The deep locus in a cluster variety

Based on joint work with Marco Castronovo, Mikhail Gorsky and José Simental Rodríguez.

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Agenda:

Part I: Definitions

What is the deep locus?

Part II: A conjecture!

A possible description of the deep locus.

Part III: Some evidence!

A few cases we can prove.

Part IV: Webs might help!

Can we prove more cases?

(1) What is a cluster algebra?

A cluster algebra A is a kind of commutative $\mathbb{C}$-algebra with a collection of elements called ***cluster variables***, which are organized into sets called ***clusters***. (This is a description, not a definition.)

A is an integral domain, each cluster is algebraically independent over $\mathbb{C}$. For each cluster $(x_1, x_2, \ldots, x_n)$, we have the Laurent phenomenon:

$$A \subseteq \mathbb{C}[x_1^{\pm}, x_2^{\pm}, \ldots, x_n^{\pm}].$$

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First example:

$$\mathbb{C}[x_1, x_2, y^{\pm}]/\langle x_1 x_2 - y - 1 \rangle = \mathbb{C}[x_1, x_2, (x_1 x_2 - 1)^{-1}].$$

Clusters: (x_1, y) and (x_2, y).

Laurent phenomenon: $x_2 = \frac{y+1}{x_1}$, $x_1 = \frac{y+1}{x_2}$.

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Second example:

$$\mathbb{C}[x_1, x_2, x_3, x_4, x_5] \Big/ \left\langle \begin{smallmatrix} x_1x_3 = x_2+1 \\ x_2x_4 = x_3+1 \\ x_3x_5 = x_4+1 \\ x_4x_1 = x_5+1 \\ x_5x_2 = x_1+1 \end{smallmatrix} \right\rangle$$

Clusters: (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_5, x_1).

Laurent phenomenon: $x_3 = \frac{x_2+1}{x_1}$, $x_4 = \frac{x_1+x_2+1}{x_1x_2}$, $x_5 = \frac{x_1+1}{x_2}$.

I've focused on very small cluster varieties, so that I can draw pictures. But cluster varieties also include many important spaces such as:

- All braid varieties.
- All Bott-Samelson varieties.
- All Grassmannians. More about this later.
- All positroid varieties.

Also, to any root system, there is an associated cluster variety.

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If x is a mutable variable in a cluster t, then there is another cluster variable x' and another cluster t' with $t \setminus \{x\} = t' \setminus \{x'\}$ and a relation of the form

$$xx' = \text{binomial in the variables of } t \cap t'.$$

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Example 1: $\qquad x_1 x_2 = y + 1.$

Example 2: $\qquad x_{j-1} x_{j+1} = x_j + 1.$

The ***cluster automorphism group***, $\mathrm{Aut}(A)$, is the group of $\mathbb{C}$-algebra automorphisms $\alpha : A \to A$ where $\alpha(x)$ is a $\mathbb{C}^*$-multiple of x for every cluster variable x. For any cluster $(x_1, x_2, \ldots, x_{n+m})$, we can think of $\mathrm{Aut}(A)$ as a subgroup of $(\mathbb{C}^*)^{n+m}$.

In our first example, where $x_1 x_2 = y + 1$, we have $\mathrm{Aut}(A) = \mathbb{C}^*$. The automorphisms are $(x_1, x_2, y) \mapsto (tx_1, t^{-1}x_2, y)$. In our second example, the automorphism group is trivial.

(2) What is a cluster variety?

Let A be a cluster algebra. Let

$$\mathcal{A} = \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(A, \mathbb{C}).$$

We call $\mathcal{A}$ a ***cluster variety***.

I write this in an abstract way to emphasize that there is a clear, unambiguous definition, even when A is infinitely generated and/or we haven't chosen a particular list of generators for A. But it has a very clear concrete meaning:

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$$A = \mathbb{C}[x_1, x_2, y^{\pm}]/\langle x_1 x_2 - y - 1\rangle$$

