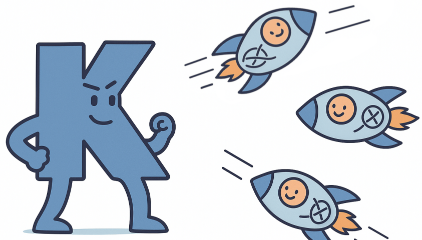
Diagrammatics for quantum symmetric pairs



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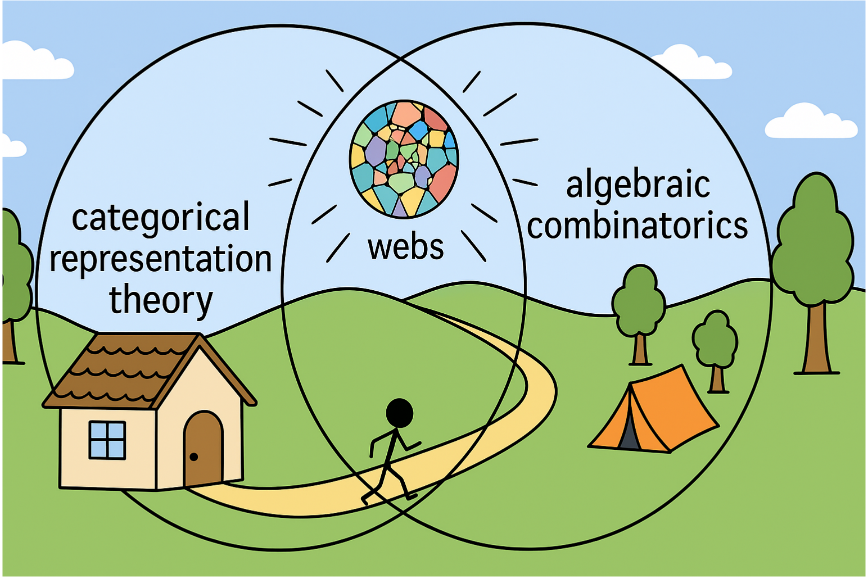
Slides: alistairsavage.ca/talks, Paper: arXiv:2507.12328

The landscape



Outline

Goal: Use diagrammatic monoidal categories and module categories to study the representation theory of quantum symmetric pairs.

Overview:

1. Categorical background: Monoidal categories, string diagrams, and module categories
2. Quantum symmetric pairs
3. The disoriented skein category
4. The iquantum Brauer category
5. Connection to quantum symmetric pairs
6. Homework!

Strict linear monoidal categories

Fix a ground field $\Bbbk$.

A strict linear monoidal category is a $\Bbbk$-linear category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbb{1}$,

such that, for objects A, B, C and morphisms f, g, h,

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$.

Remarks

- Every monoidal category is monoidally equivalent to a strict one.
- A $\Bbbk$-linear category is one where the morphism spaces are $\Bbbk$-modules, and composition and tensor product are $\Bbbk$-bilinear.

String diagrams

Fix a strict monoidal category $\mathcal{C}$.

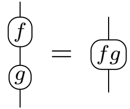
We will denote a morphism $f\colon A \to B$ by:

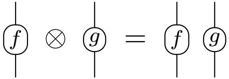
The identity map $1_A\colon A \to A$ is a string with no label:

We sometimes omit the object labels when they are clear or unimportant.

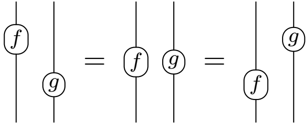
String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

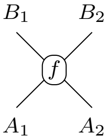




The interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:

