Skein identities at roots of unity

Vijay Higgins

University of California, Los Angeles

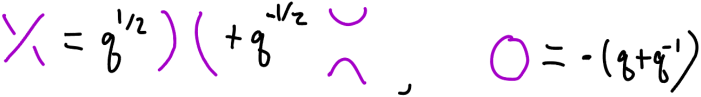
(joint with Indraneel Tambe)

ICERM, Workshop on Webs
December 11, 2025

The SL_2 skein module and skein algebra

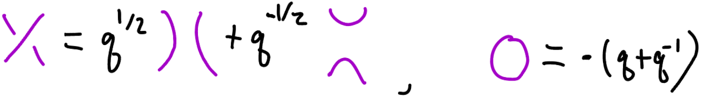
The SL_2 **skein module** of an oriented 3-manifold M is

$$\mathcal{S}_q^{SL_2}(M) = \frac{\mathbb{C}\text{-span of isotopy classes of framed links in } M}{(\text{Kauffman bracket relations})}$$



- If $M = \Sigma \times (0,1)$ is a thickened oriented surface $\mathcal{S}_q^{SL_2}(M)$ is a **skein algebra** $\mathcal{S}_q^{SL_2}(\Sigma)$ with multiplication given by stacking in the interval direction.

Basis and noncommutativity of skein algebras



- $\mathcal{S}_q^{SL_2}(\Sigma)$ has a **standard basis** of multicurves (crossingless curves on Σ with no loop bounding a disk) [Przytycki '91].
- The choice of $q \in \mathbb{C}$ influences **non-commutativity** of $\mathcal{S}_q^{SL_2}(\Sigma)$.
    - $q = 1$: $\mathcal{S}_q^{SL_2}(\Sigma)$ is commutative.
    - q generic: "trivial" center.
    - q a root of unity: rich center!

The center of the skein algebra is studied by using a **Frobenius homomorphism** for skein modules (Bonahon-Wong).

Similar Frobenius homomorphisms

- (Classical) For a commutative ring $\mathcal{R}$ of prime characteristic p, raising each **element to the pth power** $a \mapsto a^p$ gives a ring endomorphism due to the Frobenius property:

$$(a+b)^p = a^p + b^p.$$

- (Quantum algebra) If $A_q = \mathbb{C}\langle x_1, x_2 \mid x_1 x_2 = q x_2 x_1 \rangle$ for q a primitive n root of unity there is a Frobenius homomorphism $A_1 \to A_q$ by raising each **generator to the nth power** $x_i \mapsto (x_i)^n$. The generators satisfy the Frobenius property:

$$(x_1 + x_2)^n = x_1^n + x_2^n$$

  - The image is central!
- (Skein algebra) The Bonahon-Wong map goes from $\mathcal{S}_1^{SL_2}(\Sigma) \to \mathcal{S}_q^{SL_2}(\Sigma)$ by "raising" knots to "powers."

The right way to raise a knot to a power

We want a map $Fr : \mathcal{S}_1^{SL_2}(M) \to \mathcal{S}_q^{SL_2}(M)$ for $q^n = 1$ that satisfies the skein relations. Consider the easiest skein relation. Assume n is odd for this slide.

- At $q = 1$, loop $= -2$
- At $q^n = 1$, loop $= -q - q^{-1}$
- At $q^n = 1$, we want $Fr(\text{loop}) = -2 = -q^n - q^{-n}$.

Observe: $-q^n - q^{-n}$ is a polynomial in $-q - q^{-1}$

$$T_n(-q - q^{-1}) = -q^n - q^{-n},$$

where T_n is defined by $T_n(tr(A)) = tr(A^n)$ for any $A \in \mathrm{SL}_2$.

It's reasonable to expect that $Fr(\text{knot})$ is given by "replacing the knot by a polynomial of the knot."

Threading a polynomial along a knot

Definition (Focus on the picture)

K = framed knot in M, $x^i \in \mathbb{C}[x]$ = monomial, then $K^{[x^i]} \in \mathcal{S}_q^{SL_2}(M)$ obtained by <u>threading x^i along K</u> is given by taking the union of i parallel copies of K in the direction of the framing.

- If $P = \sum a_i x^i \in \mathbb{C}[x]$, then $K^{[P]} = \sum a_i K^{[x^i]} \in \mathcal{S}_q^{SL_2}(M)$.
- if $L = \cup K_j$ is a link with knot components K_j then $L^{[P]} = \cup K_j^{[P]}$.

e.g., $K^{[x^3]}$ is pictured:

Chebyshev polynomials

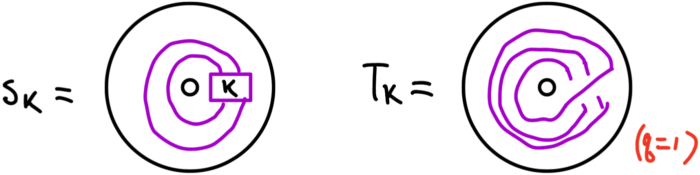
The Chebyshev polynomials $T_k, S_k \in \mathbb{Z}[x]$ are given by the following.

- (1st kind) $T_0 = 2$, $T_1 = x$, $T_k = xT_{k-1} - T_{k-2}$.
- (2nd kind) $S_0 = 1, S_1 = x, S_k = xS_{k-1} - S_{k-2}$.

They satisfy defining properties:

- $T_k(q + q^{-1}) = q^k + q^{-k}$
- $S_k(q + q^{-1}) = [k + 1]$, where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$

For core loop of annulus $x \in \mathcal{S}_q^{SL_2}(\mathcal{A}) \cong \mathbb{C}[x]$:



- S_k, T_k are involved in **positive bases** for skein algebras, unitriangularly related to the standard basis (Queffelec, Mandel-Qin, D. Thurston, Frohman-Gelca, Queffelec-Russell)

Frobenius homomorphism for SL_2 skein modules

Theorem (Bonahon-Wong '12)

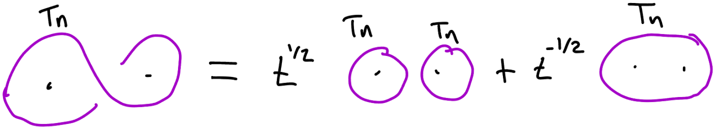
Let M be an oriented 3-manifold and q be a root of unity with $n > 0$ being the smallest integer such that $q^n \in \{1, -1\}$. Set $t := q^{n^2} \in \{1, -1\}$. Then there exists a homomorphism of skein modules

$$Fr : \mathcal{S}_t^{SL_2}(M) \to \mathcal{S}_q^{SL_2}(M)$$

defined on links by $L \mapsto L^{[T_n]}$, by threading T_n along each component of the link.

- The proof of Bonahon-Wong used the quantum trace map and a Frobenius map for Chekhov-Fock quantum torus.
- Lê '13 gave a purely skein theoretic proof.
- *Fr* can be used to characterize the centers of skein algebras and study the representation theory (Bonahon-Wong '12, Frohman–Kania-Bartoszynska–Lê '19, Ganev-Jordan-Safronov '19)

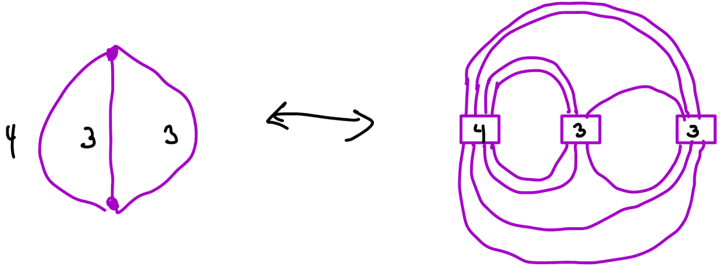
The hard thing to check is that the Frobenius homomorphism $\mathcal{S}_t^{SL_2}(M) \to \mathcal{S}_q^{SL_2}(M)$ respects the Kauffman bracket relation e.g.:



when $t^{1/2} = q^{n^2/2}$.