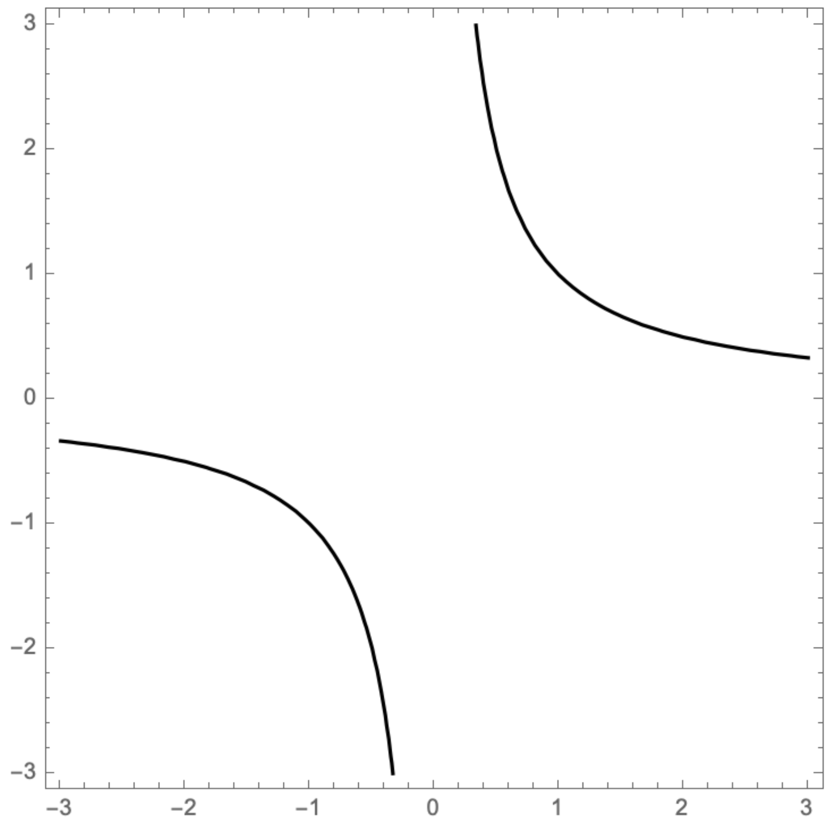
$$\mathcal{A} = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \in \mathbb{C}^3 : \boldsymbol{x}_1\boldsymbol{x}_2 = \boldsymbol{y} + 1,\ \boldsymbol{y} \neq 0\}.$$

We send each cluster variable to a complex number, obeying the cluster relations. The frozen variables must be sent to nonzero complex numbers. We think of A as $\mathbb{C}$-valued functions on $\mathcal{A}$.

Continuing with our first example:

$$A = \mathbb{C}[x_1, x_2, y^{\pm}]/\langle x_1 x_2 - y - 1\rangle$$

$$\{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \in \mathbb{C}^3 : \boldsymbol{x}_1 \boldsymbol{x}_2 = \boldsymbol{y} + 1,\ \boldsymbol{y} \neq 0\} \cong \{(\boldsymbol{x}_1, \boldsymbol{x}_2) : \boldsymbol{x}_1 \boldsymbol{x}_2 \neq 1\}.$$



How does the Laurent phenomenon come in?

Let $t = (x_1, x_2, \ldots, x_{n+m})$ be a cluster of A. We have $A[(x_1 x_2 \cdots x_{n+m})^{-1}] = \mathbb{C}[x_1^{\pm}, x_2^{\pm}, \ldots, x_{n+m}^{\pm}]$.

The geometric meaning of this is that the open set

$$T(t) := \{\boldsymbol{x}_1 \boldsymbol{x}_2 \cdots \boldsymbol{x}_{n+m} \neq 0\} \subset \mathcal{A}$$

is isomorphic to

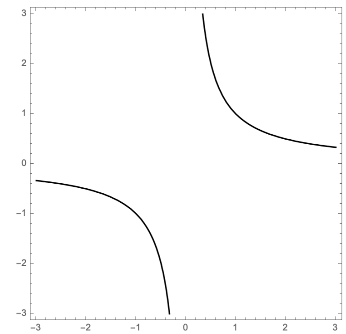
$$\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathbb{C}[x_1^{\pm}, x_2^{\pm}, \ldots, x_{n+m}^{\pm}], \mathbb{C}) = (\mathbb{C}^*)^{n+m}.$$

We call $T(t)$ a ***cluster torus*** inside $\mathcal{A}$.

