Skein and cluster quantizations of the moduli space of decorated G-local systems

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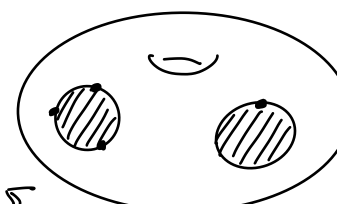
joint works w/ Hiroaki Karuo, Hironori Oya, Linhui Shen, Wataru Yuasa

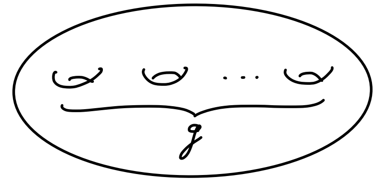
$$\begin{tikzcd}
\mathcal{A}_{G^\vee,\Sigma}(\mathbb{Z}^T) \arrow[r, "\mathbb{S}_{\mathcal{A}}"] \arrow[d, "p^T"'] \arrow[dr, phantom, "\circlearrowright"] & \overline{\mathscr{S}}^{q}_{G',(\Sigma,\mathbb{B})} \arrow[rr, "\mathrm{Tr}", "\sim"'] \arrow[d, "\Phi"'] \arrow[drr, phantom, "\circlearrowright"] & & \mathcal{O}_v(\mathcal{P}_{G',\Sigma}) \arrow[d, "p^*"] \\
\mathcal{P}_{G^\vee,\Sigma}(\mathbb{Z}^T) \arrow[r, "\mathbb{S}_{\mathcal{X}}"'] & \mathscr{S}^{q}_{G,(\Sigma,\mathbb{M})}[\partial^{-1}] \arrow[rr, "\sim", "\mathrm{Cut}"'] & & \mathcal{O}_q(\mathcal{A}^{\times}_{G,\Sigma})
\end{tikzcd}$$

Lamination | Skein | Cluster



Σ

§1. Background



$\Sigma = \Sigma_g$: closed surface $(g \geq 1)$

G : reductive group $/ \mathbb{C}$

Character variety :

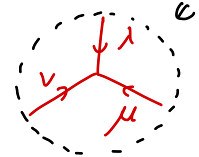
$$Ch_{G,\Sigma} := \mathrm{Hom}(\pi_1(\Sigma), G) /\!/ G \longleftrightarrow \text{moduli space of } G\text{-local systems on } \Sigma$$

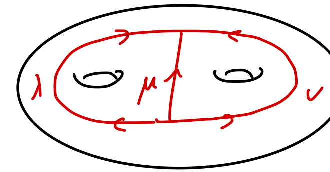
- It admits a natural symplectic form ω (Atiyah–Bott, Goldman)

Want : Quantization of $Ch_{G,\Sigma}$.

Namely, find a non-commutative deformation of $\mathcal{O}(Ch_{G,\Sigma}) = \mathcal{O}(\mathrm{Hom}(\pi, G))^G$

$\mathrm{Hom}_G(V_\lambda \otimes V_\mu \otimes V_\nu, \mathbb{C})$





$\exists$ several approaches for quantization.

Assume G : semisimple

① Skein quantization (Roughly):

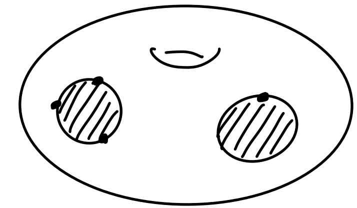
Idea : $\mathcal{O}(Ch_{G,\Sigma})$ is generated by "Trace" functions along G-colored graphs on Σ

$\rightsquigarrow$ Skein algebra

$$\mathcal{S}^q_{G,\Sigma} := \text{"}\mathbb{Z}[q,q^{-1}]\langle \underbrace{\text{tangled } G\text{-colored graphs}}_{\text{"}G\text{-webs"}} \text{ on } \Sigma \times [0,1] \rangle / \text{skein rel.}\text{"}$$

$q=1 \downarrow$

$\mathcal{O}(Ch_{G,\Sigma})$ should provide a deformation quantization



② Cluster quantization

<u>Idea</u> : $\Sigma = (\Sigma, \mathbb{M})$: a marked surface (Today $\mathbb{M} \subset \partial\Sigma$)

Consider moduli spaces $\mathcal{A}_{\underbrace{G}_{\text{simply conn.}},\Sigma}$, $\mathcal{P}_{\underbrace{G'}_{\text{adjoint}},\Sigma}$ of G-/G'-local systems on Σ
"decoration data" at $\mathbb{M}$

▷ $\exists$ Cluster structure : [Fock-Goncharov'06, Zickert'19, Le'19, Goncharov-Shen'19+]

① Cluster coordinates $(A_j^{i})_{j\in I}$, $(X_j^{i})_{j\in I}$ for each seed i (e.g. triangulation)

