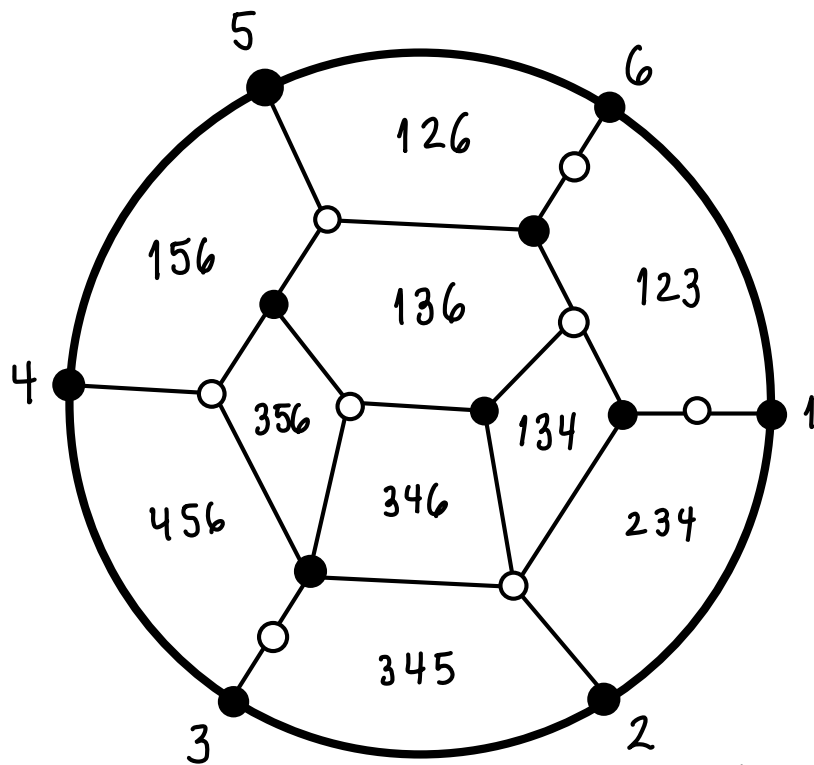
Plabic Graph to Quiver

Dualizing a plabic graph gives a quiver Q :



Plabic Graph to Quiver

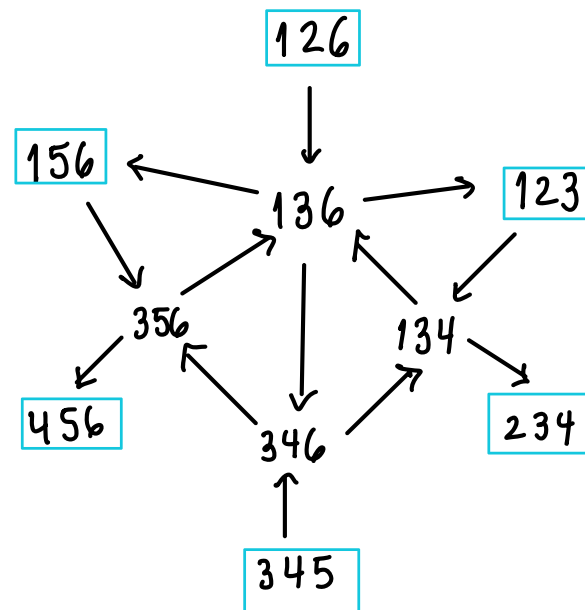
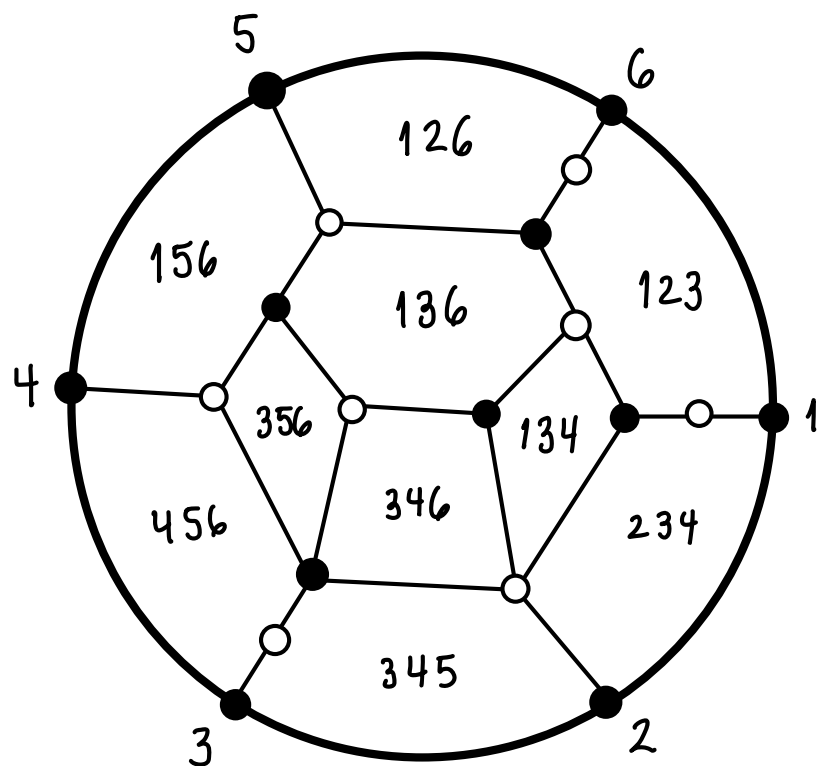
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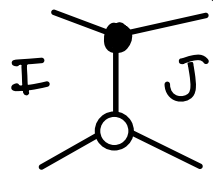
Internal faces $\longleftrightarrow$ mutable vertices
 Boundary faces $\longleftrightarrow$ frozen vertices

Plabic Graph to Quiver

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$\Delta_I \rightarrow \Delta_J$
 not both frozen

Cluster Structure on $\widehat{C[Gr(k,n)]}$

Cluster variables : homogeneous functions in Plücker coordinates obtained by mutation

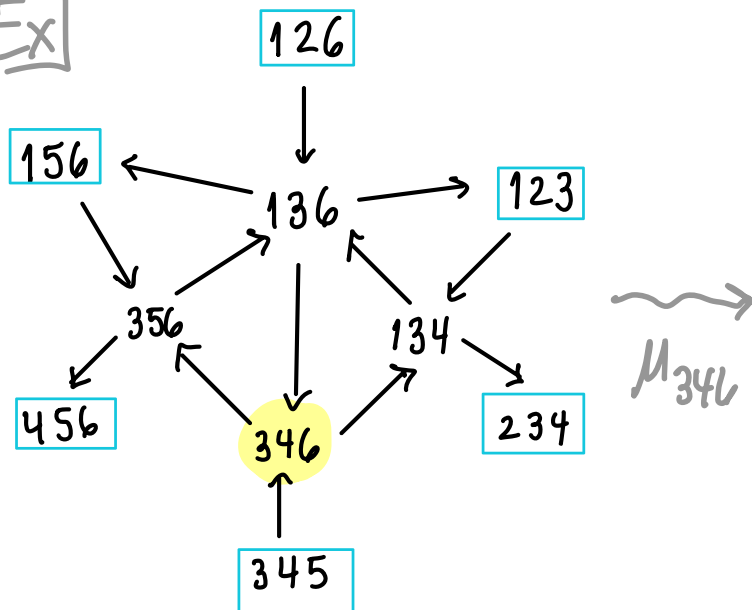
Mutation : local modification of quiver at a vertex to get new quiver & new cluster variable

Cluster Structure on $\mathcal{C}[\text{Gr}(k,n)]$

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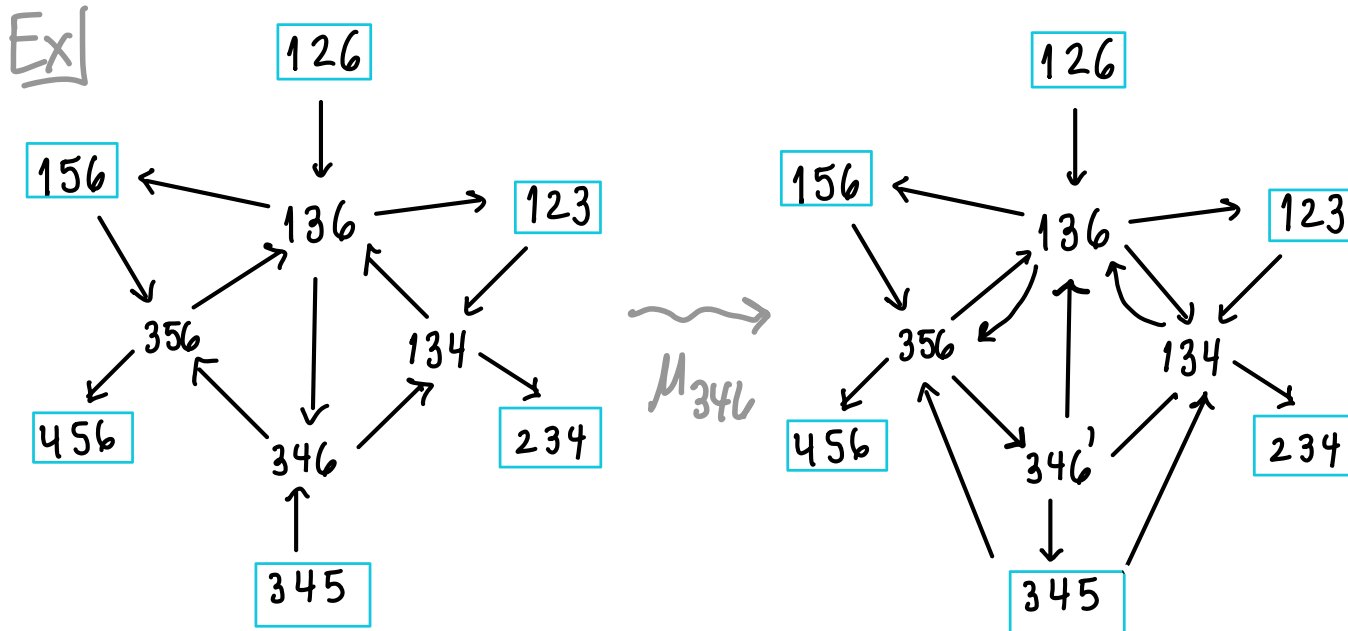
Ex]