- We will see how to prove this identity using Jones-Wenzl projectors at a root of unity, with an eye towards webs.
- The fact that T_n produces central elements can be generalized to other skein theories using multivariable polynomials:
  SL_n (Bonahon-H. '23),
  G_2 (Beaumont-Gould-Brodsky-H.-Hogan-Melby-Piazza '23),
  Sp_{2n} (H.-Wu, '25),
  PGL_2 (H.-Silva-Somers-Sundar-Stephens-Wokhlu '25).

In future work, we want to compute the image of *Fr* on an SL_2 **web** (quantum spin network) without rewriting as links.



- Partial results forthcoming in (H.-Silva-Somers-Stephens-Sundar-Wokhlu '25)
- Work of (Rose-Tubbenhauer '15) shows similar SL_2 webs called "symmetric webs" satisfy similar skein relations to Cautis-Kamnitzer-Morrison SL_n webs.

Frobenius map for SL_3 skein modules

Theorem (H. '24, Kim-Lê-Wang '25)

There is a Frobenius homomorphism for SL_3 skein modules given by threading 2-variable analogues of T_n along links.

- KLW '25 further generalizes the map to stated SL_n skein modules of manifolds with at least 1 marked point
- Our proofs use the quantum group $\mathcal{O}_q(SL_n)$ and are not completely skein theoretic.
- Little is known about the Frobenius image of a web.

Theorem (H.-Kim-Wang '25)

The image of a Kuperberg SL_3 web on a surface under the Frobenius map is a $\underline{\mathbb{Z}\text{-linear}}$ combination of elements of the Sikora-Westbury non-elliptic web basis.

Revisiting SL_2

We want to understand the Frobenius homomorphism skein theoretically in a way that hopefully helps to give:

- a purely skein theoretic way to construct the Frobenius homomorphism for SL_3 and beyond
- a way to compute the image of the Frobenius on a web

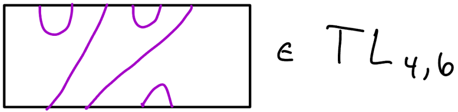
Some specific ingredients we want:

- skein identities which are somewhat local
- Relationships between the Frobenius elements and Jones-Wenzl projectors

Temperley-Lieb calculus

The TL category has

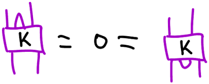
- Objects: non-negative integers
- Morphisms: The Hom space $k \to l$ is the vector space $TL_{k,l}$ spanned by (k,l)-tangles in a rectangle modulo Kauffman bracket relations, e.g.:



- The category TL is isomorphic to the full subcategory of $U_q(sl_2)$-mod with objects $V^{\otimes k}$ for the standard 2-dim rep $V = \mathbb{C}^2$.
- If $q \in \mathbb{C}^\times$ is generic, idempotent completion of TL gives $U_q(sl_2)$-mod.

Jones-Wenzl projectors

For generic q, the $U_q(sl_2)$ irreps $V_k \leftrightarrow JW_k \in TL_{k,k}$, the Jones-Wenzl projectors defined by the following axioms:

- (1) The coefficient of Id_k of JW_k in standard basis is 1
- (2) each JW_k is "uncappable": 

- Recursive definition:

- The recursion shows JW_k exists if $[2], [3], ..., [k] \neq 0$.
- Some JW projectors exist even if last point doesn't hold!