② Mutations $\mu_k : i \longrightarrow i'$ describe coordinate transformations

▷ Promote into quantum :

① Quantum tori T_A^{i}, T_X^{i}

② Quantum mutations $\mu_k^* : \mathrm{Frac}\, T_{A/X}^{i'} \xrightarrow{\sim} \mathrm{Frac}\, T_{A/X}^{i}$ (using quantum dilog)

$$\begin{cases} T_A^{i} : A_i^{i} A_j^{i} = q^{\pi_{ij}^{i}} A_j^{i} A_i^{i} \\ \\ T_X^{i} : X_i^{i} X_j^{i} = v^{2 \varepsilon_{ij}^{i}} X_j^{i} X_i^{i} \end{cases}$$

compatible pair : $\varepsilon^{i} \cdot \Pi^{i} = D \cdot Id$

$(q, v : \text{formal}, \ v = q^d)$

⇝ Quantum cluster algebras

$$\mathcal{O}_q(\mathcal{A}^{\times}_{G,\Sigma}) := \bigcap_{i} T_A^{i} \ , \quad \mathcal{O}_v(\mathcal{P}_{G',\Sigma}) := \bigcap_{i} T_X^{i}$$

… deformation quantization of $\mathcal{A}^{\times}_{G,\Sigma}$ & $\mathcal{P}_{G',\Sigma}$

$\boxed{\text{Problem: Compare the skein \& cluster quantizations}}$

graphical | explicit coordinates

Of course, the problem includes appropriate extensions of skein algebra for Σ

$(\mathcal{A}/\mathcal{X} \longleftrightarrow$ "clasped" / "stated" $)$

We expect the diagram :

"reduced stated" skein alg. [Lê'18]

(G = SL_2 case established)

$$\begin{tikzcd}
\mathcal{A}_{G'^{\vee},\Sigma}(\mathbb{Z}^T) \arrow[r, "S_A"', "\text{[I-Karuo'25]}"] \arrow[d, "p^T"'] \arrow[dr, phantom, "\circlearrowright"] & \overline{\mathcal{S}}^{q}_{G',(\Sigma,\mathbb{B})} \arrow[r, "\mathrm{Tr}", "\sim"'] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}_v(\mathcal{P}_{G',\Sigma}) \arrow[d, "p^*"] \\
\mathcal{P}_{G^{\vee},\Sigma}(\mathbb{Z}^T) \arrow[r, "S_X"'] & \mathcal{S}^{q}_{G,(\Sigma,\mathbb{M})}[\partial^{-1}] \arrow[r, "\sim", "\mathrm{Cut}"'] & \mathcal{O}_q(\mathcal{A}^{\times}_{G,\Sigma})
\end{tikzcd}$$

[Lê'18] [I.-Karuo'25] [Lê-Yu'22] [I.-Karuo'25]

[Musiker-Schiffler-Williams'13], [Thurston'14] [Muller'16]

given by

$$p_i^*(X_k^i) = \prod_{j\in I} (A_j^i)^{\varepsilon_{kj}^i + m_{kj}}$$

	$\mathfrak{g}$	$\mathfrak{g}^{\vee}$
sc	G	$G'^{\vee}$
ad	G'	$G^{\vee}$

"$\mathbb{Z}$-tropical points"

"∂-localized clasped" skein alg. [Muller'16]

- $\Phi : \overline{\mathcal{S}}^{q}_{G,(\Sigma,\mathbb{B})} \xrightarrow{\sim} \mathcal{S}^{q}_{G,(\Sigma,\mathbb{M})}[\partial^{-1}]$ <u>state-clasp correspondence</u>
- S_A, S_X correspond to give basis (many choices)
- $\mathbb{I}_A := \mathrm{Tr}\circ S_A : \mathcal{A}_{G'^{\vee},\Sigma}(\mathbb{Z}^T) \longrightarrow \mathcal{O}_v(\mathcal{P}_{G',\Sigma})$

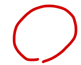
  $\mathbb{I}_{\mathcal{X}} = \mathrm{Cut}\circ S_X : \mathcal{P}_{G^{\vee},\Sigma}(\mathbb{Z}^T) \longrightarrow \mathcal{O}_q(\mathcal{A}^{\times}_{G,\Sigma})$

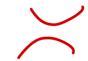
  would be <u>quantum duality map</u>.

§2. Lower rank cases $(G = SL_2, SL_3, Sp_4)$

$G = SL_2$

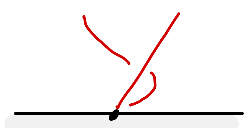
$$\boxed{\mathbb{Z}_q := \mathbb{Z}[q^{\pm 1/2}]}$$

Kauffman bracket skein algebra:  $= -(q^2 + q^{-2})$

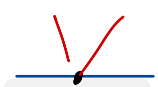
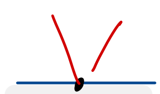
 $= q$  $+ q^{-1}$ 

clasped skein alg. [Muller'16]

$\partial\alpha \subset M \times [0,1]$

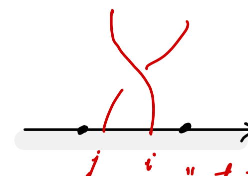


▶  $= 0$

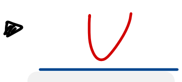
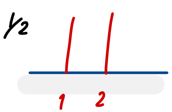
▶  $= q$ 

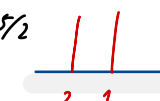
stated skein alg. [Bonahon-Wong'11, Lê'18]

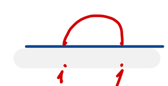
$\partial\alpha \subset \mathbb{B} \times [0,1]$, $\quad \mathbb{B} := \{\partial\text{-intervals}\}$



"states" $\in \{1, 2\}$

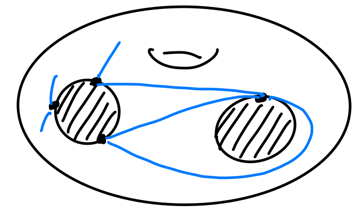
▶  $= q^{1/2}$ 

$- q^{5/2}$ 

▶ $U_{ij} :=$ 

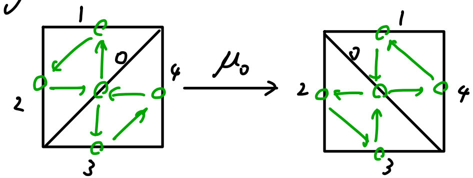
, $\quad U = \begin{pmatrix} 0 & -q^{-5/2} \\ q^{-1/2} & 0 \end{pmatrix}$

① Clasped skein alg. & $\mathcal{U}^q_{SL_2,\Sigma} := \mathcal{O}_q(\mathcal{A}^\times_{SL_2,\Sigma})$ [Muller'16]

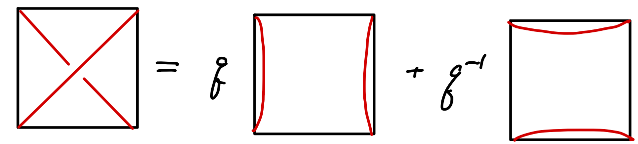


<u>clusters</u>: ideal triangulations of Σ

<u>mutations</u>: flips



$$A_0 A_0' = q A_2 A_4 + q^{-1} A_1 A_3$$



$$\mathcal{A}^q_{SL_2,\Sigma} = \frac{\langle A_\alpha \mid \alpha : \text{ideal arc} \rangle}{\text{mutations}} [A_\beta^{-1}]_{\beta : \partial\text{-arc}}$$

quantum cluster alg.

<u>Theorem</u> ([Muller'16])

If $|M| \geq 2$, then

$$\mathcal{S}^q_{SL_2,(\Sigma,M)}[\partial^{-1}] \xrightarrow{\sim} \mathcal{A}^q_{SL_2,\Sigma} = \mathcal{U}^q_{SL_2,\Sigma}$$

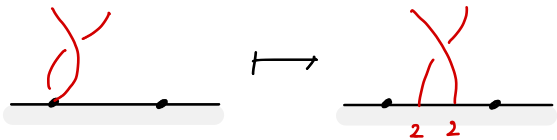
$$[\alpha] \longleftrightarrow A_\alpha$$

Laurent phenomenon [Berenstein-Zelevinsky'04]
+ local acyclicity

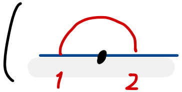
② State-clasp correspondence

reduced

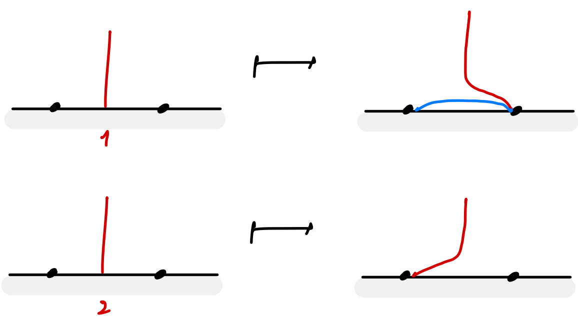
$\mathcal{S}^{\xi}_{SL_2,(\Sigma,\mathbb{M})} \hookrightarrow \mathcal{S}^{\xi}_{SL_2,(\Sigma,\mathbb{B})}$ $\rightsquigarrow$ $\Phi^{-1} : \mathcal{S}^{\xi}_{SL_2,(\Sigma,\mathbb{M})}[\partial^{-1}] \xrightarrow{\sim} \overline{\mathcal{S}}^{\xi}_{SL_2,(\Sigma,\mathbb{B})}$ [Lê-Yu'22]



$\overline{\mathcal{S}}^{\xi}_{SL_2,(\Sigma,\mathbb{B})} = \mathcal{S}^{\xi}_{SL_2,(\Sigma,\mathbb{B})} / \langle \text{bad arcs} \rangle$

( $= 0$)

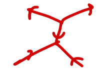
$\Phi : \overline{\mathcal{S}}^{\xi}_{SL_2,(\Sigma,\mathbb{B})} \xrightarrow{\sim} \mathcal{S}^{\xi}_{SL_2,(\Sigma,\mathbb{M})}[\partial^{-1}]$ is also explicit :



- $\mathcal{S}^{\xi}_{PGL_2,(\Sigma,\mathbb{B})} < \mathcal{S}^{\xi}_{SL_2,(\Sigma,\mathbb{B})}$: spanned by W s.t. $i(W,\alpha) \in 2\mathbb{Z}$ for any ideal arc α

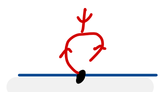
⑩ $G = SL_3$ ~~9/18~~

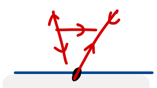
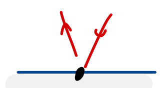
Kuperberg's SL_3-web :  $= q^2 + 1 + q^{-2}$

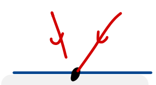
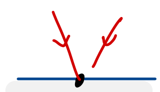
 $= q^2$  $+ \; q^{-1}$  etc.

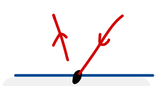
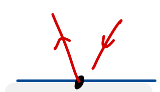
$\boxed{(\rightarrow)^* := \leftarrow \quad \text{alg. autom}}$

clasped [Frohman–Sikora '22, I.–Yuasa '23]

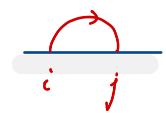
▶  $= 0$,  $= 0$,

 $=$  , etc.

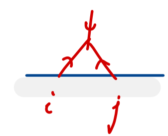
▶  $= q^2$  ,

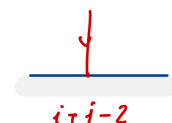
 $= q$  , etc.

stated [Higgins '23] states $\in \{1, 2, 3\}$

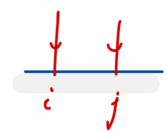
▶ 

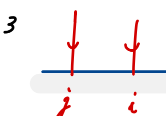
$=: U_{ij}$, $U = \begin{pmatrix} & & q^{-7} \\ & -q^{-4} & \\ q^{-1} & & \end{pmatrix}$

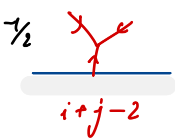


$=: V_{ij}$ 

, $V = \cdots$

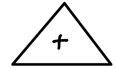
▶ 

$= q^3$ 

$+ \; q^{-1/2}$ 

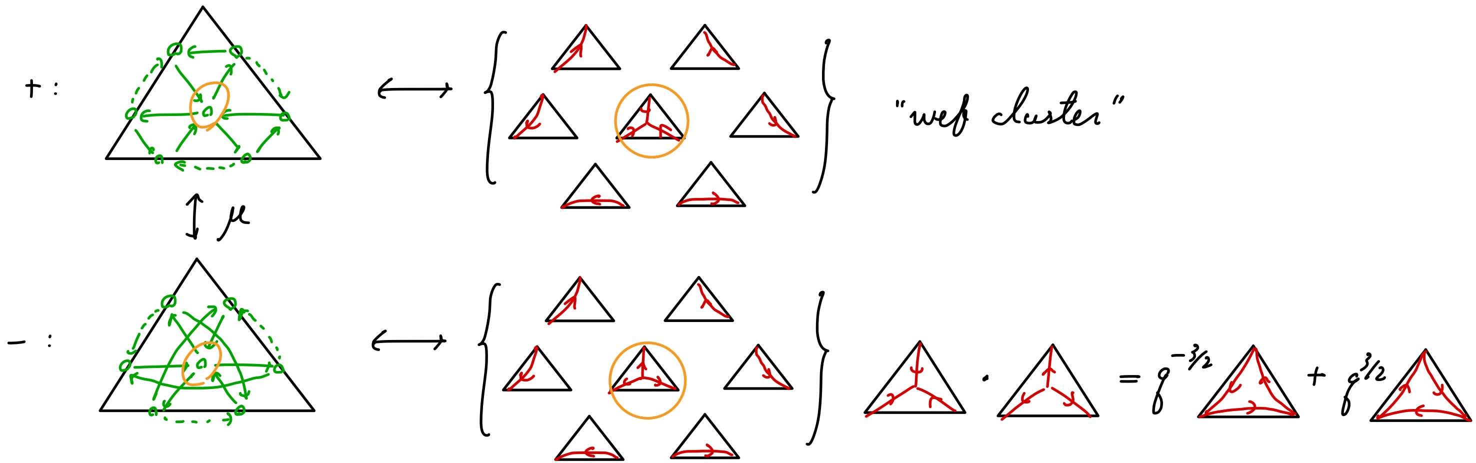
$(i<j)$

etc.

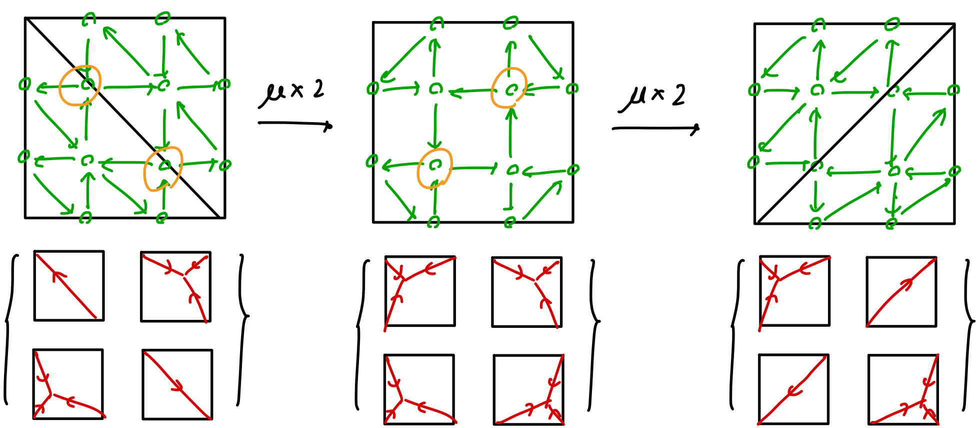
clusters : decorated triangulations 

or 

, and more ... (created via mutations)



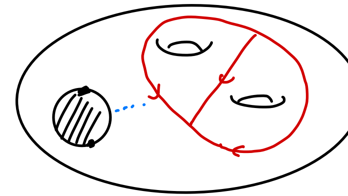
Flip :



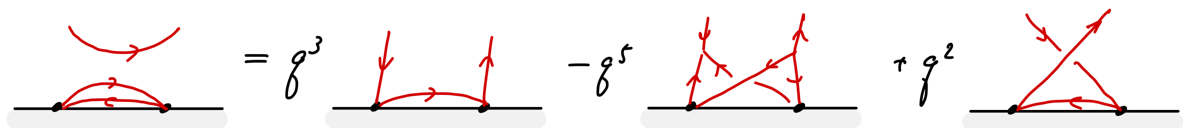
Theorem (I.–Yuasa '23)

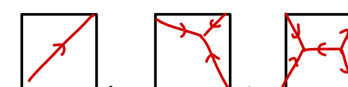
If $|\mathbb{M}| \geq 2$, then

$$\mathscr{S}^{q}_{SL_3,(\Sigma,\mathbb{M})}[\partial^{-1}] \subseteq \mathscr{A}^{q}_{SL_3,\Sigma} \subseteq \mathscr{U}^{q}_{SL_3,\Sigma}$$



"sticking trick":



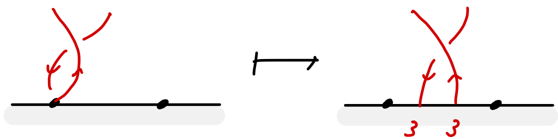
$\rightsquigarrow$ generate $\mathscr{S}^{q}_{SL_3,(\Sigma,\mathbb{M})}[\partial^{-1}]$ by cluster variables 

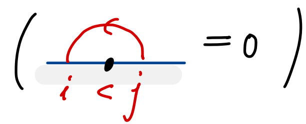
Rem All the equalities hold if

- Σ = 3-gon, 4-gon, 5-gon (finite type)
  (A_1) (D_4) (E_7)
- $q=1$ [I.–Oya–Shen '23] (from geometry of $\mathcal{A}^{\times}_{SL_3,\Sigma}$)