Cluster Structure on $\mathcal{C}[\text{Gr}(k,n)]$

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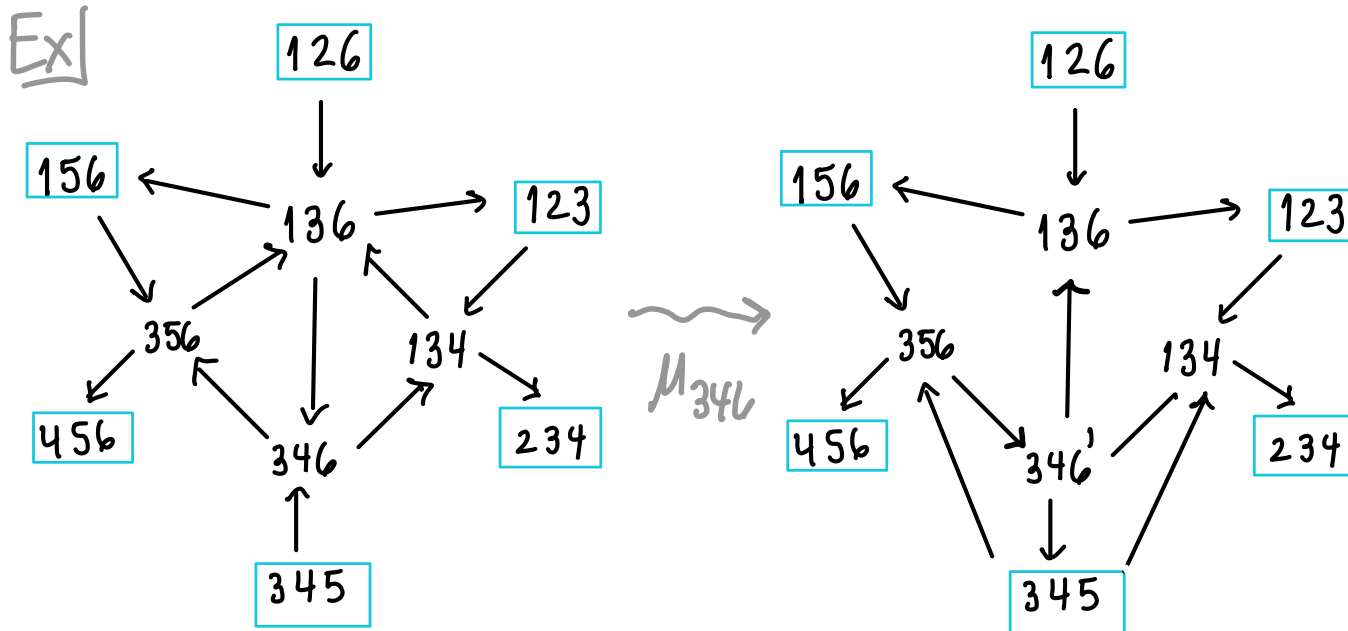
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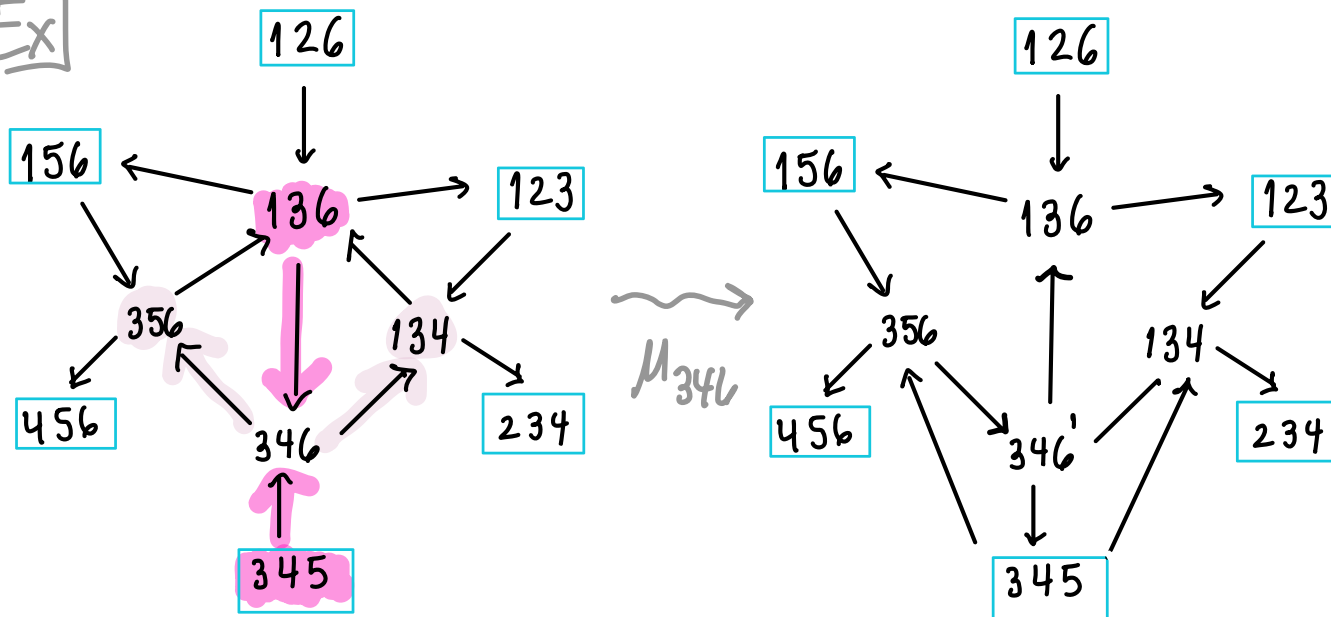


Cluster Structure on $\mathbb{C}[\text{Gr}(k,n)]$

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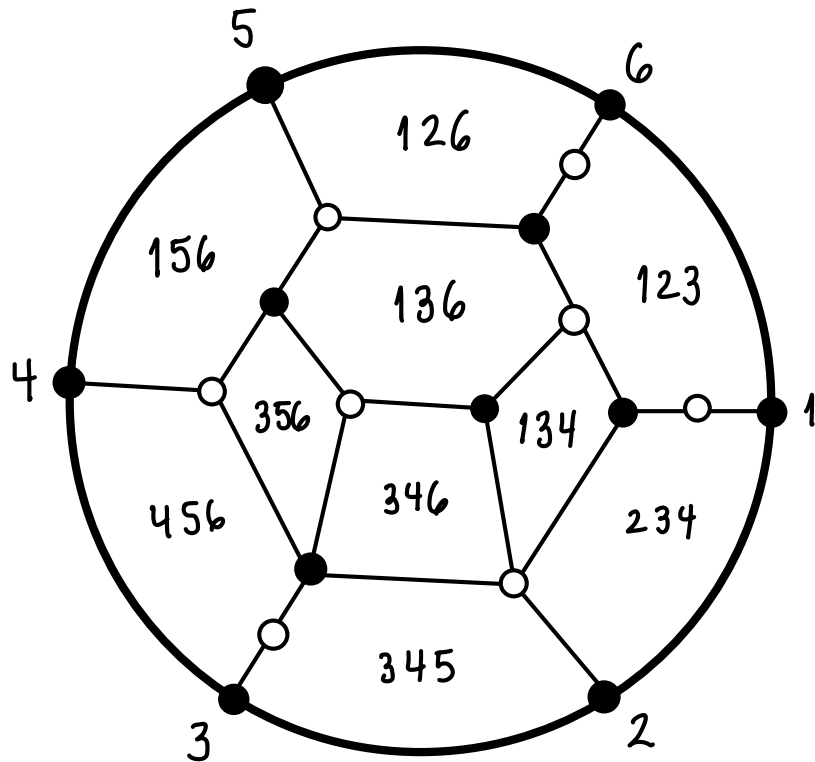
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Ex



$$\begin{aligned} \Delta_{346}' &= \frac{\prod_{v \rightarrow \Delta_{346}} v + \prod_{w \leftarrow \Delta_{346}} w}{\Delta_{346}} \\ &= \frac{\Delta_{136} \Delta_{345} + \Delta_{134} \Delta_{356}}{\Delta_{346}} \\ &= \dots = \Delta_{135} \end{aligned}$$

Example $\widehat{ClGr(3,6)}$



Cluster Variables :

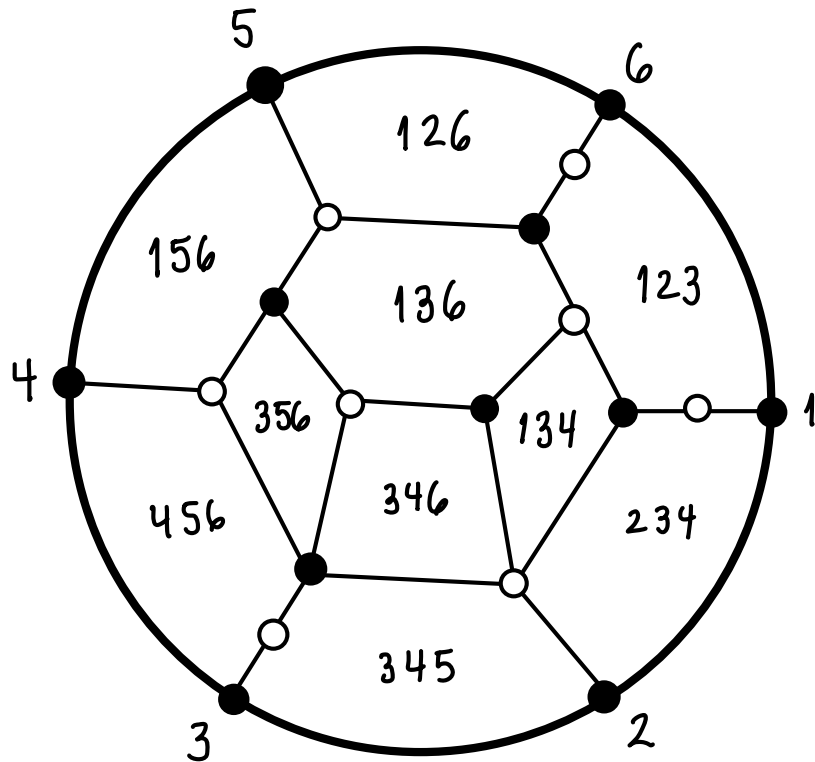
$\Delta_{136}, \Delta_{134}, \Delta_{346}, \Delta_{356}$

,

,

,

Example $\widehat{ClGr(3,6)}$



Cluster Variables :

$$\Delta_{136}, \Delta_{134}, \Delta_{346}, \Delta_{356}$$

$$\Delta_{124}, \Delta_{125}, \Delta_{135}, \Delta_{235},$$

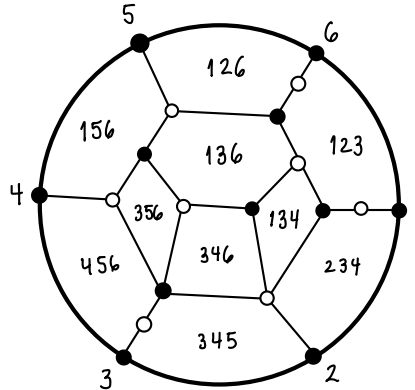
$$\Delta_{236}, \Delta_{345}, \Delta_{145}, \Delta_{146},$$

$$X = \Delta_{124} \Delta_{356} - \Delta_{123} \Delta_{456}$$

$$Y = \Delta_{145} \Delta_{236} - \Delta_{123} \Delta_{456}$$

Cluster Structure on $\mathbb{C}[\text{Gr}(k,n)]$

(some)
Seeds:



plabic graphs

Mutation:



quiver mutation

cluster
variables:

$$\Delta_{134} \Delta_{256} - \Delta_{234} \Delta_{156}$$

homogeneous functions
in Plücker coordinates
obtained by mutation

Central Questions in Cluster Theory

What are
cluster variables?

Without mutation,
can we find expansion
formula for cluster
variables?

Fomin - Pylyavskyy Conjectures

Fomin - Pylyavskyy Conjectures: [2012]

Cluster variables in $\widehat{\mathbb{C}[Gr(3,n)]}$ correspond to certain SL_3 -webs.

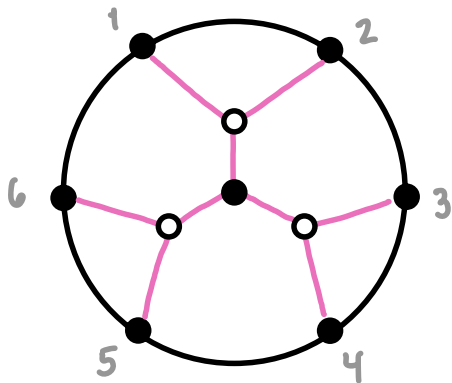
Fomin - Pylyavskyy Conjectures

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Cluster variables in $\widehat{\mathbb{C}[\text{Gr}(3,n)]}$ correspond to certain SL_3 -webs.