The cluster tori inside our first example:

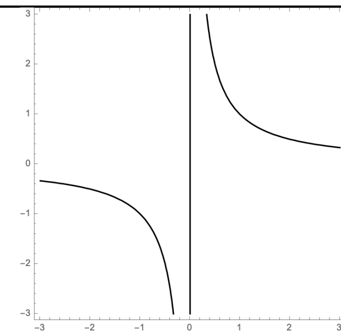
$$A = \mathbb{C}[x_1, x_2, y^{\pm}]/(x_1 x_2 = y + 1)$$

$$\mathcal{A} = \{(\boldsymbol{x}_1, \boldsymbol{x}_2) : \boldsymbol{x}_1 \boldsymbol{x}_2 - 1 \neq 0\}$$



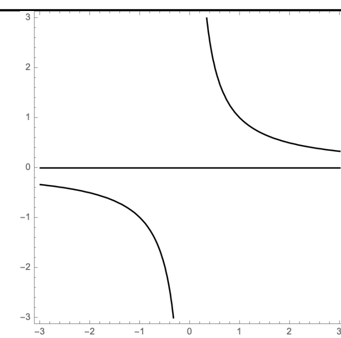
$$A[x_1^{-1}] = \mathbb{C}[x_1^{\pm}, y^{\pm}]$$

$$\begin{aligned} \mathcal{A} \quad &= \{(\boldsymbol{x}_1, \boldsymbol{x}_2) : \boldsymbol{x}_1(\boldsymbol{x}_1 \boldsymbol{x}_2 - 1) \neq 0\} \\ &= \{(\boldsymbol{x}_1, \boldsymbol{y}) : \boldsymbol{x}_1 \boldsymbol{y} \neq 0\} \end{aligned}$$



$$A[x_2^{-1}] = \mathbb{C}[x_2^{\pm}, y^{\pm}]$$

$$\begin{aligned} \mathcal{A} \quad &= \{(\boldsymbol{x}_1, \boldsymbol{x}_2) : \boldsymbol{x}_2(\boldsymbol{x}_1 \boldsymbol{x}_2 - 1) \neq 0\} \\ &= \{(\boldsymbol{x}_2, \boldsymbol{y}) : \boldsymbol{x}_2 \boldsymbol{y} \neq 0\} \end{aligned}$$



How does the cluster automorphism group fit in?

For every cluster $(x_1, x_2, \ldots, x_{m+n})$, a cluster automorphism α acts on $\mathbb{C}[x_1^{\pm}, \ldots, x_{n+m}^{\pm}]$ by $\alpha(x_i) = \zeta_i x_i$.

Geometrically, this corresponds to translation by $(\zeta_1, \ldots, \zeta_{n+m})$ in the torus $T(t) \cong (\mathbb{C}^*)^{n+m}$.

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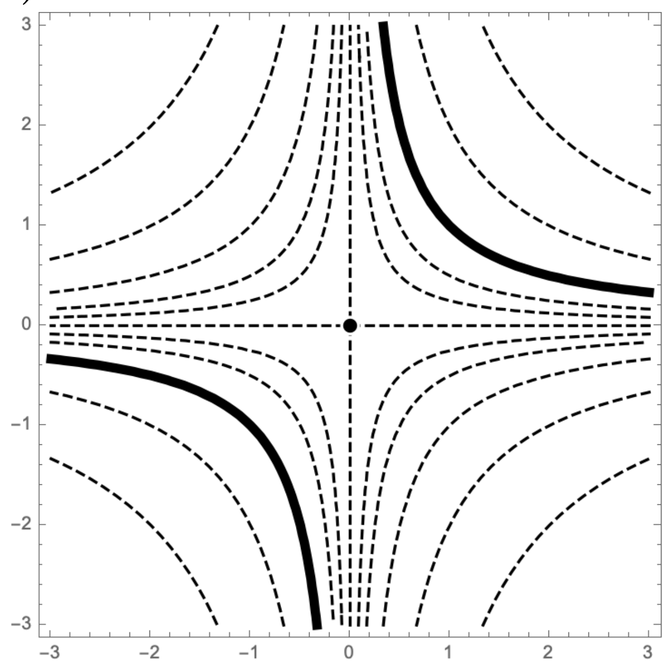
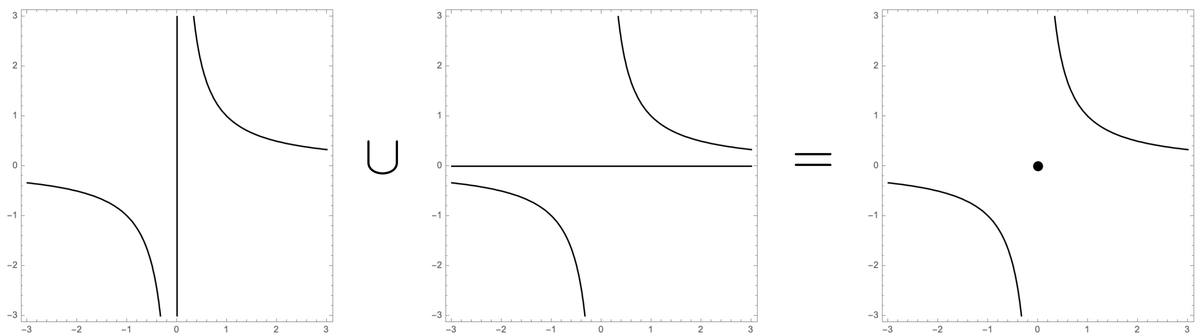