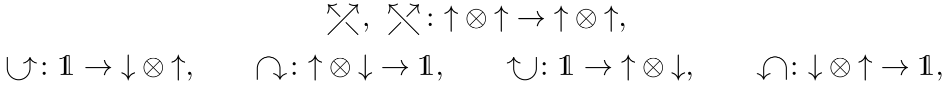


The oriented skein category

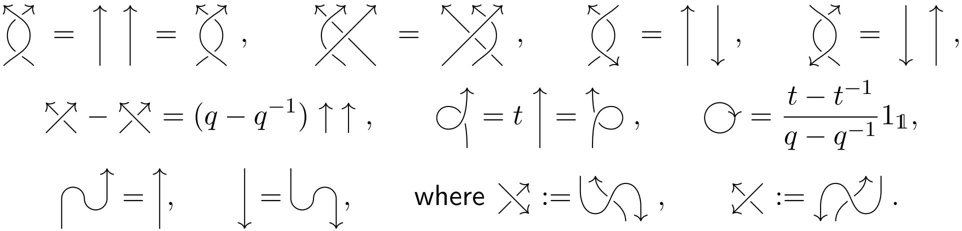
The framed HOMFLYPT skein category, or oriented skein category for short, $\mathcal{OS}(q,t)$, is the $\mathbb{C}(q,t)$-linear strict monoidal category generated by objects

$$\uparrow \qquad \text{and} \qquad \downarrow,$$

and morphisms

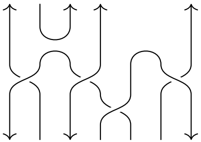


subject to the relations



The oriented incarnation functor

Morphisms in $\mathcal{OS}(q,t)$ are linear combinations of diagrams such as

 $: \downarrow \otimes \uparrow \otimes \downarrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \downarrow \to \uparrow \otimes \downarrow \otimes \uparrow \otimes \uparrow \otimes \uparrow.$

There is a full $\mathbb{C}(q)$-linear monoidal functor

$$\mathcal{OS}(q,q^m) \to U_q(\mathfrak{gl}(m))\text{-mod}, \qquad \uparrow \mapsto V, \qquad \downarrow \mapsto V^*,$$

where V is the quantum analogue of the natural module.

Connection to webs

The $U_q(\mathfrak{gl}_n)$ web category is a partial idempotent completion of $\mathcal{OS}(q,q^n)$ (plus some additional relations).

Module categories

Suppose $\mathcal{C}$ is a strict monoidal category. A category $\mathcal{M}$ is a right $\mathcal{C}$-module category, or $\mathcal{C}$-module for short, if we have a functor

$$\mathcal{M} \times \mathcal{C} \to \mathcal{M}, \qquad (M, C) \mapsto M \otimes C, \quad (f, g) \mapsto f \otimes g,$$

such that

$$M \otimes \mathbb{1} = M, \qquad f \otimes 1_{\mathbb{1}} = f,$$
$$(M \otimes C) \otimes D = M \otimes (C \otimes D), \qquad (f \otimes g) \otimes h = f \otimes (g \otimes h),$$

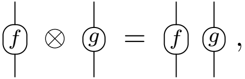
for all objects M and morphisms f in $\mathcal{M}$, and objects C, D and morphisms g, h in $\mathcal{C}$.

There is a natural notion of a morphism of $\mathcal{C}$-modules. These can be strict or not strict.

Diagrammatics for module categories

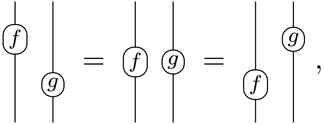
Suppose $\mathcal{C}$ is a monoidal category and $\mathcal{M}$ is a $\mathcal{C}$-module.

We have a string diagram calculus for $\mathcal{M}$, except that we can only juxtapose on the right with morphisms in $\mathcal{C}$:



$$f \in \mathrm{Mor}(\mathcal{M}),\ g \in \mathrm{Mor}(\mathcal{C}).$$

We have a form of the interchange law:



$$f \in \mathrm{Mor}(\mathcal{M}),\ g \in \mathrm{Mor}(\mathcal{C}).$$

Important:

- Morphisms in $\mathcal{M}$ can only appear on the left of a diagram.
- No horizontal juxtaposition of morphisms in $\mathcal{M}$.

Quantum symmetric pairs

Non-quantum case

Consider $\mathfrak{so}(m) \subseteq \mathfrak{gl}(m)$. Passing to universal enveloping algebras,

$$U(\mathfrak{so}(m)) \text{ is a Hopf subalgebra of } U(\mathfrak{gl}(m)).$$

Thus, restriction

$$U(\mathfrak{gl}(m))\text{-mod} \to U(\mathfrak{so}(m))\text{-mod}$$

is a monoidal functor.

Quantum case

Things are not as nice in the quantum case.

Problem: $U_q(\mathfrak{so}(m))$ is not naturally a Hopf subalgebra of $U_q(\mathfrak{gl}(m))$.