Important Connection: $\widehat{\mathbb{C}[\text{Gr}(k,n)]}_{\lambda=(1\dots 1)} \cong \text{Inv}_{SL_k}(V^{\otimes n})$

Idea:



SL_3 -web

$$\rightsquigarrow \Delta_{134} \Delta_{256} - \Delta_{234} \Delta_{156}$$

cluster variable

Web Bases

An SL_k -web is a planar bipartite graph embedded in a disk such that each internal vertex has valence k .

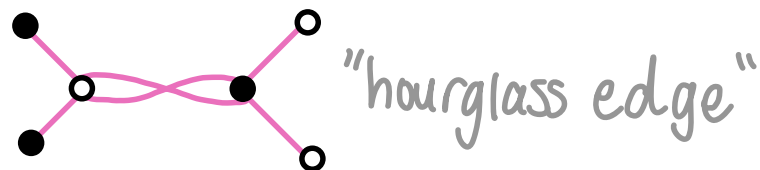
For $k \leq 4$, webs give diagrammatic bases of $\text{Inv}_{SL_k}(V^{\otimes n})$ with desirable properties e.g. rotation invariance.

Thrm: [Kuperberg '96]

Webs with no 2- or 4-faces form a basis of $\text{Inv}_{SL_3}(V^{\otimes n})$.

Thrm: [Gaetz - Pechenik - Pfannerer - Striker - Swanson '23]

SL_4 -hourglass webs form a basis of $\text{Inv}_{SL_4}(V^{\otimes n})$.



Goal of Project

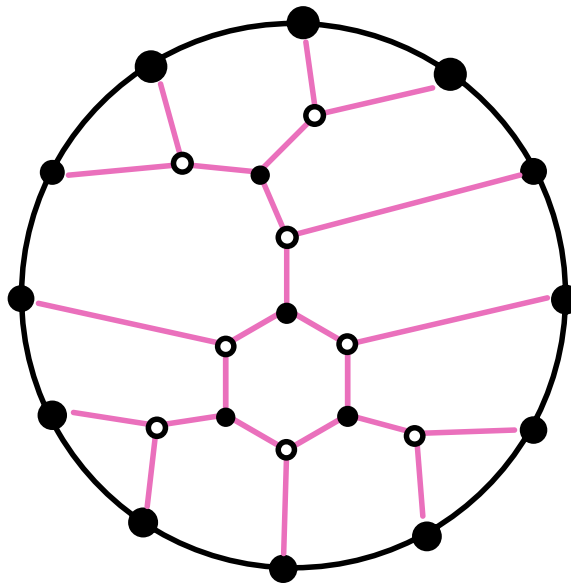
Use the combinatorics of SL_3 & SL_4 webs to understand & classify small degree cluster variables in $\mathbb{C}[\widehat{Gr}(3, n)]$.

Overview of Main Machinery:

1	2	3	9
4	5	6	11
7	8	10	12

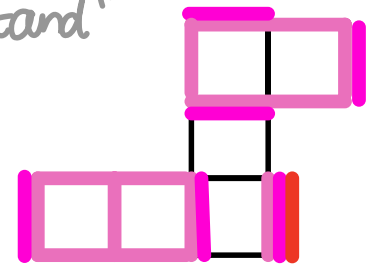
tableau

"classify"



web

"understand"



higher dimer cover

Expansion Formula

For special small cases, we can understand cluster variables S as **dimer partition functions**:

$$\tau^*(S) = \sum_D \text{wt}(D)$$

up to a twist τ^*

where D is a set of certain **dimer covers** on a plabic graph governed by combinatorics associated to S .

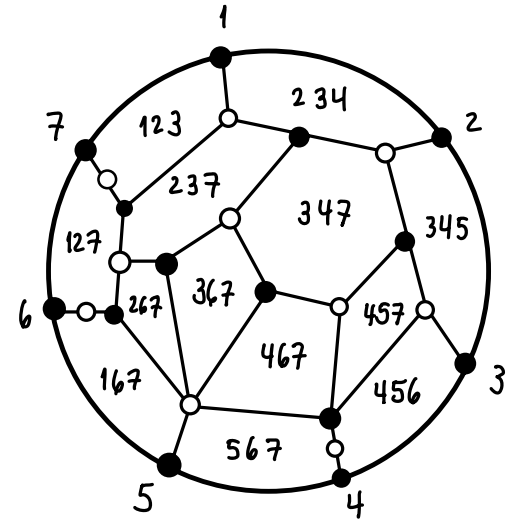
harder with larger degree
cluster variables

[Marsh-Scott, Muller-Speyer, Elkin-Musiker-W']

Expansion Formula Example

Take $S = \Delta_{346} \in \widehat{\mathbb{C}[\text{Gr}(3,7)]}$.

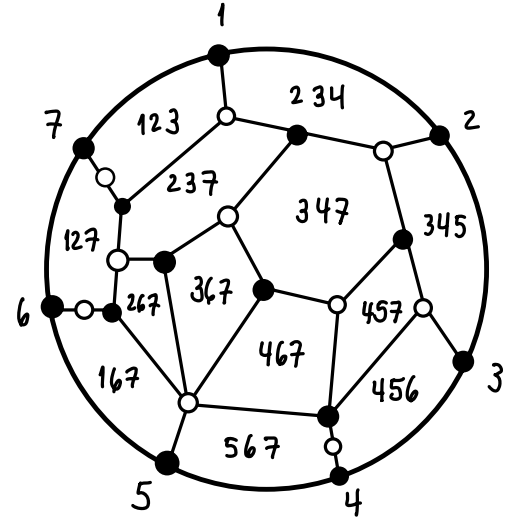
Consider the initial seed given by :



Expansion Formula Example

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Consider the initial seed given by :



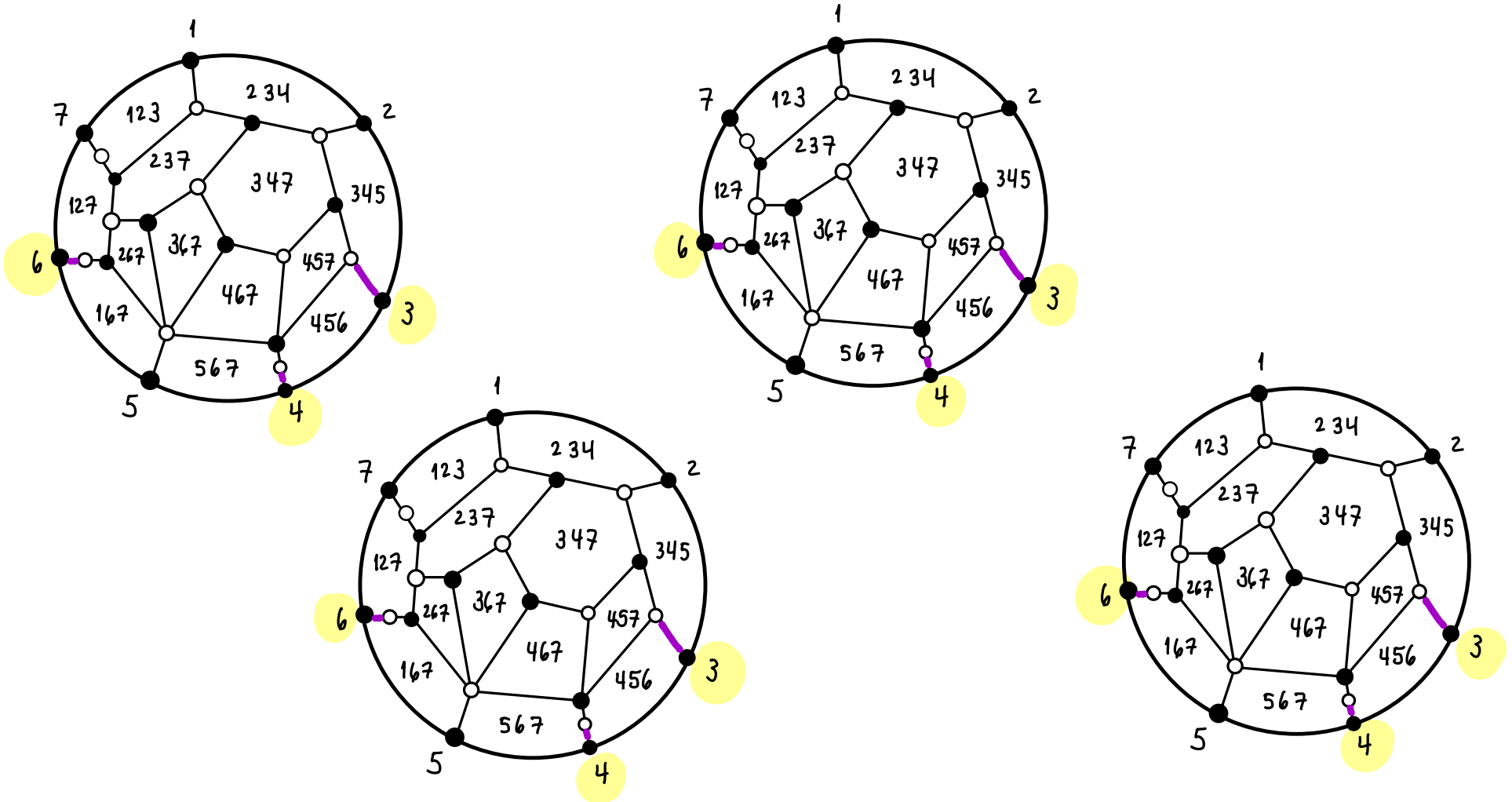
We compute the Laurent expansion for $\tau^*(S)$:

$$\tau^*(\Delta_{346}) = \Delta_{456} \Delta_{571} - \Delta_{556} \Delta_{471} = \Delta_{456} \Delta_{157}$$