$$\{(\boldsymbol{x}_1, \boldsymbol{x}_2) : \boldsymbol{x}_1 \boldsymbol{x}_2 \neq 1\} \qquad (\boldsymbol{x}_1, \boldsymbol{x}_2) \mapsto (t \cdot \boldsymbol{x}_1, t^{-1} \cdot \boldsymbol{x}_2).$$

What is the deep locus?

People often informally say that the cluster variety is covered by the cluster tori. But this isn't right!



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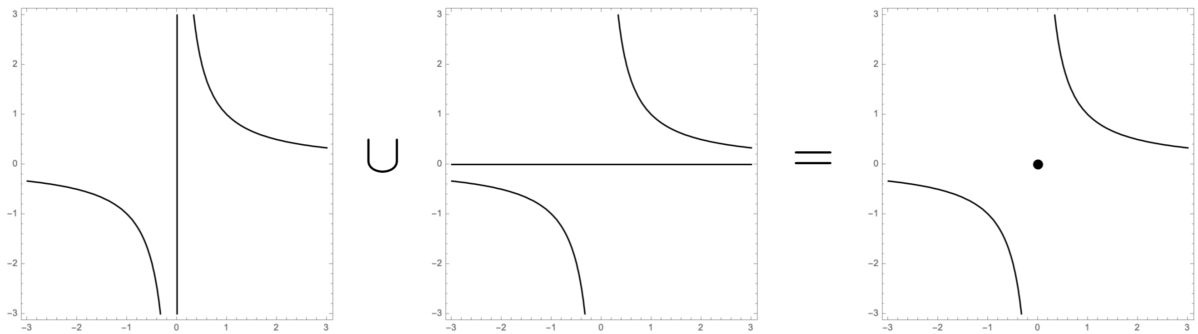

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The deep locus is important in mirror symmetry. There is a function $W : \mathcal{A} \to \mathbb{C}$ called the ***superpotential***, and mirror theorists want to compute its critical points. Mirror theorists can compute $W|_{T(t)}$. If there are critical points of W in $\mathcal{D}(\mathcal{A})$, then this won't see them.

The deep locus and the stabilizer locus

Recall that $\mathrm{Aut}(\mathcal{A})$ acts on every torus $T(t)$ by translation. Thus, if $x \in \mathcal{A}$ lies in some cluster torus, then the stabilizer of x for the $\mathrm{Aut}(A)$ action is trivial. We define the ***stabilizer locus***, $\mathcal{S}(\mathcal{A})$, to be the closed sub-variety of $\mathcal{A}$ consisting of points with non-trivial stabilizer for the $\mathrm{Aut}(A)$ action. So $\mathcal{S}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$.

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Conjecture: Let $\mathcal{A}$ be a locally acyclic cluster variety. Then $\mathcal{A}$ has no mysterious points.

This conjecture emerged in private conversation between me and Vivek Shende in May 2016, and is stated publicly the paper with M. Castronovo, M. Gorsky and J. Simental.

Conjecture: Let $\mathcal{A}$ be a locally acyclic cluster variety. Then $\mathcal{A}$ has no mysterious points.

To prove this conjecture, we must take an arbitrary point $z \in \mathcal{A}$ and either

(1) Find a cluster torus $T(t)$ containing z or

(2) Find a cluster automorphism α stabilizing z.

Generalities about the deep locus

Every singular point of $\mathcal{A}$ is in the deep locus. It is not clear whether every singular point is in the stabillizer locus.

Let $\Psi : \mathcal{A}_2 \to \mathcal{A}_1$ be a cluster quasi-homomorphism, in the sense of Fraser. Then $\Psi(\mathcal{D}(\mathcal{A}_2)) \subseteq \mathcal{D}(\mathcal{A}_1)$. If $\mathcal{A}_2$ has no mysterious points, then neither does $\mathcal{A}_1$.

In particular, let $\mathcal{A}_1$ correspond to the exchange matrix $\widetilde{B}$, and let $\mathcal{A}_2$ correspond to an exchange matrix of the form $\left[\begin{smallmatrix} \widetilde{B} \\ d \end{smallmatrix}\right]$, with an extra frozen row. Then, if $\mathcal{A}_2$ has no mysterious points, then neither does $\mathcal{A}_1$.

More generally, let $\widetilde{B}_1$ and $\widetilde{B}_2$ be two exchange matrices of the form $\left[\begin{smallmatrix} B \\ C_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} B \\ C_2 \end{smallmatrix}\right]$, such that $\widetilde{B}_1^T \mathbb{Z}^{n+m_1} \subseteq \widetilde{B}_2^T \mathbb{Z}^{n+m_2}$. If $\mathcal{A}_2$ has no mysterious points, then neither does $\mathcal{A}_1$. If $\widetilde{B}_1^T \mathbb{Z}^{n+m_1} = \widetilde{B}_2^T \mathbb{Z}^{n+m_2}$, then $\mathcal{A}_2$ has no mysterious points if and only if $\mathcal{A}_1$ has no mysterious points.

(3) What can we can prove?

Theorem: (Castronovo-Gorsky-Rodriguez-S.) The "no mysterious points conjecture" holds for all 2-strand braid varieties $B(\sigma_1^k)$ and for 3-strand braid varieties of the form $X(k,\ell) := B(\sigma_1^k(\sigma_2\sigma_1)^\ell)$.

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- $B(\sigma_1^k)$ has cluster type A_{k-2}, the same as $G(2, k+1)$.
- $X(k, 4)$ has cluster type D_{k+3}.
- $X(1, 5)$, $X(2, 5)$ and $X(1, 6)$ have cluster types E_6, E_7, E_8.
- $X(1, k)$ has the same cluster type as $G(3, k+2)$.

Thus, we have proved the conjecture for all simply laced finite types (A, D, E_6, E_7, E_8) and for Grassmannians of type $G(2, n)$ and $G(3, n)$.

(4) Why might webs help? Grassmannians and positroid varieties:

The Grassmannian $G(k,n)$ is the space of k-planes in n-space. We record a point of $G(k,n)$ as the row span of a $k \times n$ matrix M. Replacing M by gM, for $g \in \mathrm{GL}_k$, gives the same point of the Grassmannian.

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We write $v_1, v_2, \ldots, v_n$ for the columns of M. So we are interested in n-tuples of vectors, $v_1, v_2, \ldots, v_n$, up to GL_k action. We'll work with the affine cone over the Grassmannian, which means we are working with SL_k-invariants.

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For $1 \leq i_1 < i_2 < \cdots i_k \leq n$, we set

$$\Delta(i_1, i_2, \ldots, i_k) = \det(v_{i_1}, v_{i_2}, \ldots, v_{i_k}).$$

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$$\Delta(i_1, i_2, \ldots, i_k) = \det(v_{i_1}, v_{i_2}, \ldots, v_{i_k}).$$

People casually say that "the Grassmannian has a cluster structure". The actual cluster algebra in question is the Plücker algebra generated by the $\Delta(I)$'s, with the coordinates $\Delta(12\cdots k)$, $\Delta(23\cdots(k+1))$, ..., $\Delta(n12\cdots(k-1))$ inverted. The corresponding geometric space is n-tuples of vectors v_1, v_2, ..., v_n modulo SL_k, such that $\det(v_i v_{i+1} \cdots v_{i+k-1}) \neq 0$ for every $1 \leq i \leq n$.

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The cluster automorphism group is $(\mathbb{C}^*)^n$, corresponding to rescaling the vectors by $(v_1, v_2, \ldots, v_n) \mapsto (t_1 v_1, t_2 v_2, \ldots, t_n v_n)$

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We'll say that $(v_1, \ldots, v_n)$ is ***disconnected*** if we can split $[n]$ as $I \sqcup J$ such that $\mathbb{C}^k = \bigoplus_{i \in I} v_i \oplus \bigoplus_{j \in J} v_j$.

For example, $\left[\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right]$ is disconnected. The point $(v_1, \ldots, v_n)$ has a nontrivial stabilizer if and only if it is disconnected.

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I don't know, but our proof (which goes through braid varieties) does not stay in the world of Plücker variables.

The Grassmannian is stratified into postiroid subvarieties, of which the one which we have described is the largest one. For other positroid varieties, we definitely need to go to webs.

For example, consider $3k$-tuples of vectors $(u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k)$ obeying the conditions

$$\begin{aligned} \mathrm{rank}(u_{p+1}, \ldots, u_k, v_1, v_2, \ldots, v_q) &= \max(k-p, q) \\ \mathrm{rank}(u_{p+1}, \ldots, u_k, w_1, w_2, \ldots, w_r) &= \max(k-p, r) \ . \\ \mathrm{rank}(v_{q+1}, \ldots, v_k, w_1, w_2, \ldots, w_r) &= \max(k-q, r) \end{aligned}$$

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\operatorname{rank}(u_{p+1},\dots,u_k,v_1,v_2,\dots,v_q) &= \max(k-p,q)\\
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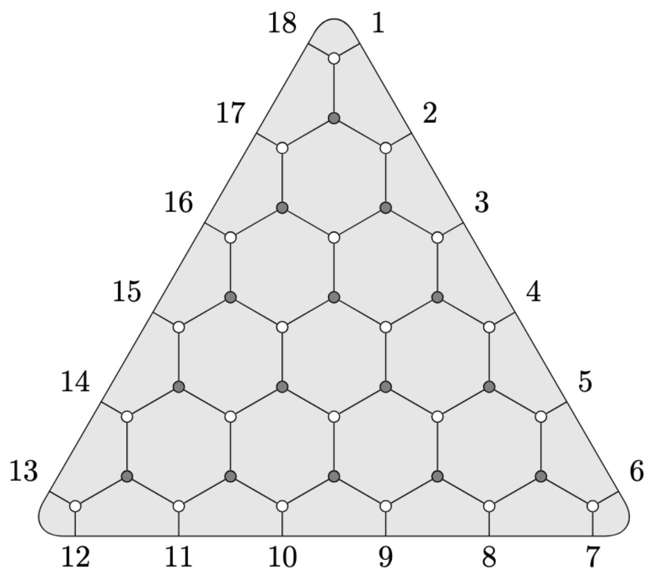
\operatorname{rank}(v_{q+1},\dots,v_k,w_1,w_2,\dots,w_r) &= \max(k-q,r)
\end{aligned}$$

This is a postroid variety, the corresponding plabic graph is



Since the plabic graph has no moves, there is only one cluster of Plücker variables. It is

$$\{\det(u_1,u_2,\dots,u_a,v_1,v_2,\dots,v_b,w_1,w_2,\dots,w_c) := a+b+c=n\}$$

So, if we were to prove the "no mysterious points" conjecture for the Grassmannian, we need to take v_1, v_2, ..., v_n which cannot be split up into a direct sum, and build many webs not vanishing on them, so that the webs form a cluster.

The Fomin-Pylyavskyy conjectures on when webs form a cluster seem very hard. So here is a variant conjecture that only mentions webs:

Given a SL_k-web $\mathcal{W}$ on n boundary vertices with clasp type Sym^{d_1}, Sym^{d_2}, …, $\mathrm{Sym}\, d_n$, let $\Delta_{\mathcal{W}}(v_1, v_2, \ldots, v_n)$ be the web evaluated on $(v_1^{\otimes d_1}, v_2^{\otimes d_2}, \ldots, v_n^{\otimes d_n})$. