Skein Algebras of Surfaces and quantum groups

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Kauffman bracket, Jones polynomial

$D \subset \mathbb{R}^2$: link diagram $\longrightarrow \langle D \rangle \in \mathbb{Z}[q^{\pm 1}]$

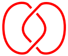
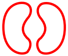
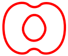
 $=\ q$  $+\ q^{-1}$  (1)

 $=\ (-q^2 - q^{-2})$ (2)

- Example:

$\langle$  $\rangle = d := -q^2 - q^{-2}$

$\langle$  $\rangle = q\langle$  $\rangle + q^{-1}\langle$  $\rangle = qd + q^{-1}d^2 = -q^{-3}d$

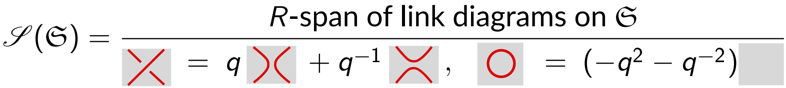
$\langle$  $\rangle = q^2\langle$  $\rangle + \langle$  $\rangle + \langle$  $\rangle + q^{-2}\langle$  $\rangle$

$$= q^2 d^2 + 2d + q^{-2}d^2 = q^6 + q^2 + q^{-2} + q^{-6}$$

- $\langle D \rangle$ invariant of framed links. Framed Jones polynomial.

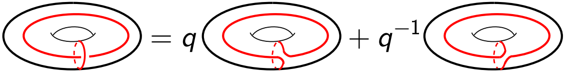
Kauffman bracket skein module

Ground ring $R = \mathbb{Z}[q, q^{-1}]$, or $R = \mathbb{C} \ni q \neq 0$.
$\mathfrak{S}$ oriented surface. Skein module (Przytycki, Turaev)



Convention: $\emptyset$ is a link diagram.
Link diagrams = framed links in $\widetilde{\mathfrak{S}} := \mathfrak{S} \times (-1, 1)$.

- Example:



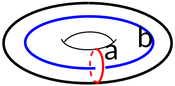
- Przytycki: $\{$simple diagrams$\} = R$-basis

(simple=no crossings, no trivial knot).
Simplifying relations $+$ diamond lemma.

Algebra structure

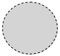
$\mathscr{S}(\mathfrak{S})$ is an **associative algebra with unit** (Turaev)

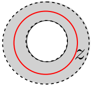
$$\alpha_1\alpha_2 = \begin{array}{c}\boxed{\alpha_1}\\ \boxed{\alpha_2}\end{array}$$

example: $a.b =$ 

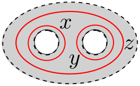
- unit $=$ empty link.
- $\mathscr{S}(\mathfrak{S})$ non-commutative in general.
- There are $\mathfrak{S} \neq \mathfrak{S}'$ with $\tilde{\mathfrak{S}} = \tilde{\mathfrak{S}}'$. But as algebra $\mathscr{S}(\mathfrak{S}) \neq \mathscr{S}(\mathfrak{S}')$.

Examples: Simple surfaces

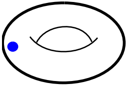
$\Sigma_{0,1}$  $\mathscr{S} \cong R = \mathbb{Z}[q^{\pm 1}], \qquad L \to \langle L \rangle$

$\Sigma_{0,2}$ 

$\mathscr{S} \cong R[z]$

$\Sigma_{0,3}$ 

$\mathscr{S} \cong R[x, y, z]$

$\Sigma_{1,1}$  $\mathscr{S} \cong R\langle x, y, z \rangle / Rel$

$q^2 - q^{-2}$ is invertible. *Rel* is

$$[x, y]_q = z, [y, z]_q = x, [z, x]_q = y$$

Here $[x, y]_q = qxy - q^{-1}yx$. (Bullock-Przytycki; quantum $SO(3)$.)

Quantization of Character variety

If $R = \mathbb{C}, q = \pm 1$, then $\mathscr{S}_{\pm 1}(M)$ is commutative

$(q = -1)$

Turaev, Przytycki-Sikora, Bullock-Frohman-Kania-Bartoszynska:

$$\mathscr{S}_{-1}(\mathfrak{S}) \cong \mathbb{C}[\chi_{SL_2}(\mathfrak{S})], \quad \chi_{SL_2}(\mathfrak{S}): \text{ character variety:}$$
$$\chi_{SL_2}(\mathfrak{S}) := \mathrm{Hom}(\pi_1(\mathfrak{S}) \to SL_2(\mathbb{C}))//SL_2(\mathbb{C})$$

- quantization along the Atiyah-Bott-Goldman bracket.
- connects Jones polynomial and classical topology.
- Used in TQFT.
- Helps proving AJ conjecture for many knots.
- Closely related to (quantum) Teichmüller space, cluster algebras.

$\mathfrak{g}$-skein algebra

Generalization of Jones polynomial (Reshetikhin-Turaev):
Ribbon category $\mathcal{C} \rightsquigarrow$ operator invariants of ribbon graphs, including (colored) framed oriented links.

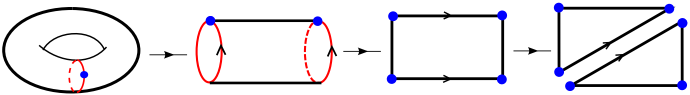
$\rightsquigarrow$ $\mathcal{C}$-skein algebra (Walker)

$$\mathscr{S}_{\mathcal{C}}(\mathfrak{S}) = \frac{R\langle \text{ribbon graphs in } \mathfrak{S} \times (0,1) \rangle}{\text{local RT operator relations}}$$

• Main examples of ribbon categories: $\mathfrak{g}$ simple Lie algebra
$\rightsquigarrow$ $U_q(\mathfrak{g})$ quantized enveloping algebra
$\mathrm{Rep}(U_q(\mathfrak{g}))$ is a ribbon category $\rightsquigarrow$ $\mathscr{S}_{\mathfrak{g}}(\mathfrak{S})$
• quantization of the $G(\mathfrak{g})$-character variety;
Atiyah-Bott-Goldman's Poisson structure.
• $\mathfrak{g} = sl_2 \rightsquigarrow$ Kauffman bracket skein modules.
• For root of unity q, there are many versions of $\mathrm{Rep}(U_q(\mathfrak{g}))$.

Cutting surfaces

Most punctured surfaces can be cut into ideal triangles:



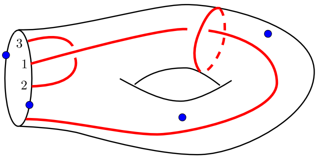
$$\mathfrak{S} = \bigsqcup \tau_i / \text{(edge identications)}$$

Can we study $\mathscr{S}(\mathfrak{S})$ using the triangulation?

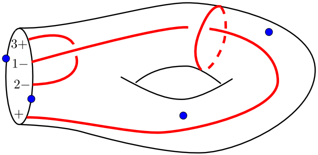
- Understand behavior of $\mathscr{S}(\mathfrak{S})$ under cutting of surfaces.
- Need to extend skein algebra to involve the boundary.

Extension of skein algebras: Tangle diagrams

Goal: Extend $\mathscr{S}(\mathfrak{S})$ to involve boundary edges.
Assumption: each component of $\partial\mathfrak{S}$ is $(0,1)$.
Links in $\widetilde{\mathfrak{S}} = \mathfrak{S} \times (-1,1)$ can “come" to boundary: Tangles in $\widetilde{\mathfrak{S}}$.



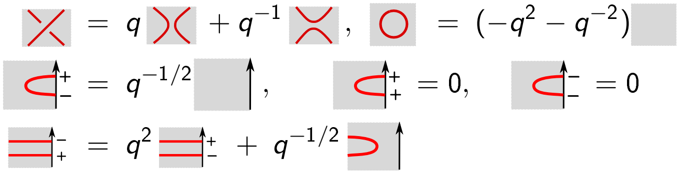
Tangle diagram α on $\mathfrak{S}$: closed curves and arcs with endpoints on $\partial\mathfrak{S}$.
In interior of $\mathfrak{S}$: like a link diagram
Each boundary edge b of $\mathfrak{S}$: a linear (height) order on $b \cap \partial\alpha$



State $s : \partial\alpha \to \{\pm\}$

Stated skein algebra (L. 2016, $\mathfrak{g} = sl_2$)

$$\mathscr{S}(\mathfrak{S}) := \frac{R\text{-span of stated tangles in } \widetilde{\mathfrak{S}}}{(1),\text{ boundary rels } (2)\ \&\ (3)}$$



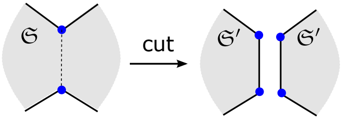
(1)

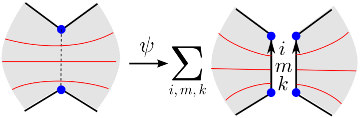
(2)

(3)

• RT sl_2-operator invariant, (dual)canonical basis.
Equ (2): Bonahon-Wong work on quantum trace.
• Geometric Basis: simple diagrams with **increasing states** in counterclockwise direction. (Simplifying relations + diamond lemma.)
• product $\alpha\beta$: α is above β, higher on each boundary edge.

Cutting homomorphism





Theorem (L. 2016)

ψ is an algebra algebra homomorphism $\psi : \mathscr{S}(\mathfrak{S}) \to \mathscr{S}(\mathfrak{S}')$.
Injective (any ground ring).

The exact image is known. (Hochschild cohomology, [CL,KQ])

$$\text{triangulation } \lambda \rightsquigarrow \Psi : \mathscr{S}(\mathfrak{S}) \to \bigotimes_{\tau:\text{faces}} \mathscr{S}(\tau)$$

A presentation of $\mathscr{S}(\tau)$ is known.
$\rightsquigarrow$ many useful facts; quantum trace map (Bonahon-Wong sl_2; L.-Yu sl_n; Kim sl_3. Quantization of Fock-Goncharov map sl_n).

General Lie algebras

Stated skein algebra

• $\mathfrak{g} = sl_3$: V. Higgins. (Kuperberg's relations, reduction rules) Basis. Cutting homomorphism is injective.

• $\mathfrak{g} = sl_n$: L.-Sikora.

• Costantino-Korinman-L.: General Tannakian ribbon category.

• over $\mathbb{Q}(q)$: related to factorization homology, lattice field theory of Alekseev-Grosse-Schomerus and Buffenoir-Roche, skein category (work of Ben-Zvi-Brochier-Jordan, Cooke, Haioun).

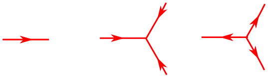
• Important problems: sl_n with $n \geq 4$

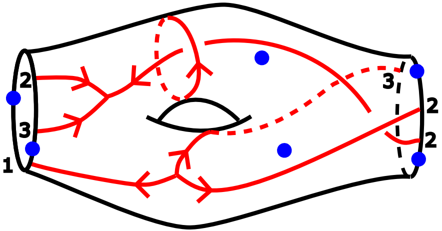
(1) natural geometric bases?

(2) Is cutting homomorphism injective? (over $\mathbb{Z}[q, q^{\pm 1}]$)

Generators of $\mathscr{S}_{sl_n}(\mathfrak{S})$

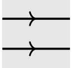
• *n*-web-diagram on $\mathfrak{S}$: 1-dimensonal & oriented; locally either a smooth point (including boundary point), or an n-valent sink or source (Example: $n = 3$)

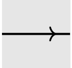




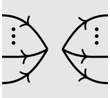
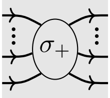
Height order on each boundary edge. States are from $1, \ldots, n$. Relations coming from local identities of the RT operators.

Interior Relations for SL_n-skein algebra

$q^{\frac{1}{n}}$  $-q^{-\frac{1}{n}}$  $= (q - q^{-1})$ ,

 $= (-1)^{n-1} q^{n-1/n}$ ,

 $= (-1)^{n-1} [n]_q$, $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

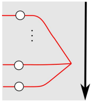
 $= (-q)^{\binom{n}{2}} \cdot \sum_{\sigma \in S_n} (-q^{(1-n)/n})^{\ell(\sigma)}$ 

.

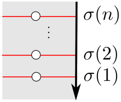
($\ell(\sigma)$ length, σ_+ positive braid)

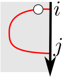
(Sikora 2005; twisted version)

Boundary Relations for sl_n

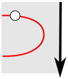


$= q^{\frac{(1-n)(2n+1)}{4}} \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)}$

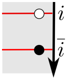




$= \delta_{\bar{j},i}\, c_i,$ where $c_i = (-1)^{n-i} q^{\frac{2n^2+n-1}{2n} - i}$



$= \sum_{i=1}^{n} (c_{\bar{i}})^{-1}$

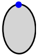


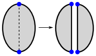
where $\bar{i} = n + 1 - i$

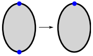
Here  stands for  or .

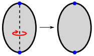
- Can be defined using MOY graphs or CKM graphs, if $[n]_q!$ is invertible. (A. Poudel, $\partial\Sigma = \emptyset$)

Monogon $\mathbb{P}_1$ and bigon $\mathbb{P}_2$

$\mathbb{P}_1$  $\mathscr{S} = R = \mathbb{Z}[\hat{q}^{\pm 1}] \ni r \to r\,\emptyset.$

 $\mathscr{S}(\mathbb{P}_2) \to \mathscr{S}(\mathbb{P}_2) \otimes \mathscr{S}(\mathbb{P}_2)$, *coproduct*

 $\mathscr{S}(\mathbb{P}_2) \to \mathscr{S}(\mathbb{P}_1) = R$ (*twisted*), *counit*

 $\mathscr{S}(\mathbb{P}_2) \to \mathscr{S}(\mathbb{P}_2)$ (*twisted*), *antipode*

} Hopf algebra

- L.-Sikora: $\mathscr{S}_{sl_n}(\mathbb{P}_2) \cong \mathcal{O}_q(sl_n)$ (integral version; Krein-Tannaka reconstruction using coend).

($n = 2$, Costantino-L., Korinman-Quesney; $n = 3$ Higgins)

- $\mathcal{O}_q(sl_2) = \mathbb{Z}[q^{\pm 1}]$-algebra generated by entries of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\text{Relations} \begin{cases} ab = qba, ac = qca, bd = qdb, cd = qdc, bc = cb \\ ad - da = (q - q^{-1})bc, \quad \det_q(A) := ad - qbc = 1. \end{cases}$$

Quantized coordinate ring of $SL_2(\mathbb{C})$. Hopf dual of $\mathcal{U}_q(sl_2)$.

(Dual) canonical basis

Kashiwara, Lusztig: $\mathcal{O}_q(sl_n)$ has **dual canonical basis** B^*.
Dual to the **canonical basis** of $\dot{\mathcal{U}}_q(sl_n)$.
Important roles in representation theory.
Explicit Formulas? $n = 2$ (Kashiwara), $n = 3$ (Zhang, Skandera, Rhoades).

• sl_2: $\mathcal{O}_q(sl_2)$ is generated by entries of $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$B^* = \{b^m a^k c^l\} \cup \{b^m d^k c^l\}$$

Two clusters $\{a, b, c\}$ and $\{d, b, c\}$; all cluster monomials.
• sl_3: $\mathcal{O}_q(sl_3)$ is generated by entries of $(u_{ij})_{i,j=1}^3$.
There are 50 clusters. Rhoades, Skandera:

$$B^* = \bigcup \{\text{cluster monomials}\}$$

One cluster $\{u_{13}, u_{31}, (12|23), (23|12), u_{23}, u_{33}, (12|13), (132)\}$
$(ij|kl) = (ij \times kl)$-q-minor; $(132) = (12|12)u_{23} - q(23|12)u_{23}$

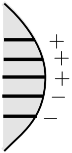
Dual canonical basis of $\mathcal{O}_q(sl_2) = \mathscr{S}(\mathbb{P}_2)$

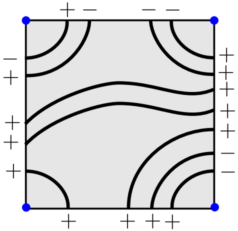
L.: Defining relations for $sl_2 \rightsquigarrow$ reductions of skeins

$\xrightarrow{\text{diamond lemma}}$ **Geometric Basis** of $\mathscr{S}(\mathfrak{S})$ (any surface)

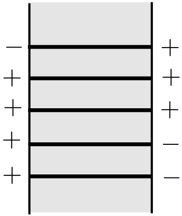
{ simple diagrams; counterclockwise increasing states}

simple: no crossings, no trivial loops, no trivial arcs.





in a square



in a bigon

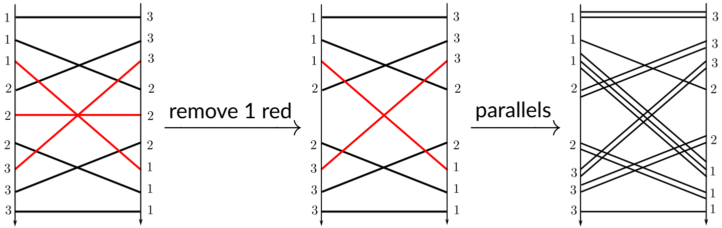
L.-Costantino: For bigon

Geometric basis = Dual canonical basis

$$= \{b^m a^k c^l\} \cup \{b^m d^k c^l\}$$

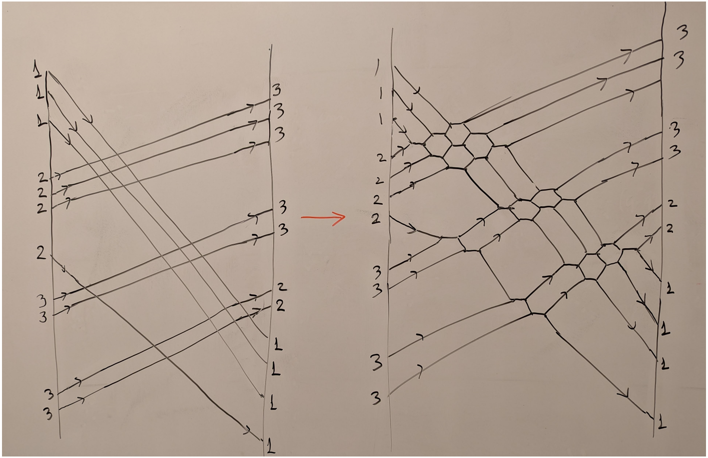
Dual canonical basis of $\mathcal{O}_q(sl_3) = \mathscr{S}_{sl_3}(\mathbb{P}_2)$

$\partial\mathfrak{S} \neq \emptyset$, Higgins: Skein relations, diamond lemma $\rightsquigarrow$ basis.
No similar basis known for sl_n, $n \geq 4$.
L.-Sikora: Dual canonical basis = modified Higgin' basis.
$\{$irred diagrams; increasing states, near $\partial\mathbb{P}_2$ direction is left to right$\}$.



Convert crossings to 3-valent vertices:

Dual canonical basis: $\mathcal{O}_q(sl_3)$



Set of all such diagrams is the dual canonical basis.

THANK YOU!