↑ generalized
cross-product

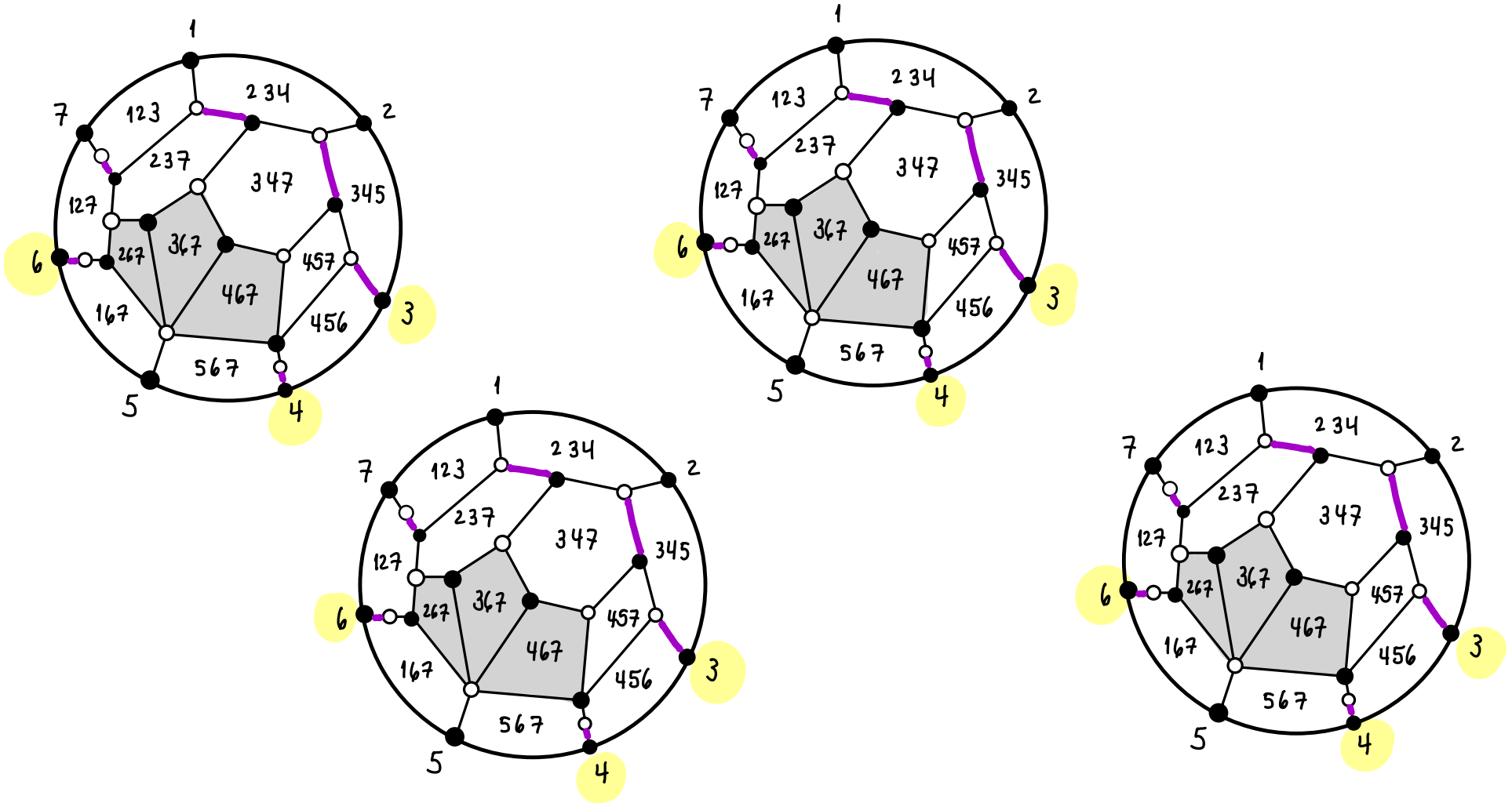
Expansion Formula Example

Find all single dimers matching vertices 3, 4 & 6:



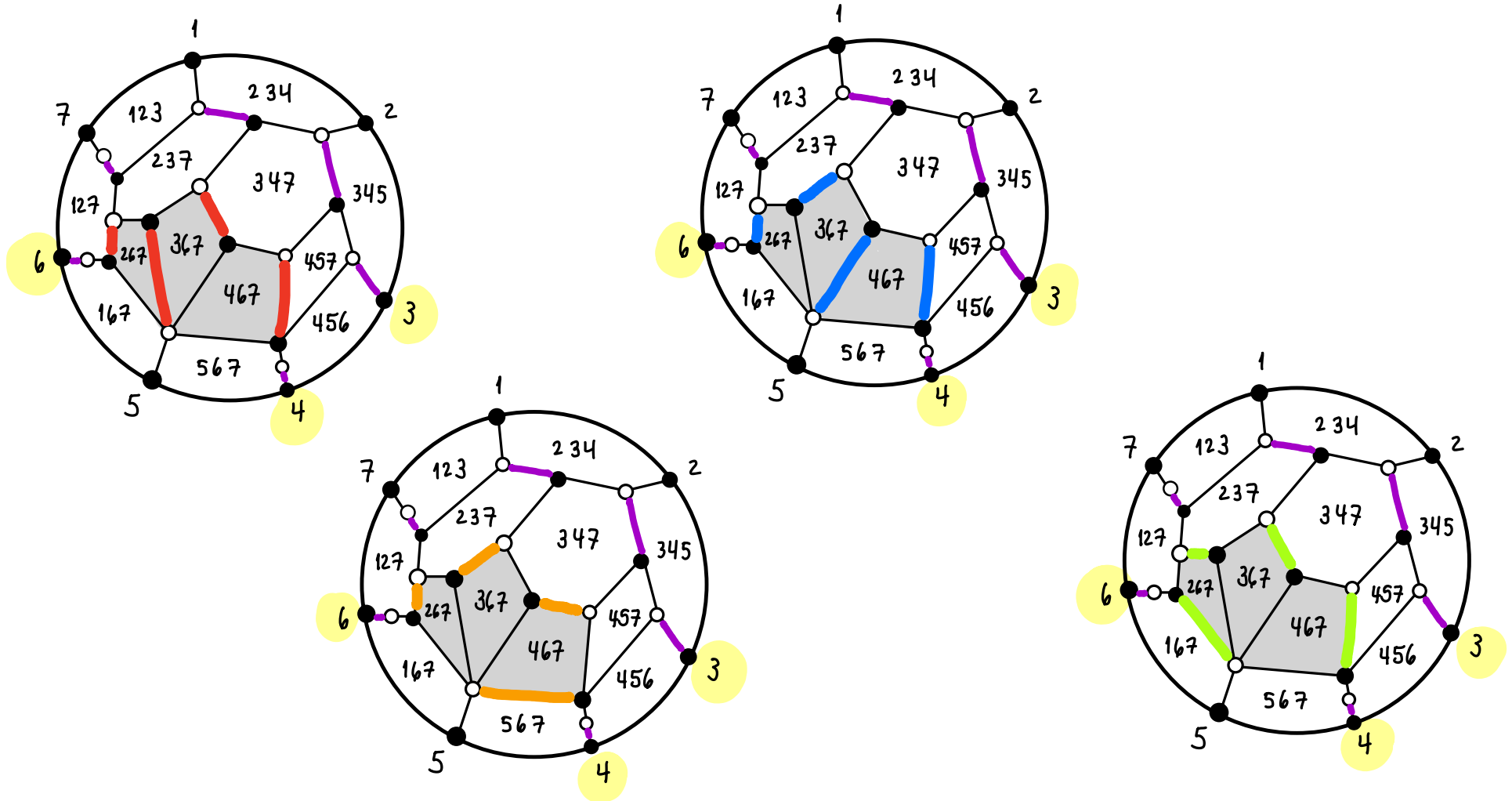
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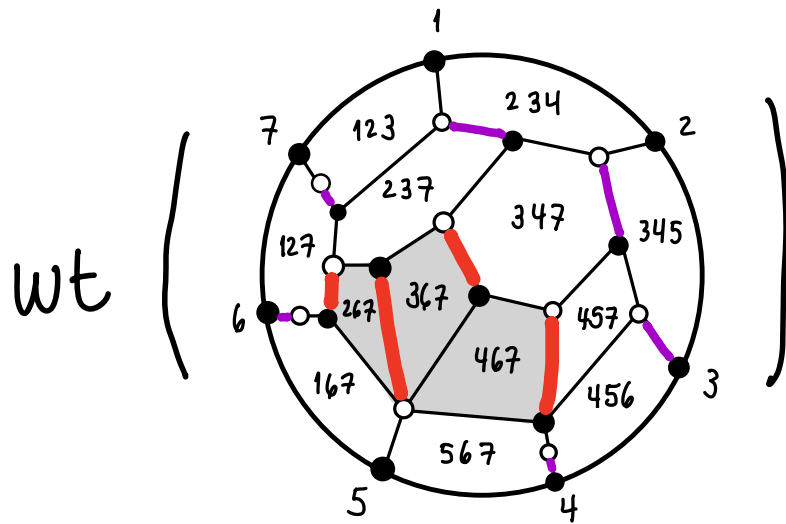
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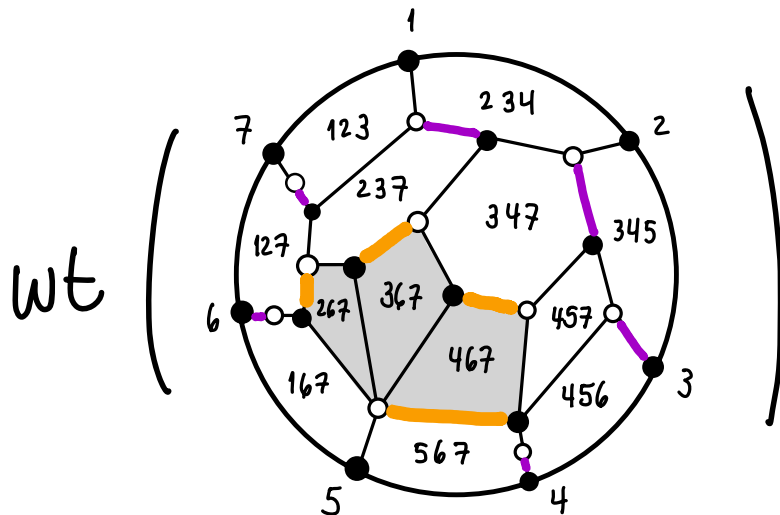


Expansion Formula Example

We assign a **weighting** to each face, then to each dimer.



$$= \frac{\Delta_{456} \Delta_{237} \Delta_{167} \Delta_{567}}{\Delta_{267} \Delta_{367}}$$



$$= \frac{\Delta_{456} \Delta_{457} \Delta_{167}}{\Delta_{467}}$$

Expansion Formula Example

The terms of the Laurent expansion for $\tau^*(\Delta_{346}) = \Delta_{157}$ are given by single dimers using boundary vertices 3, 4, 6

$$\tau^*(\Delta_{346}) = \text{wt} \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) + \text{wt} \left(\begin{array}{c} \text{Diagram 2} \end{array} \right) + \text{wt} \left(\begin{array}{c} \text{Diagram 3} \end{array} \right) + \text{wt} \left(\begin{array}{c} \text{Diagram 4} \end{array} \right)$$

up to multiplication by frozen Δ_{456}

Higher Dimers

Most cluster variables are higher degree functions in Plücker's.

An analogous expansion formula requires more difficult combinatorics involving webs!

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degree r cluster
variable



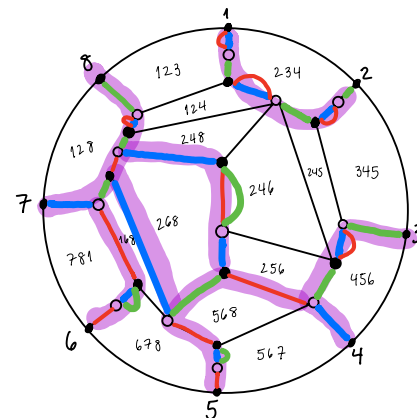
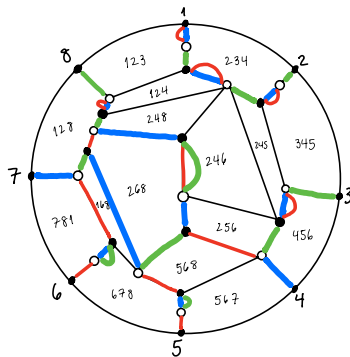
higher dimer
cover on
plabic graph



SL_k -web

S

$$k \cdot r = n$$



General Expansion Formula

Thrm: [Banaian - Catania - Gaetz - Moore - Musiker - W '25]

Let S be a degree k cluster variable & $n = kr$. We understand S as a higher dimer partition function:

$$\tau^*(S) = \sum_{D: r\text{-dimer cover}} \langle W_D, \cdot \rangle \text{wt}(D)$$

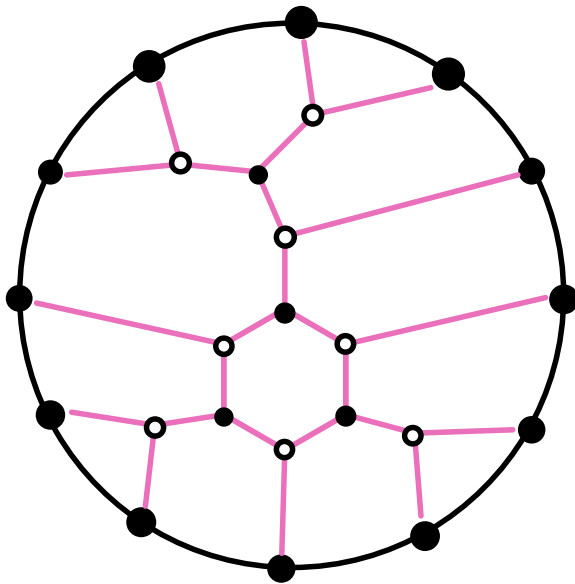
where $\langle W_D, \cdot \rangle$ is the Fraser-Lam-Le inner product that takes in an SL_r & SL_k -web & outputs a complex number.