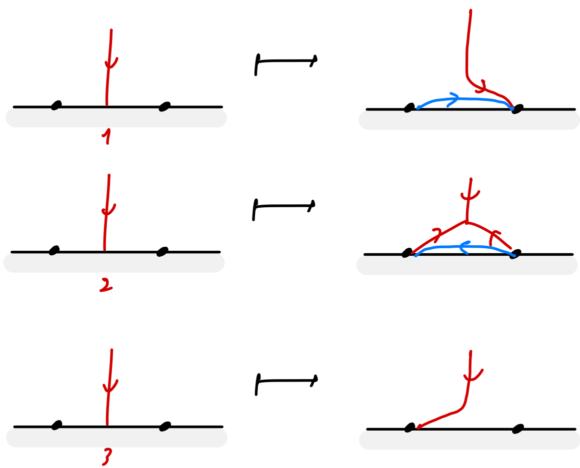
· State-clasp correspondence [I.-Yuasa '25+?]

$$\Phi^{-1} : \mathcal{S}^q_{\mathfrak{sl}_3,(\Sigma,\mathbb{M})}[\partial^{-1}] \xrightarrow{\sim} \overline{\mathcal{S}}^q_{\mathfrak{sl}_3,(\Sigma,\mathbb{B})}$$



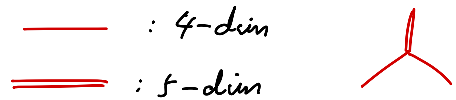


$$\Phi : \overline{\mathcal{S}}^q_{\mathfrak{sl}_2,(\Sigma,\mathbb{B})} \xrightarrow{\sim} \mathcal{S}^q_{\mathfrak{sl}_2,(\Sigma,\mathbb{M})}[\partial^{-1}]$$

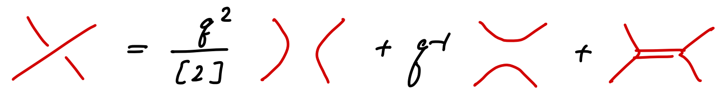


(iii) $G = Sp_4$

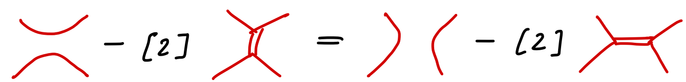
Kuperberg's Sp_4-web :



$R_q := \mathbb{Z}[q^{\pm 1}, \frac{1}{[2]}]$



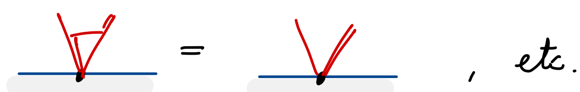
$\leftarrow \frac{1}{[2]}$ appears!

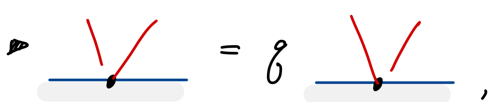


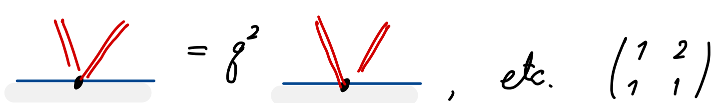
$\leftarrow$ dependence!

clasped [I.-Yuasa'24]

▶ 



▶ 

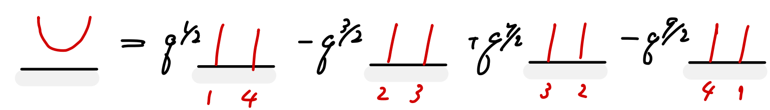


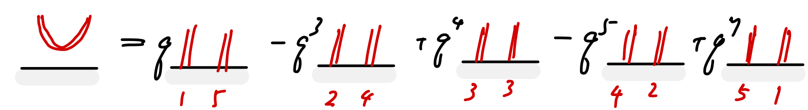
stated [IY'25+?] states $\in \{1,2,3,4\}$

[Bonahon-Higgins'25+?] $\{1,2,3,4,5\}$

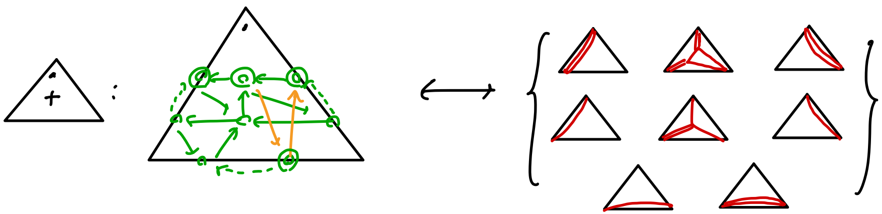
▶ U, V given

▶ sticking rel. :



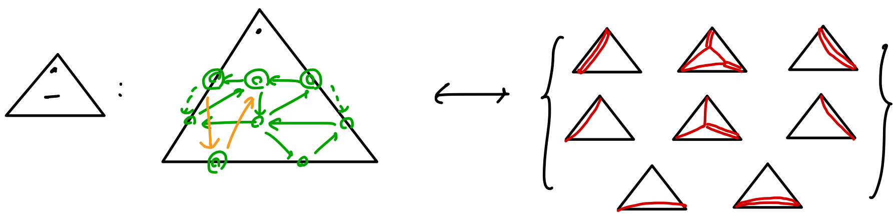


clusters : decorated triangulations  [Zickert '19, Le '19, Goncharov-Shen '19+]

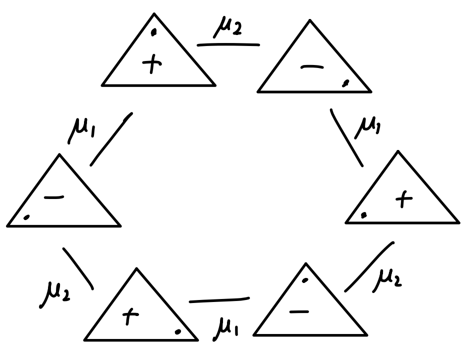


$Q = (I, \sigma, d)$ weighted quiver

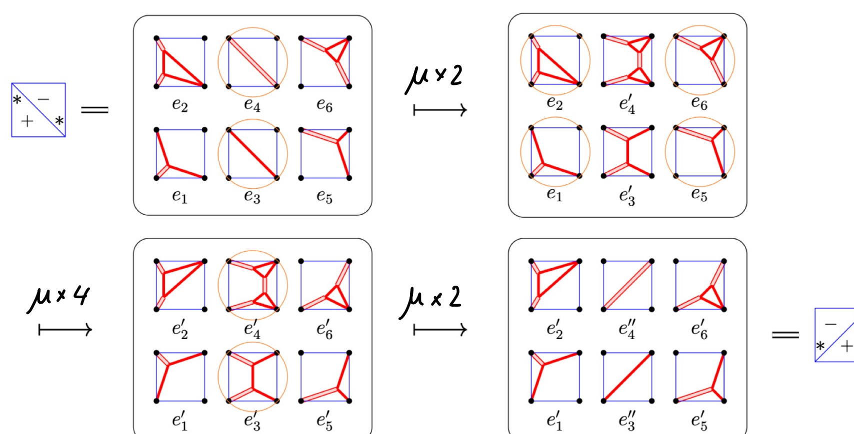
$$\varepsilon_{ij} = \frac{d_i \sigma_{ij}}{\gcd(d_i, d_j)}$$



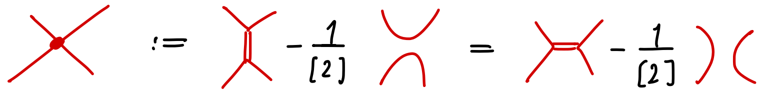
6 clusters on triangle :



Flip :



(III) Integral form



: crossroad

$\mathrm{BWeb}_{Sp_4,\Sigma} := \{$ flat crossroad web w/o elliptic faces $\}$: R-basis of $\mathscr{S}^q_{Sp_4,(\Sigma,\mathbb{M})}$

<u>Def</u> $\mathscr{S}^{\mathbb{Z}_q}_{Sp_4,(\Sigma,\mathbb{M})} := \mathrm{span}_{\mathbb{Z}_q} \mathrm{BWeb}_{Sp_4,\Sigma} \subset \mathscr{S}^q_{Sp_4,(\Sigma,\mathbb{M})}$ $\quad \mathbb{Z}_q := \mathbb{Z}[q^{\pm 1}]$

($\mathbb{Z}_q$-subalgebra)

<u>Theorem</u> (I.-Yuasa '25)

If $|\mathbb{M}| \geq 2$, then

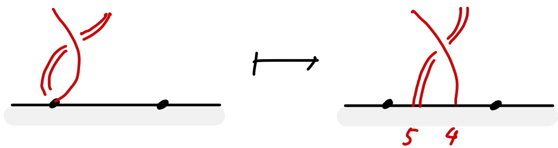
$$\mathscr{S}^{\mathbb{Z}_q}_{Sp_4,(\Sigma,\mathbb{M})}[\partial^{-1}] \subseteq \mathscr{A}^q_{Sp_4,\Sigma} \subseteq \mathscr{U}^q_{Sp_4,\Sigma}$$

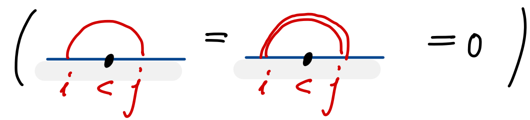
sticking trick

"=" at $q=1$. [IOS'25]

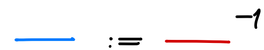
- State-clasp correspondence (over $\mathcal{R}' := \mathbb{Z}[q^{\pm 1/2}, [2]^{\pm 1/2}]$) [I.-Yuasa '25+?]