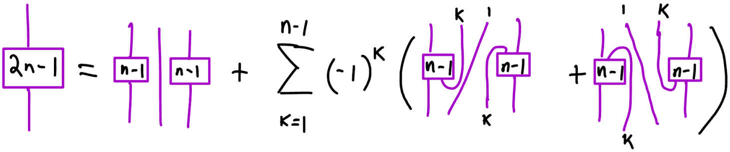
Temperley-Lieb category at roots of unity

Let $q \in \mathbb{C}$ be a root of unity such that n is the smallest positive integer for which $q^n \in \{-1, 1\}$, i.e. $[n] = 0$ but $[k] \neq 0$ for $k < n$.

- The idempotent completion of TL is the category of tilting modules of $U_q(sl_2)$, direct summands of $V^{\otimes k}$ (Elias '15)
- The JW recursion implies $JW_1, ..., JW_{n-1} \in TL$ exist.
- JW_n does not exist but $JW_{2n-1}, JW_{3n-1}, ...$ exist.

JW_{2n-1} at a root of unity

When n is the smallest positive integer such that $q^n \in \{-1, 1\}$, the projector JW_{2n-1} is given by the following formula (Martin-Spencer '21).



- It is fast to check that this expression satisfies the axioms for JW_{2n-1}.
- You can obtain the same expression by writing JW_{2n-1} in generic q and canceling the denominators of $[n]$ before specializing q.
- These JW projectors and other idempotents for tilting modules are constructed in Sutton-Tubbenhauer-Wedrich-Zhu '21, Martin-Spencer '21, Goodman-Wenzl '93,...

The quantum group at roots of unity

The following provides context for the skein identities we want.

Let $t = q^{n^2} = \pm 1$. There is a functor of ribbon categories

$$Fr : \mathrm{Rep}_t(SL_2) \to \mathrm{Rep}_q(SL_2)$$

coming from Lusztig's Frobenius map $U_q(sl_2) \to U_t(sl_2)$.

- For the standard rep $V^{(t)}$ of $U_t(sl_2)$, $Fr(V^{(t)})$ is not a tilting module. There is no idempotent in TL corresponding to it.
- There is a Steinberg tensor product formula (Andersen-Wen '92):

$$V^{(q)}_{n-1} \otimes Fr(V^{(t)}) \cong V^{(q)}_{2n-1}$$

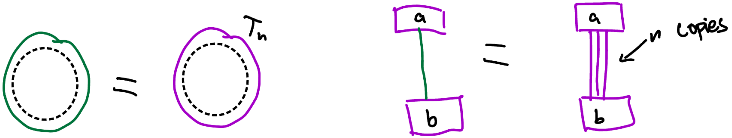
- $V^{(q)}_{n-1} \leftrightarrow JW_{n-1}$ (the Steinberg module) and $V^{(q)}_{2n-1} \leftrightarrow JW_{2n-1}$.

Frobenius strands

Although $Fr(V^{(t)})$ doesn't have a corresponding idempotent in TL, the representation still manifests itself in skein module theory in two common ways:

- A knot threaded by T_n is the image of the knot under the Bonahon-Wong Frobenius map
- n parallel copies of an arc is the image of an arc under certain Frobenius maps, e.g. for stated skein modules or Muller skein modules (Lê-Paprocki '18, Bloomquist-Lê '20)

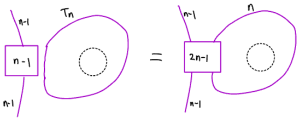
We will later use the following notation to use a green strand to label either type of Frobenius strand:

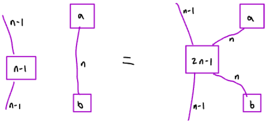


Steinberg skein identities

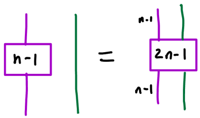
Theorem (H.-Tambe ’25)

The following skein identities hold when n is the smallest positive integer such that $q^n \in \{-1, 1\}$.