Web Duality

We give a combinatorial interpretation of the FLL-inner product in certain cases.

Idea:



Given a cluster variable S with web W_S , find which webs W' have

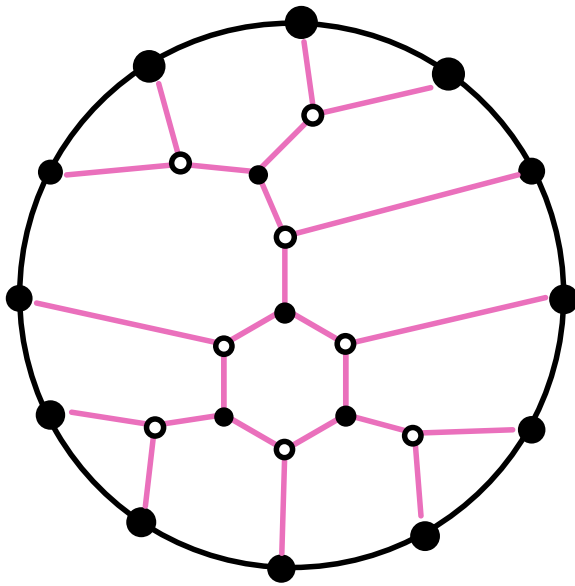
$$\langle W_S, W' \rangle \neq 0$$

(ie) contribute to expansion
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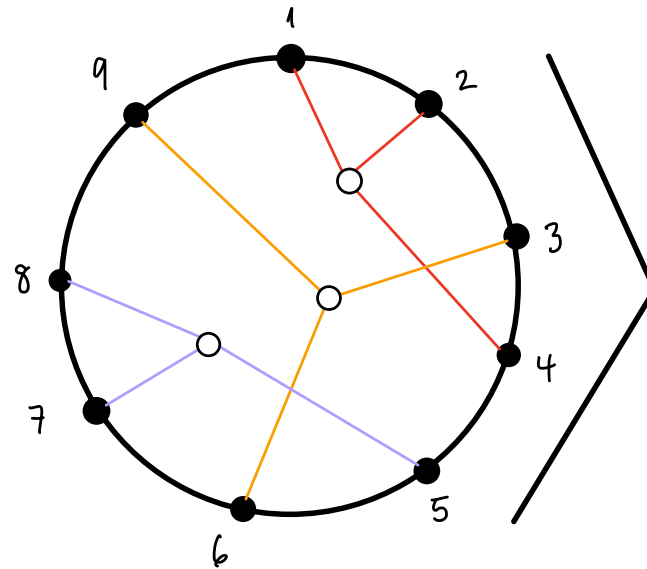
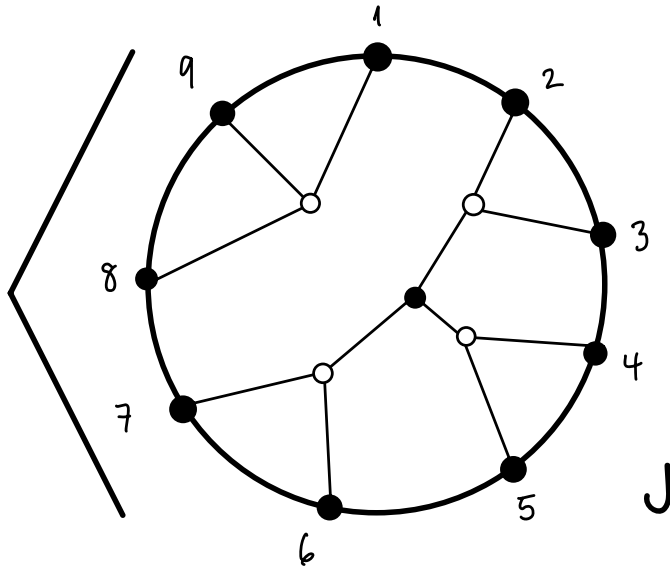
(ie) contribute to expansion
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Answer: We discover a unique dual web W_S^* such that $\langle W_S, W_S^* \rangle \neq 0$.

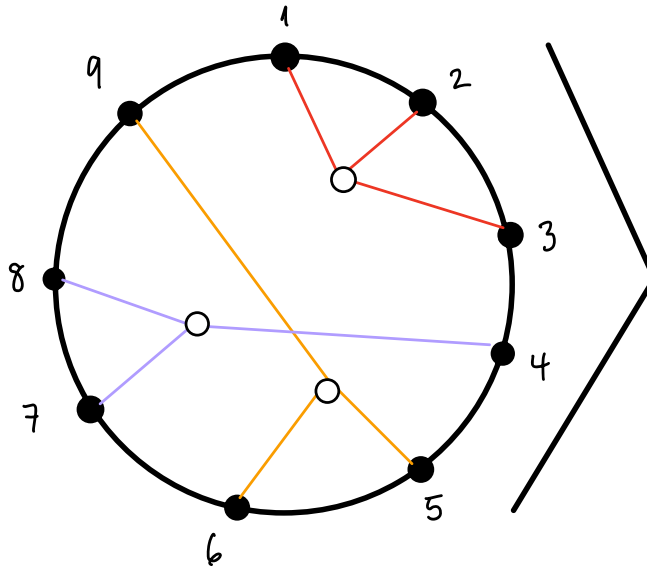
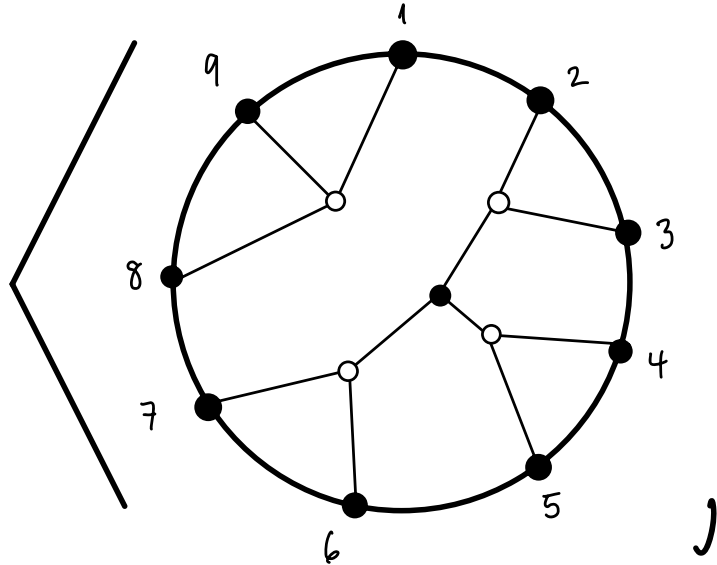
FLL-Pairing Plücker's

$$\langle W, \Delta_{I_1}, \dots, \Delta_{I_r} \rangle = \# \left\{ \text{ways to color } W \text{ with } \begin{matrix} \text{colors } I_1, I_2, \dots, I_r \end{matrix} \right\}$$

Ex



= ?

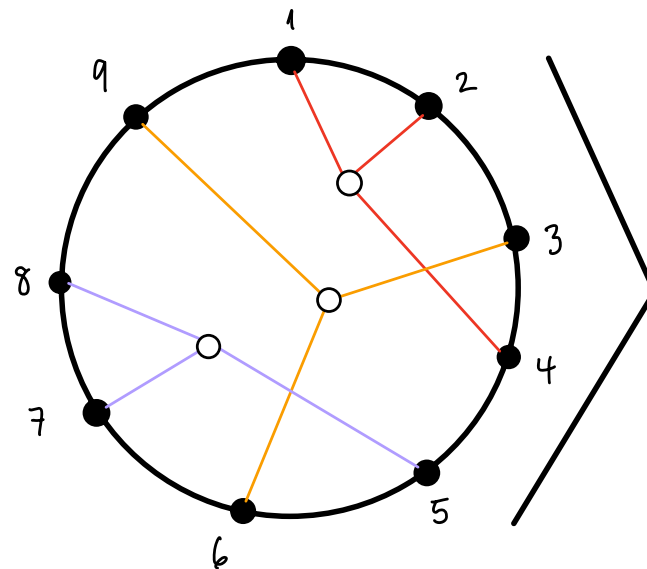
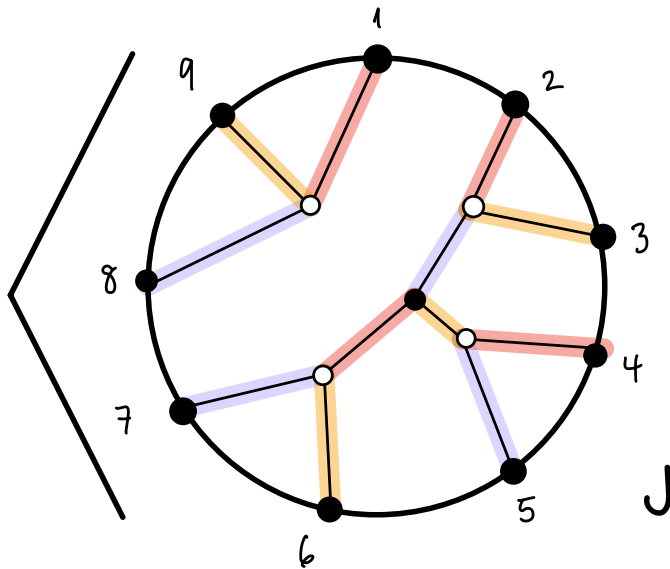


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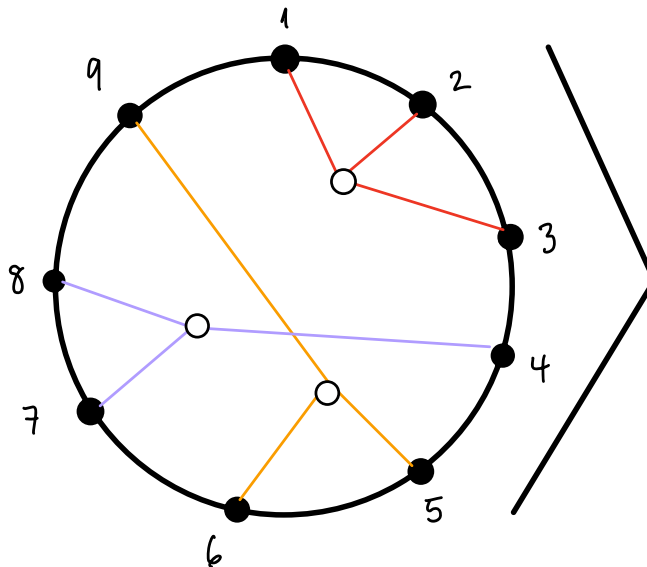
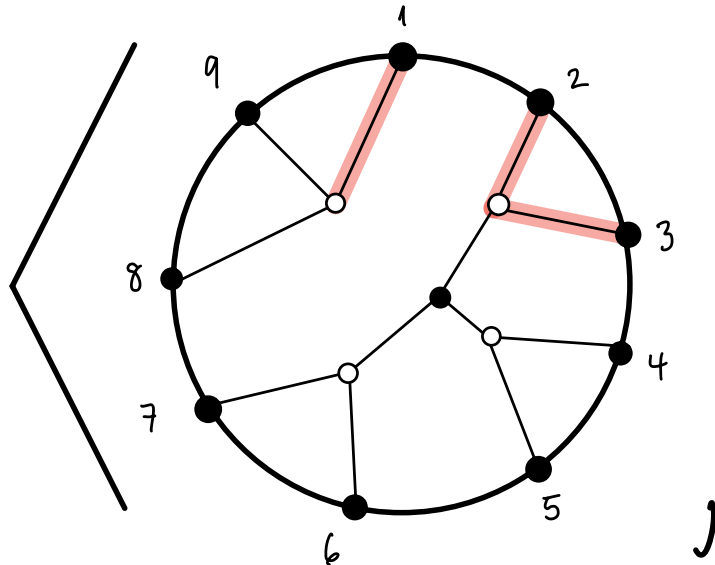
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Ex