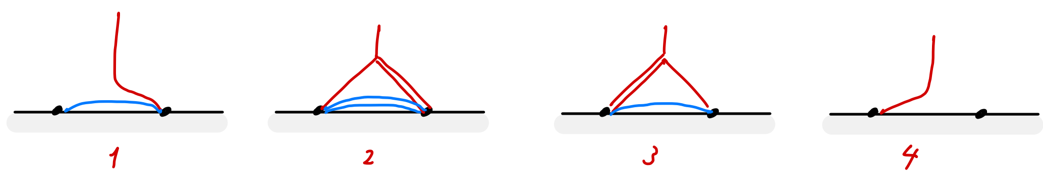
$\Phi^{-1} : \mathcal{S}^{q}_{Sp_4,(\Sigma,\mathbb{M})}[\partial^{-1}] \xrightarrow{\sim} \overline{\mathcal{S}}^{q}_{Sp_4,(\Sigma,\mathbb{B})}$

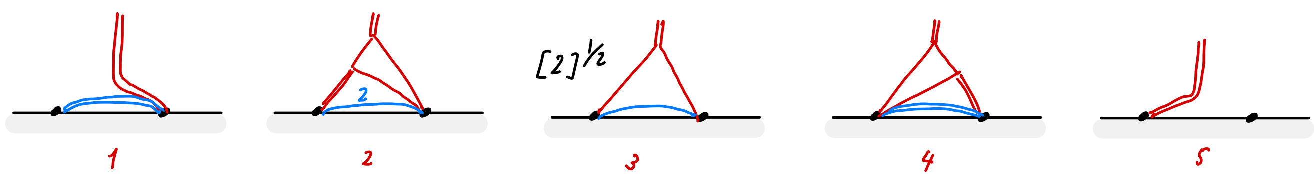




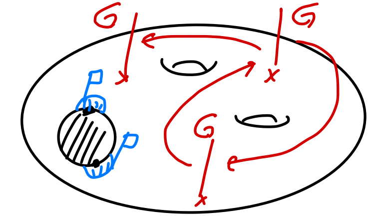
$\Phi : \overline{\mathcal{S}}^{q}_{Sp_4,(\Sigma,\mathbb{B})} \xrightarrow{\sim} \mathcal{S}^{q}_{Sp_4,(\Sigma,\mathbb{M})}[\partial^{-1}]$







§3. State-clasp formula from geometry



G : simply connected semisimple alg. group $/\mathbb{C}$

$\cup$

U^+ : maximal unipotent subgroup

$\mathcal{A}_{G,\Sigma}$: moduli space of decorated twisted G-local systems on Σ

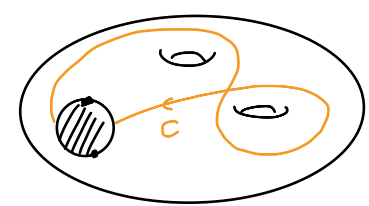
$$\simeq \left[\mathrm{Hom}(\pi_1(\Sigma), G) \times (G/U^+)^{M} \big/ G \right]$$

e.g. Σ : triangle $\Rightarrow \mathcal{O}(\mathcal{A}_{G,\Sigma}) \cong \bigoplus_{\lambda,\mu,\nu\in P_+} (V_\lambda \otimes V_\mu \otimes V_\nu)^G$ $\Rightarrow$ cluster variables $A_{\lambda,\mu,\nu}$ w/ appropriate (λ,μ,ν)

$\exists\ \mathcal{A}^\times_{G,\Sigma} \overset{\text{open}}{\subset} \mathcal{A}_{G,\Sigma}$ "generic part"

We have $g_{[c]} : \mathcal{A}^\times_{G,\Sigma} \longrightarrow G$ "Wilson lines" [IOS'23]

associated to oriented arcs $[c]$



Theorem ([IOS'23] + [Oya'25+])

$|\mathbb{M}| \geq 2$ & $G \neq F_4$

$$\Rightarrow \mathscr{A}_{G,\Sigma} = \mathscr{U}_{G,\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$$

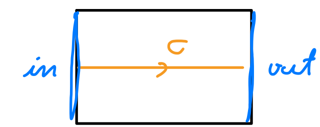
$$\begin{array}{ccc} \| & & \| \\ \langle \text{cluster variables} \rangle & \longleftrightarrow & \langle \text{matrix coeff. of Wilson lines} \rangle \end{array}$$

state-clasp relation?

GS cluster variable

Proposition $$\Delta_{w\varpi_s, w'\varpi_s}(g_{[c]}) = \frac{A_{w\varpi_s, w'\varpi_s}}{A_{in}^{[w\varpi_s]_+^*} A_{out}^{[w'\varpi_s]_+}}$$

frozen in/out



Example

$G = SL_3$

$g_{[c]} =$