The iquantum enveloping algebra

Definition

The iquantum enveloping algebra $U^\imath = U^\imath(\mathfrak{so}(m))$ is the $\mathbb{C}(q)$-subalgebra of $U = U_q(\mathfrak{gl}(m))$ generated by

$$B_i = F_i - q^{-1} E_i K_i^{-1}, \qquad 1 \le i \le m-1,$$

where E_i, F_i, and K_i are the usual generators of U.

Remarks

Taking $q \rightsquigarrow 1$ recovers $U(\mathfrak{so}(m)) \subseteq U(\mathfrak{gl}(m))$. So, $U^\imath$ is a quantum analogue of $U(\mathfrak{so}(m))$.

The iquantum enveloping algebra

Important

$U^\imath$ is not a Hopf subalgebra of U. Instead, $U^\imath$ is a coideal subalgebra:

$$\Delta(U^\imath) \subseteq U^\imath \otimes U.$$

Thus, $U^\imath$-mod is a right U-module category.

So, if $M, N \in U^\imath$-mod and $X \in U$-mod, then

- we can form $M \otimes X \in U^\imath$-mod,
- we cannot form $M \otimes N$,
- we have the dual $X^* \in U$-mod,
- we do not have the dual M^*,
- we have a trivial module triv $\in U$-mod,
- we have a trivial module $\mathrm{Res}^U_{U^\imath}$(triv) $\in U^\imath$-mod.

Important motivating observation

Let $V \in U$-mod be the quantum analogue of the natural module for $U = U_q(\mathfrak{gl}(m))$.

Then

$$V^* \ncong V \qquad \text{as } U\text{-modules},$$

but

$$V^* \cong V \qquad \text{as } U^\imath\text{-modules},$$

as in the non-quantum setting.

So, to pass from U-mod to $U^\imath$-mod, we need to add this isomorphism.

Oversimplified foreshadowing

To go from U-mod to $U^\imath$-mod, we only need to add an isomorphism

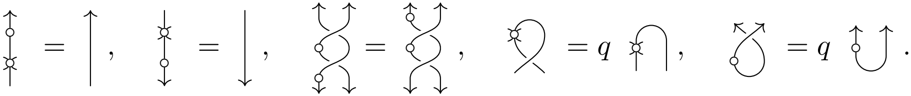
$$\varphi \colon V^* \xrightarrow{\cong} V.$$

The disoriented skein category

The disoriented skein category $\mathcal{DS}(q,t)$ is the $\mathcal{OS}(q,t)$-module generated by the morphisms

 $: \downarrow \to \uparrow$,  $: \uparrow \to \downarrow$,

which we call toggles, subject to the relations



Guiding philosophy

The toggles impose the condition that $\uparrow$ and $\downarrow$ are isomorphic.

Warning!

$\mathcal{DS}(q,t)$ is not a monoidal category!

We don't have arbitrary juxtaposition. We can only juxtapose on the right with an $\mathcal{OS}(q,t)$ diagram.

The disoriented incarnation functor

Recall that there is a full $\mathbb{C}(q)$-linear monoidal functor

$$\mathbf{R}_{\mathcal{OS}} \colon \mathcal{OS}(q, q^m) \to U\text{-mod}, \qquad \uparrow \mapsto V, \qquad \downarrow \mapsto V^*,$$

and that we have an isomorphism

$$\varphi \colon V^* \xrightarrow{\cong} V \qquad \text{of } U^\imath\text{-modules.}$$

The composition

$$U^\imath\text{-mod} \times \mathcal{OS}(q, q^m) \xrightarrow{\mathrm{id} \times \mathbf{R}_{\mathcal{OS}}} U^\imath\text{-mod} \times U\text{-mod} \xrightarrow{\otimes} U^\imath\text{-mod}$$

makes $U^\imath$-mod a $\mathcal{OS}(q, q^m)$-module.

Theorem (Salmasian–S.–Shen)

There is a full morphism of $\mathcal{OS}(q, q^m)$-modules

$$\mathbf{R}_{\mathcal{DS}} \colon \mathcal{DS}(q, q^m) \to U^\imath\text{-mod},$$

$\mathsf{B}_0 \mapsto \mathrm{Res}^U_{U^\imath}(\text{triv})$,  $\mapsto \varphi$,  $\mapsto \varphi^{-1}$.

The big picture (to be continued)

The following diagram commutes:

$$\begin{tikzcd}
\mathcal{DS} \times \mathcal{OS} \arrow[rr, "\otimes"] \arrow[dr, "\mathbf{R}_{\mathcal{DS}} \times \mathbf{R}_{\mathcal{OS}}"'] & & \mathcal{DS} \arrow[dl, "\mathbf{R}_{\mathcal{DS}}"] \\
& U^\imath\text{-mod} \times U\text{-mod} \xrightarrow{\otimes} U^\imath\text{-mod} &
\end{tikzcd}$$

Missing from the picture

Others have studied endomorphism algebras of $U^\imath$-modules. How does that work fit into the above picture?

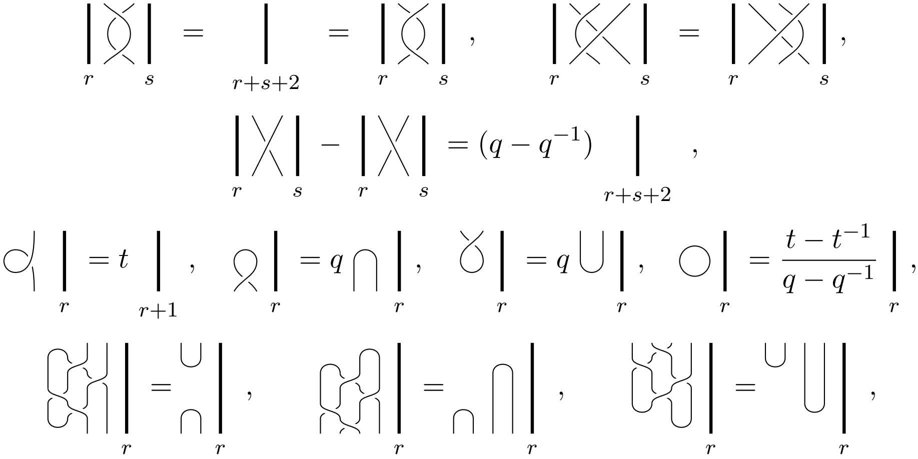
The iquantum Brauer category, $\mathcal{B}(q,t)$, is a $\mathbb{C}(q,t)$-linear category with objects B_r, $r \in \mathbb{N}$, and generating morphisms

$$\cup \ \underset{r}{\big|} : \mathsf{B}_r \to \mathsf{B}_{r+2}, \qquad \cap \ \underset{r}{\big|} : \mathsf{B}_{r+2} \to \mathsf{B}_r, \qquad r \in \mathbb{N},$$

$$\underset{r}{\big|} \ \times \ \underset{s}{\big|}\,, \quad \underset{r}{\big|} \ \times \ \underset{s}{\big|} : \mathsf{B}_{r+s+2} \to \mathsf{B}_{r+s+2}, \qquad r, s \in \mathbb{N}.$$

The iquantum Brauer category

We impose the following relations on morphisms, for all $r, s \in \mathbb{N}$:



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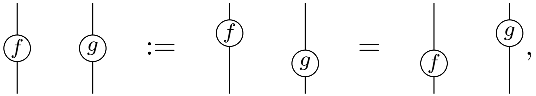
and



where f runs over all generating morphisms and g only runs over the generating morphisms

 and .

Remarks

- The iquantum Brauer category is not a monoidal category.
- One cannot horizontally juxtapose diagrams or use the general interchange law for monoidal categories.
- Caps and cups can only appear on the left-hand side of diagrams.
- The category and its endomorphism algebras were developed by Molev, Wenzl, and Satori–Tubbenhauer. They are sometimes called the q-Brauer category and q-Brauer algebras.

The big picture (to be continued)

There is a $\mathbb{C}(q)$-linear functor

$$\mathbf{R}_{\mathcal{B}} \colon \mathcal{B}(q, q^m) \to U^\imath\text{-mod}.$$

So now we have

$$\begin{tikzcd}
\mathcal{DS} \times \mathcal{OS} \arrow[rrr, "\otimes"] \arrow[dr, "\mathbf{R}_{\mathcal{DS}} \times \mathbf{R}_{\mathcal{OS}}"] & & & \mathcal{DS} \arrow[dl, "\mathbf{R}_{\mathcal{DS}}"'] \\
& U^\imath\text{-mod} \times U\text{-mod} \arrow[r, "\otimes"] & U^\imath\text{-mod} & \\
\mathcal{B} \times \mathcal{OS} \arrow[rrr, "\otimes"] \arrow[ur, "\mathbf{R}_{\mathcal{B}} \times \mathbf{R}_{\mathcal{OS}}"] & & & \mathcal{B} \arrow[ul, "\mathbf{R}_{\mathcal{B}}"']
\end{tikzcd}$$

Missing from the picture

It is natural to expect that $\mathcal{B}(q,t)$ should be a right $\mathcal{OS}(q,t)$-module category.

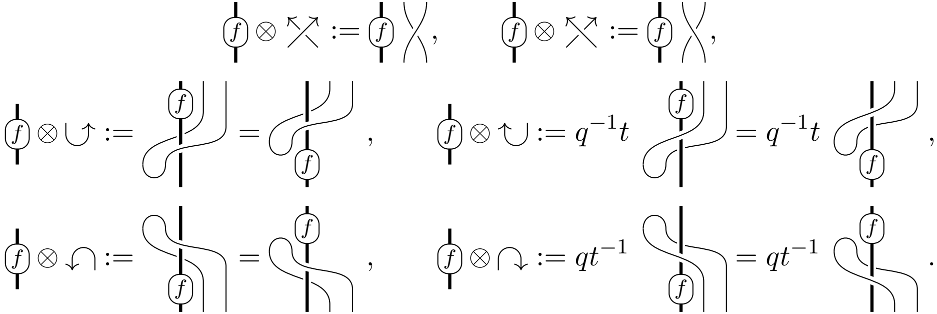
Module category structure on $\mathcal{B}(q,t)$ [Salmasian-S.-Shen]

We can make $\mathcal{B}(q,t)$ a $\mathcal{OS}(q,t)$-module as follows.

On objects, we define

$$\mathsf{B}_r \otimes \uparrow := \mathsf{B}_{r+1}, \qquad \mathsf{B}_r \otimes \downarrow := \mathsf{B}_{r+1}.$$

On morphisms, we define



The big picture (to be continued)

So now we have

$$
\begin{tikzcd}
\mathcal{DS} \times \mathcal{OS} \arrow[rrr, "\otimes"] \arrow[dr, "\mathbf{R}_{\mathcal{DS}} \times \mathbf{R}_{\mathcal{OS}}"] & & & \mathcal{DS} \arrow[dl, "\mathbf{R}_{\mathcal{DS}}"'] \\
& U^\imath\text{-mod} \times U\text{-mod} \arrow[r, "\otimes"] & U^\imath\text{-mod} & \\
\mathcal{B} \times \mathcal{OS} \arrow[rrr, "\otimes"] \arrow[ur, "\mathbf{R}_{\mathcal{B}} \times \mathbf{R}_{\mathcal{OS}}"] & & & \mathcal{B} \arrow[ul, "\mathbf{R}_{\mathcal{B}}"']
\end{tikzcd}
$$

Remarks

- The top square commutes.
- The bottom square only commutes up to natural isomorphism.

Missing from the picture

What is the connection between $\mathcal{B}(q,t)$ and $\mathcal{DS}(q,t)$?

Functor from $\mathcal{DS}$ to $\mathcal{B}$

There is a strict morphism of $\mathcal{OS}(q,t)$-modules

$$\mathbf{F}\colon \mathcal{DS}(q,t) \to \mathcal{B}(q,t),$$

$\mathbf{F}(\mathbb{1}) = \mathsf{B}_0, \qquad \mathbf{F}($  $) = 1_{\mathsf{B}_1} = \mathbf{F}($  $).$

Essentially, $\mathbf{F}$ forgets orientation and erases toggles.

One can explicitly define a (not strict!) morphism of $\mathcal{OS}(q,t)$-modules

$$\mathbf{G}\colon \mathcal{B}(q,t) \to \mathcal{DS}(q,t).$$

Theorem (Salmasian–S.–Shen)

The functors $\mathbf{F}$ and $\mathbf{G}$ are equivalences of $\mathcal{OS}(q,t)$-modules. More precisely,

$$\mathbf{FG} = \mathrm{id}_{\mathcal{B}(q,t)}, \qquad \mathbf{GF} \cong \mathrm{id}_{\mathcal{DS}(q,t)}.$$

Thus, the disoriented skein category is equivalent to the iquantum Brauer category as right $\mathcal{OS}(q,t)$-module categories.

The big picture

Now we have

$$\begin{tikzcd}
\mathcal{DS} \times \mathcal{OS} \arrow[rrr, "\otimes"] \arrow[dr, "\mathbf{R}_{\mathcal{DS}} \times \mathbf{R}_{\mathcal{OS}}"] & & & \mathcal{DS} \arrow[dl, "\mathbf{R}_{\mathcal{DS}}"'] \\
& U^\imath\text{-mod} \times U\text{-mod} \arrow[r, "\otimes"] & U^\imath\text{-mod} & \\
\mathcal{B} \times \mathcal{OS} \arrow[rrr, "\otimes"] \arrow[ur, "\mathbf{R}_{\mathcal{B}} \times \mathbf{R}_{\mathcal{OS}}"] \arrow[uu, "\mathbf{G} \times \mathrm{id}"] & & & \mathcal{B} \arrow[ul, "\mathbf{R}_{\mathcal{B}}"'] \arrow[uu, "\mathbf{G}"']
\end{tikzcd}$$

- $\mathbf{G}$ is an equivalence of categories.
- The left and right triangles commute ($\mathbf{R}_{\mathcal{B}} = \mathbf{R}_{\mathcal{DS}}\mathbf{G}$).
- The top rectangle commutes ($\mathbf{R}_{\mathcal{DS}}$ is strict).
- The bottom rectangle does not commute ($\mathbf{R}_{\mathcal{B}}$ is not strict).

Disoriented skein category versus iquantum Brauer category

The disoriented skein category $\mathcal{DS}$ has advantages over the iquantum Brauer category:

- commutativity of the diagram (incarnation functor is strict morphism of module categories),
- cups and caps in arbitrary positions,
- easier diagrammatics.

These advantages arise from the fact that $\mathcal{DS}$ contains the restrictions of the natural U-module and its dual, even though these are isomorphic as $U^\imath$-modules.

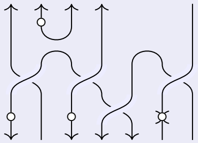
This gives extra flexibility, allowing the incarnation functor to be a strict morphism of $\mathcal{OS}(q,t)$-modules.

Other results

Basis theorems

One can describe explicit bases for the morphism spaces of $\mathcal{DS}(q,t)$, similar to the bases for Brauer, Kauffman, or oriented skein categories.

After defining toggles on arbitrary strands, the bases contain diagrams like:



As a corollary, one gets explicit bases for morphism spaces of $\mathcal{B}$ as well.

Symplectic and orthosymplectic

There are symplectic analogues of the above results. More generally, we can replace the orthogonal Lie algebra by the orthosymplectic Lie algebra.

Work in progress: Module subcategories

For each $d \in \mathbb{Z}$, we have incarnation functors

$$\mathbf{R}_{\mathcal{DS}}^{m,2n} \colon \mathcal{DS}(q, q^d) \to U^\imath(\mathfrak{osp}(m|2n))\text{-smod}, \qquad m - 2n = d.$$

Conjecture

The kernels of the $\mathbf{R}_{\mathcal{DS}}^{m,2n}$, $m - 2n = d$, give a complete set of $\mathcal{OS}(q, q^d)$-submodules of $\mathcal{DS}(q, q^d)$.

These submodules form a chain:

$$\cdots \supseteq \ker \mathbf{R}_{\mathcal{DS}}^{m,2n} \supseteq \ker \mathbf{R}_{\mathcal{DS}}^{m+2,2n+2} \supseteq \cdots .$$

Other work in progress

Diagrammatics for other quantum symmetric pairs.

Homework for spiders

Step 1: Take your favorite model, result, or theorem for webs (viewed as a monoidal category).

Step 2: Consider module categories over your webs, generated by an element corresponding to the K-matrix (the toggle).

Step 3: Profit. (Obtain web model, result, or theorem for quantum symmetric pairs.)