= 1

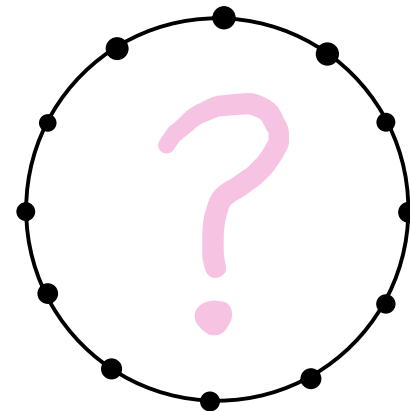
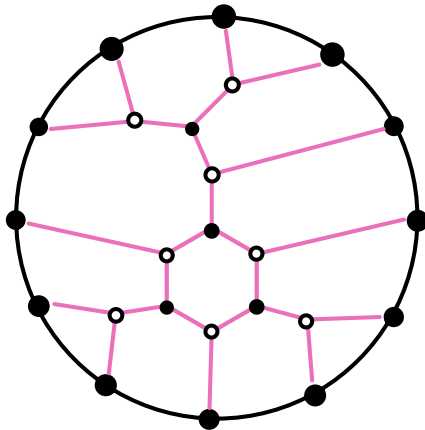


= 0

Web Duality & Tableaux

Observation: [FLL'19, BCGMMW'25]

$\langle w, w' \rangle \neq 0$ when the tableau associated to w is the transpose of tableau associated to w' .

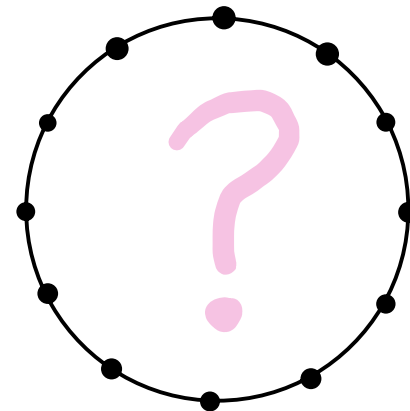
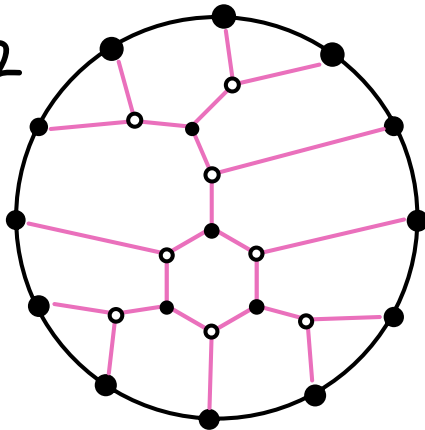


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7	8	10	12



Khovanov -
Kuperberg '99
+
Tymoczko '12

Web Duality & Tableaux

Observation: [FLL'19, BCGMMW'25]

$\langle W, W' \rangle \neq 0$ when the tableau associated to W is the transpose of tableau associated to W' .

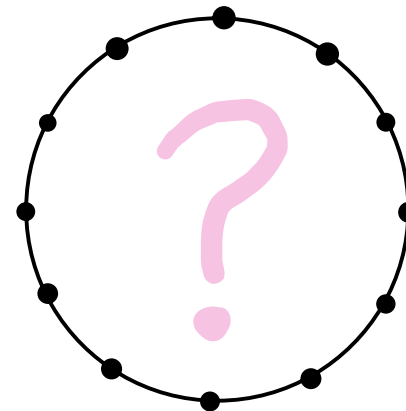
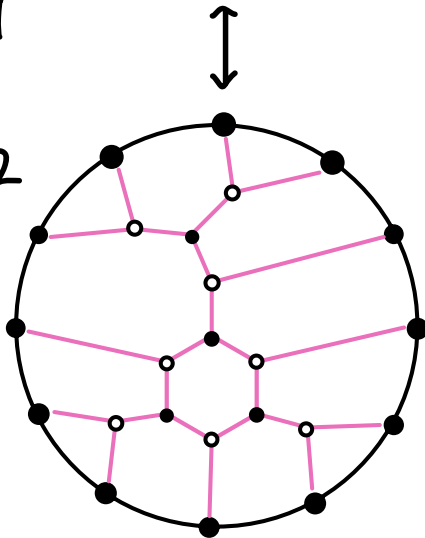
Khovanov -
Kuperberg '99
+

Tymoczko '12

1	2	3	9
4	5	6	11
7	8	10	12

transpose
←→

1	4	7
2	5	8
3	6	10
9	11	12



Web Duality & Tableaux

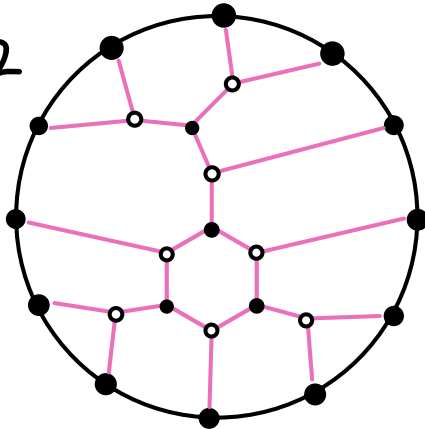
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Khovanov -
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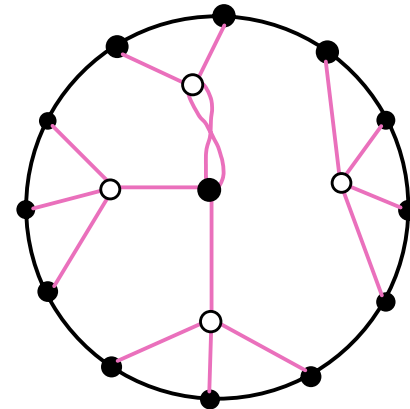


transpose

1	4	7
2	5	8
3	6	10
9	11	12



GPSS '23



duality

Classification* of Cluster Variables

We show this duality extends to $\widehat{Gr}(3,12)$ using SYT.

★

1	2	3	9
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7	8	10	12

There are 462 3×4 SYT!

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★ promotion on tableau $\longleftrightarrow$ rotation of web

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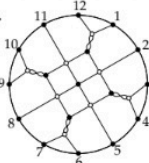
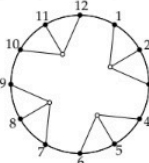
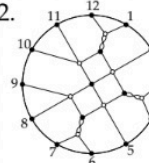
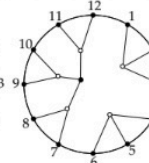
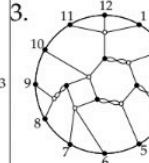
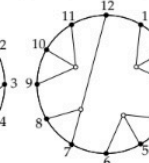
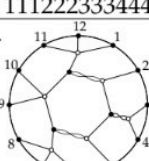
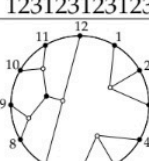
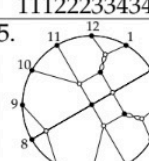
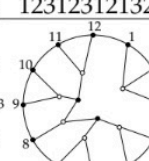
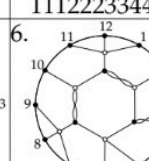
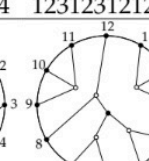
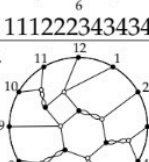
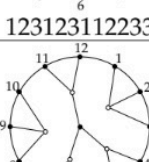
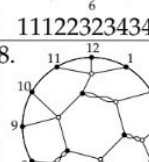
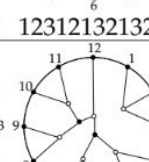
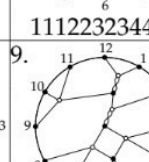
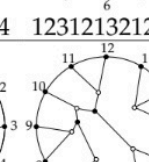
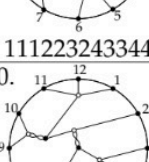
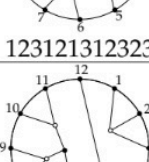
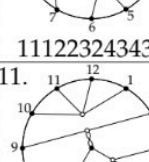
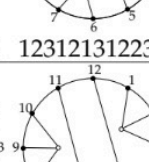
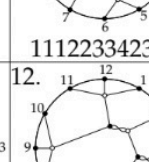
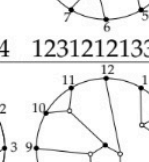
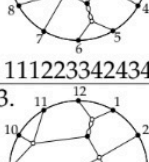
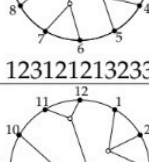
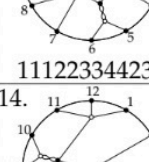
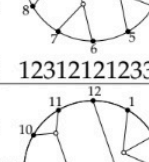
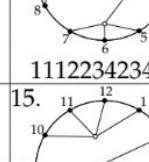
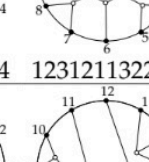
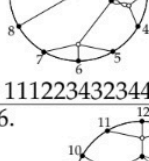
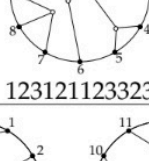
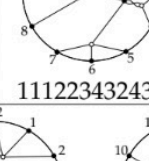
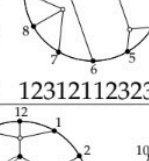
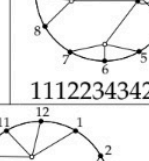
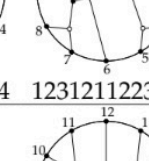
★ promotion on tableau $\longleftrightarrow$ rotation of web

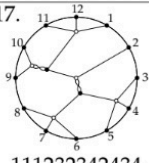
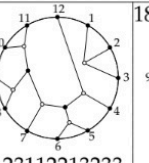
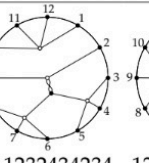
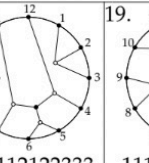
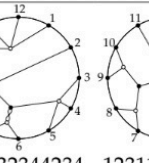
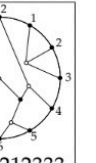
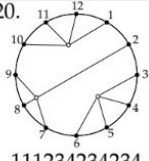
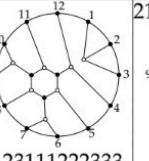
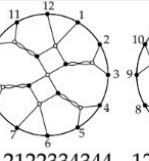
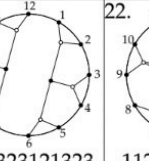
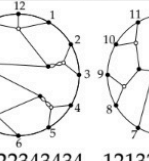
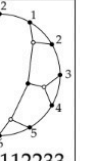
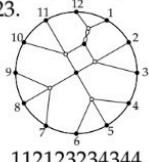
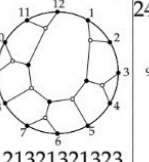
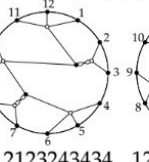
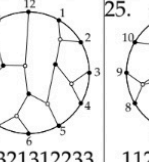
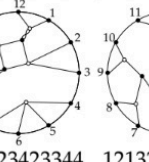
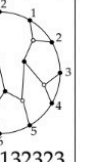
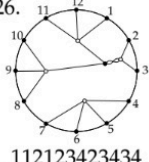
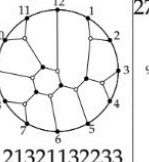
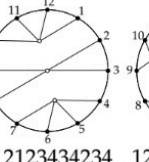
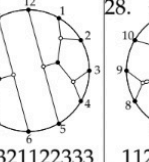
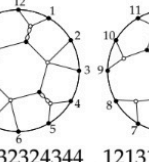
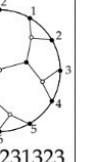
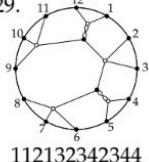
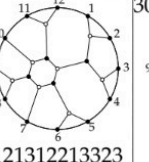
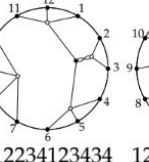
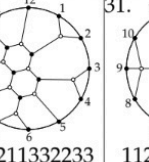
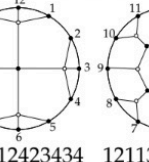
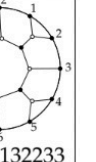
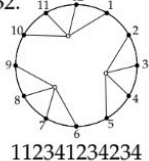
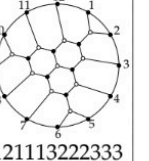
★ we can group into 32 orbits up to rotation/reflection