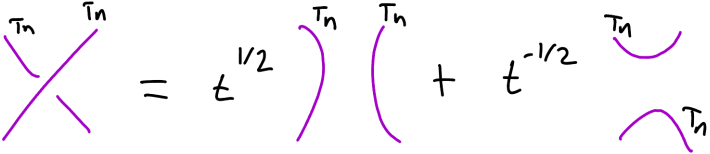
Using the green strand notation, both identities are encoded by the local identity



- We view the identities as incarnations of the Steinberg tensor product formula $V_{n-1} \otimes Fr(V) \cong V_{2n-1}$.

Threading map of Bonahon-Wong

To show that $L \mapsto L^{[T_n]}$ gives a well-defined skein module homomorphism $\mathcal{S}_t^{SL_2}(M) \to \mathcal{S}_q^{SL_2}(M)$, the main identity we need to show is:



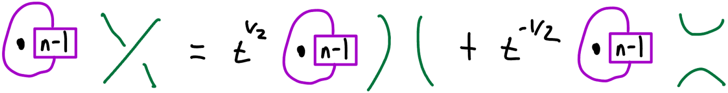
in $\mathcal{S}_q^{SL_2}(M)$ when $t^{1/2} = (q^{1/2})^{n^2}$.

- It will suffice to consider the identity in a thickened surface.
- It will suffice to assume the thickened surface is punctured.
- This identity doesn't seem to involve JW_{n-1}, so we need a trick in order to use our skein identities.

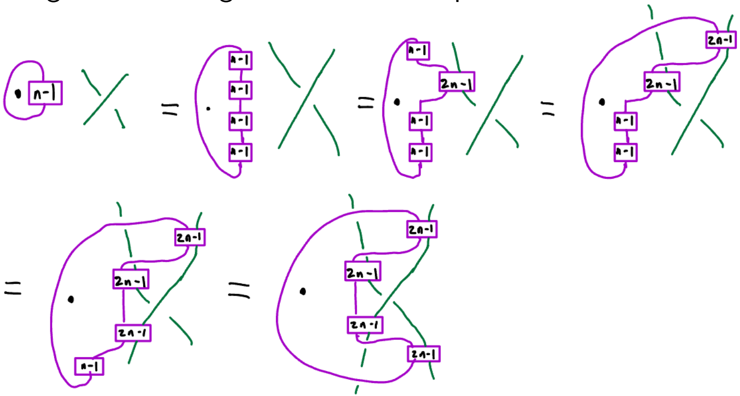
The trick: multiply by 1

- JW_{n-1} closed around a puncture is not a zero divisor in the skein algebra of the surface. (Think of multiplying in the standard basis)

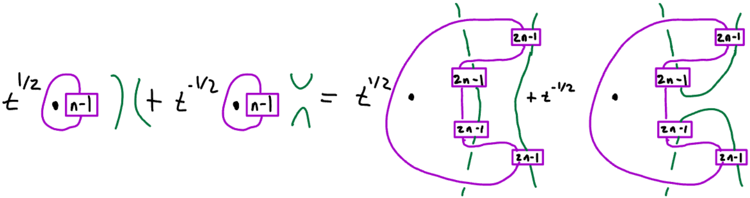
It will suffice to show the following identity:



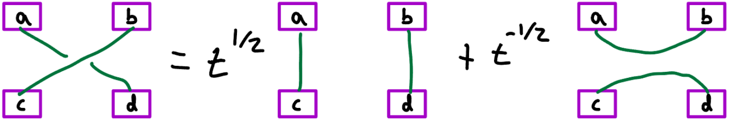
Using the Steinberg identities we compute



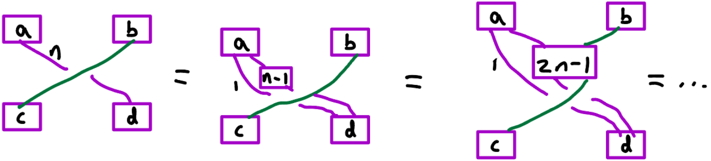
Similarly,



It remains to see



- This follows from rather well-known identities.
- The Steinberg identities give us a new proof that starts:



Thank you