We compute $\langle w, - \rangle$ for a web w from each orbit to check

web duality = transposition of tableaux

Web Duality Tables

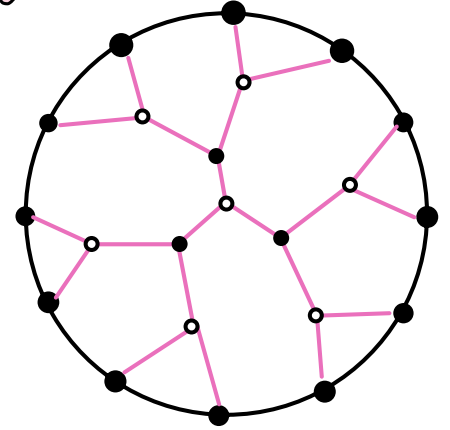
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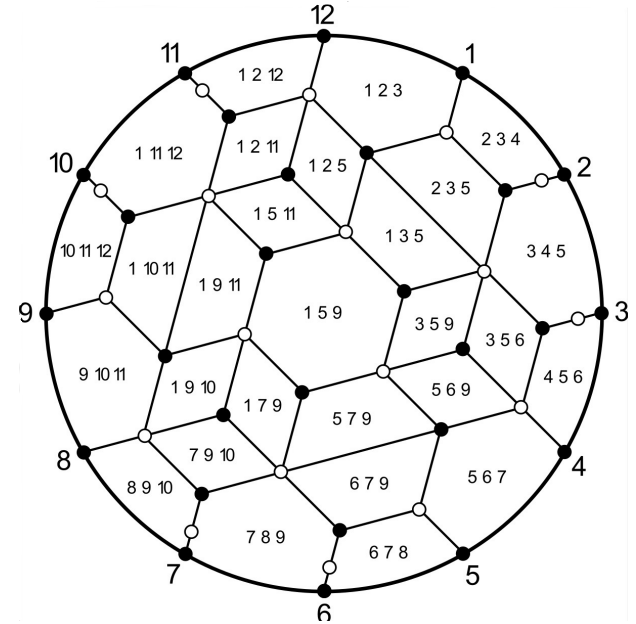
Higher Dimer Example

Consider S to be the function represented by the following SL_3 -web:

(cluster variable in $\widehat{\mathbb{C}[Gr(3,12)]}$)



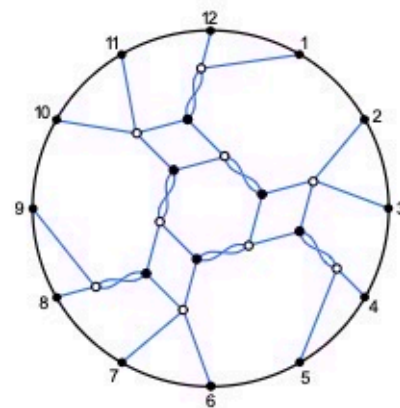
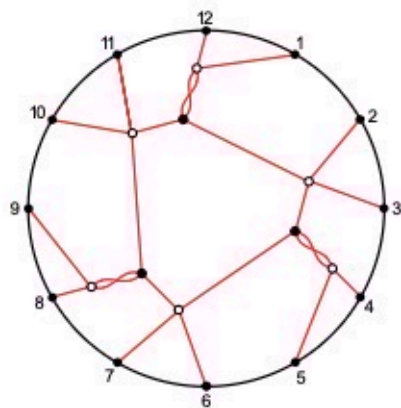
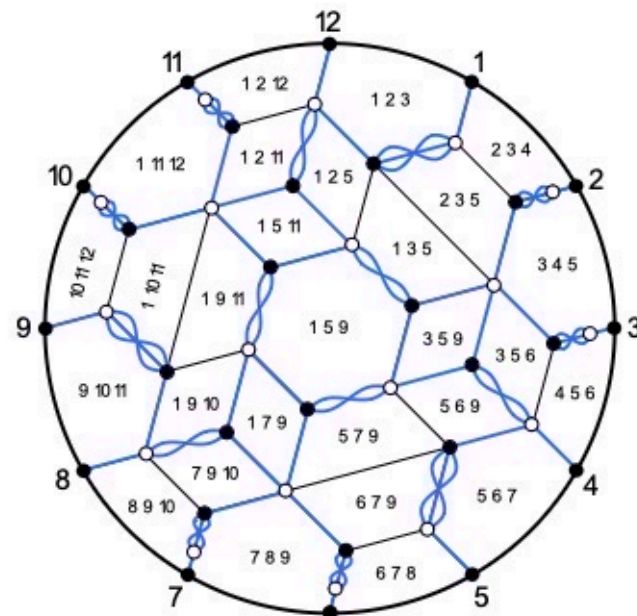
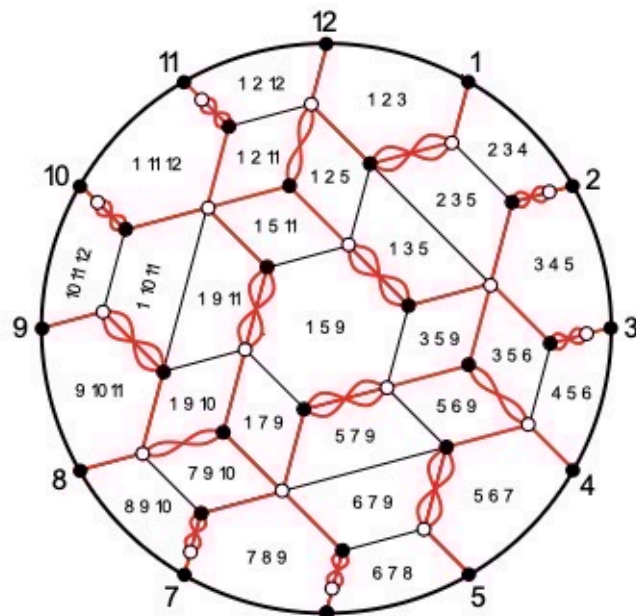
Consider the initial seed given by :



We compute the Laurent expansion for $\tau^*(S)$ using
quadruple dimers with SL_4 -web S^* .

Higher Dimer Example

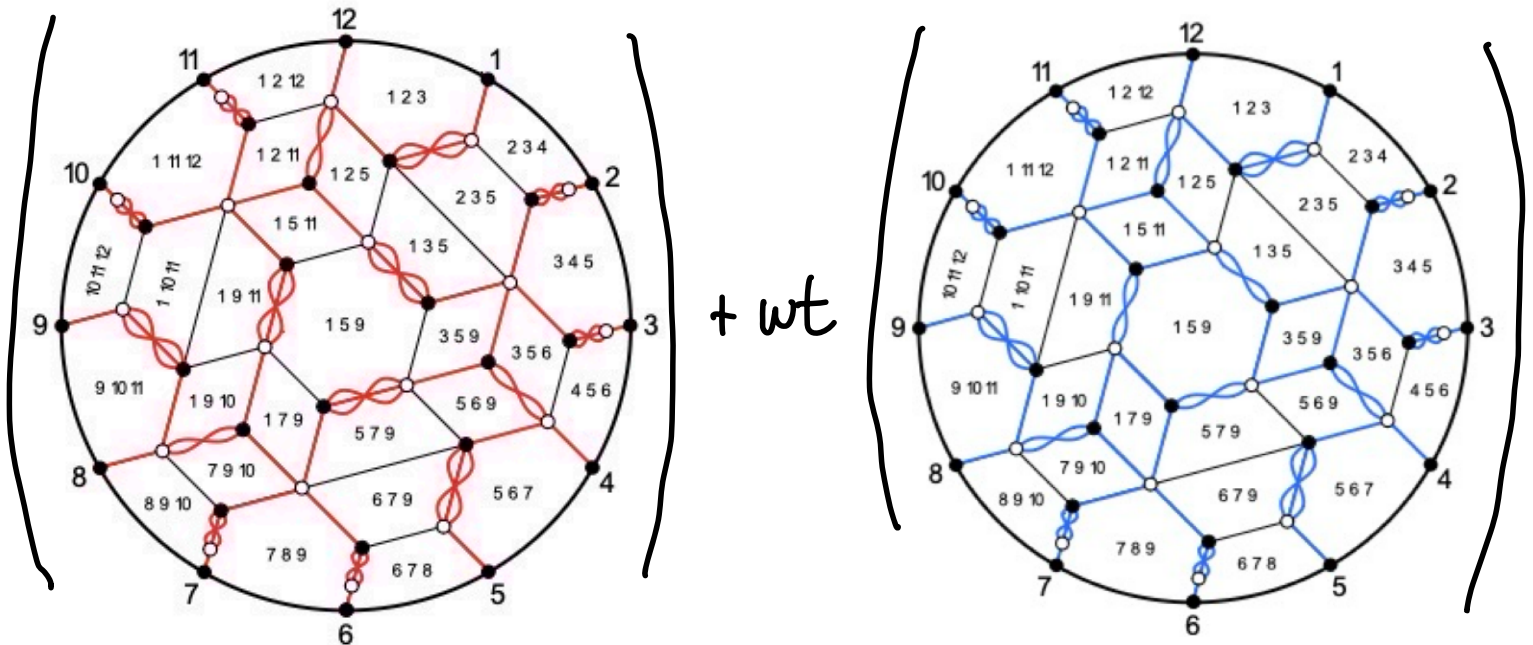
Find all quadruple dimers with weblike subgraph S^* :



have to reduce
via SL_4 - skein
relations
[GPPSS]

Higher Dimer Example

The terms of the Laurent expansion for $\tau^*(S)$ are given by :

$$\tau^*(S) = \text{wt} \left(\text{Diagram 1} \right) + \text{wt} \left(\text{Diagram 2} \right)$$


The image shows two circular diagrams, each representing a term in the Laurent expansion of $\tau^*(S)$. Both diagrams have 12 vertices labeled 1 to 12 around the perimeter. The internal structure consists of a network of edges connecting vertices. In the first diagram (left), the edges are colored red. In the second diagram (right), the edges are colored blue. The diagrams are enclosed in large parentheses, and the expression $\tau^*(S) = \text{wt} (\text{Diagram 1}) + \text{wt} (\text{Diagram 2})$ is written to the left of the first diagram.

Clustering Clusters Conjectures

Conjecture: [Cheung - Dechant - He - Heyes - Hirtz - Li '22]

A machine learning model enumerates the # of cluster variables...

$$\#\{\text{degree 3 in } \widehat{\mathbb{C}[\text{Gr}(3,n)]}\} = 24 \binom{n}{8} + 9 \binom{n}{9}$$

$$\#\{\text{degree 4 in } \widehat{\mathbb{C}[\text{Gr}(3,n)]}\} = 288 \binom{n}{9} + 400 \binom{n}{10} + 264 \binom{n}{11} + 52 \binom{n}{12}$$

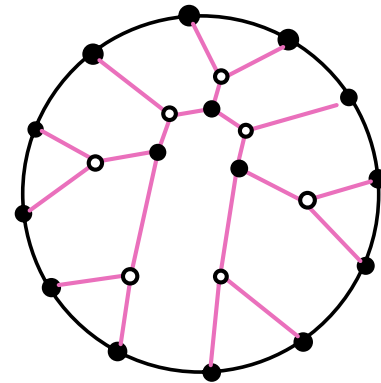
$$\begin{aligned} \#\{\text{degree 3 in } \widehat{\mathbb{C}[\text{Gr}(4,n)]}\} &= 174 \binom{n}{8} + 855 \binom{n}{9} + 1285 \binom{n}{10} \\ &\quad + 698 \binom{n}{11} + 123 \binom{n}{12} \end{aligned}$$

Connecting Conjectures

Assuming the Fomin-Pylyavskyy conjectures hold, we need to find "indecomposable & arborizable" webs. 

Idea: find all "indecomposable
& arborizable" webs
in $\mathbb{C}[\text{Gr}(3,12)]$

52



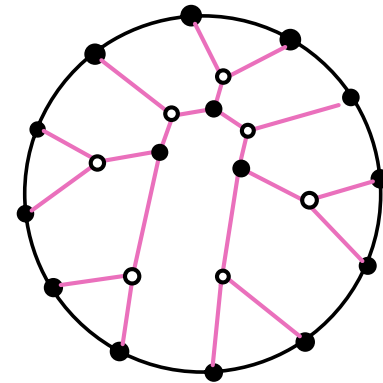
$$\#\{\text{degree 4 in } \mathbb{C}[\widehat{\text{Gr}(3,n)}]\} = 288 \binom{n}{9} + 400 \binom{n}{10} + 264 \binom{n}{11} + 52 \binom{n}{12}$$

Connecting Conjectures

Assuming the Fomin-Pilyavskyy conjectures hold, we need to find "indecomposable & arborizable" webs. 

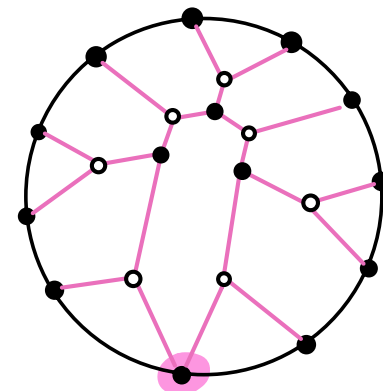
Idea: find all "indecomposable & arborizable" webs in $\mathbb{C}[\widehat{\text{Gr}}(3,12)]$

52



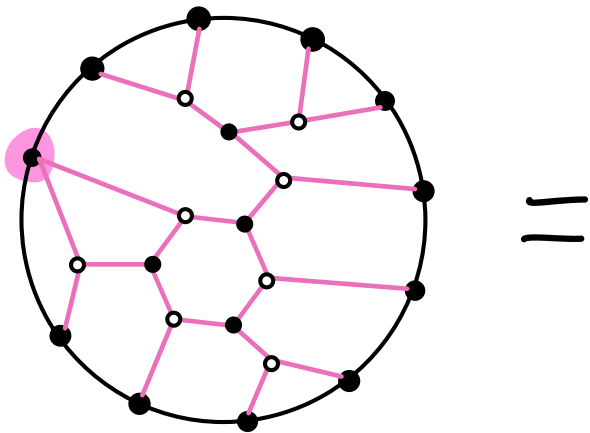
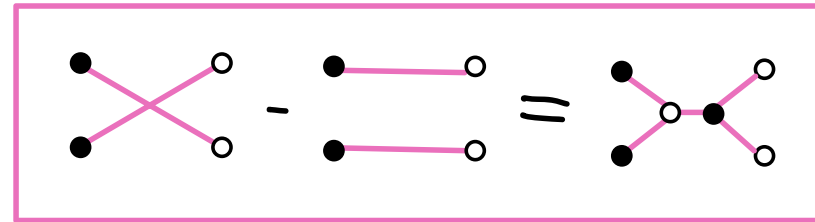
clasp two boundary vertices, then arborize to obtain web in $\mathbb{C}[\widehat{\text{Gr}}(3,11)]$

264

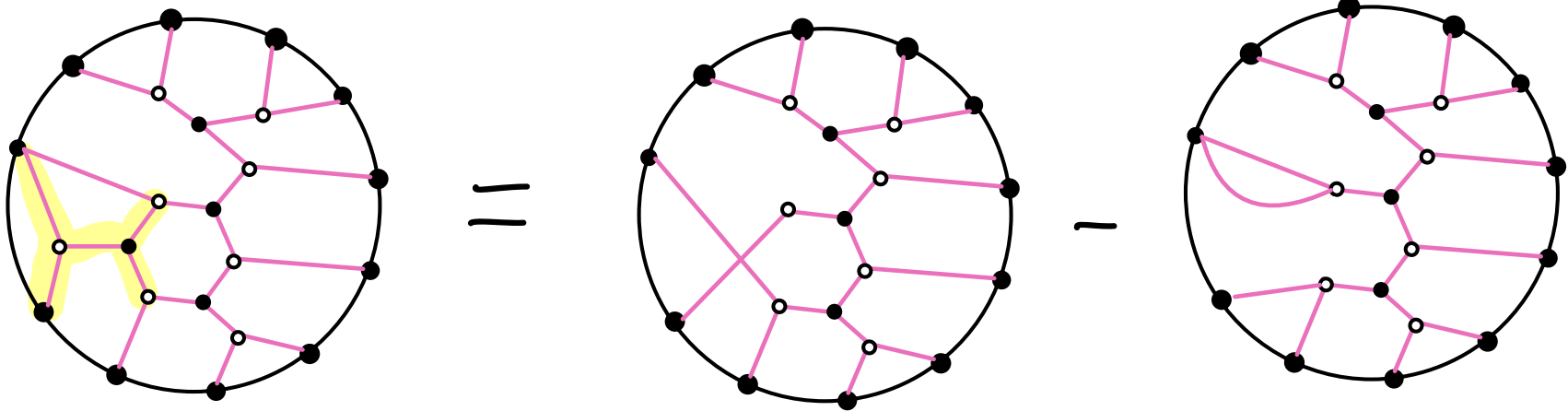
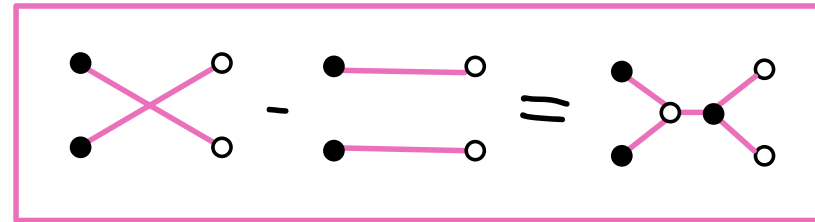


$$\#\{\text{degree 4 in } \mathbb{C}[\widehat{\text{Gr}}(3,n)]\} = 288 \binom{n}{9} + 400 \binom{n}{10} + 264 \binom{n}{11} + 52 \binom{n}{12}$$

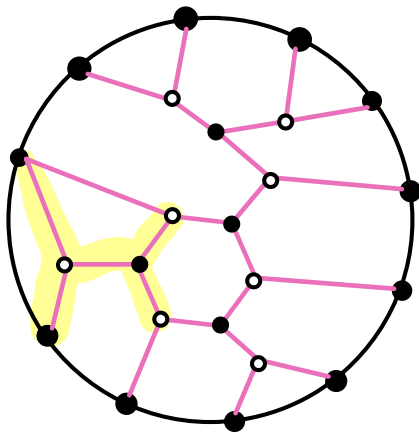
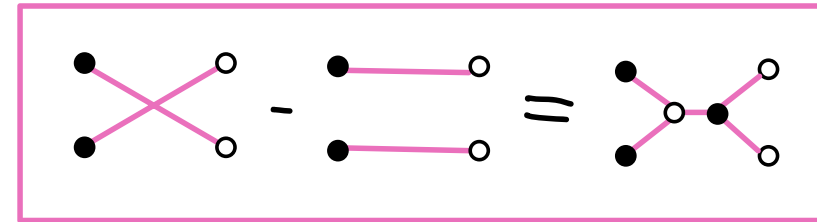
Arborization



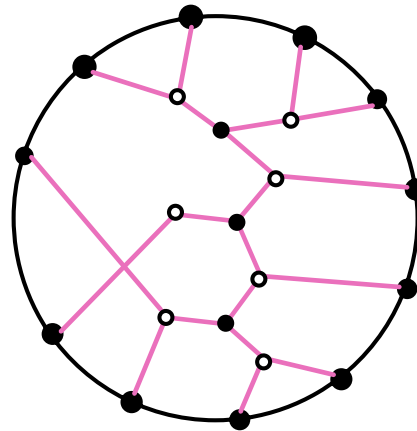
Arborization



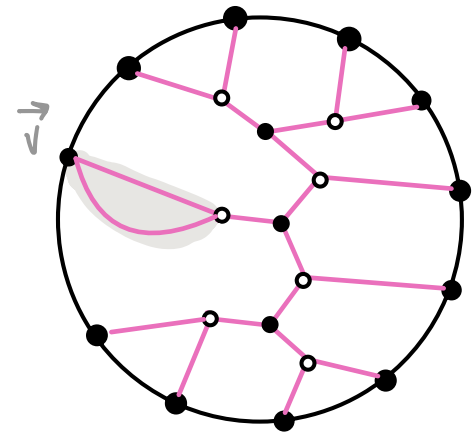
Arborization



$=$

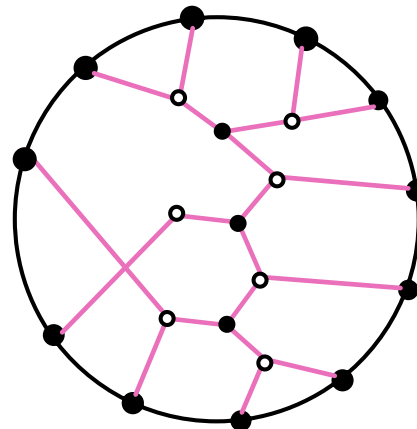


$-$



$$\det \begin{pmatrix} \vdots & \vec{v} & \vdots & \vec{v} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = 0$$

$=$



connected tree



$\Rightarrow$ cluster variable
in $\mathbb{C}[Gr(3,11)]$

Arborization Affirms Conjecture

Using clasping & arborization, we are able to verify the enumerative conjecture:

Conjecture:

$$\# \{ \text{degree 3 in } \widehat{\mathbb{C}[\text{Gr}(3,n)]} \} = 24 \binom{n}{8} + 9 \binom{n}{9} \quad \checkmark$$

$$\# \{ \text{degree 4 in } \widehat{\mathbb{C}[\text{Gr}(3,n)]} \} = 288 \binom{n}{9} + 400 \binom{n}{10} + 264 \binom{n}{11} + 52 \binom{n}{12} \quad \checkmark$$

$$\begin{aligned} \# \{ \text{degree 3 in } \widehat{\mathbb{C}[\text{Gr}(4,n)]} \} &= 174 \binom{n}{8} + 855 \binom{n}{9} + 1285 \binom{n}{10} \\ &\quad + 698 \binom{n}{11} + 123 \binom{n}{12} \end{aligned}$$

Thank you

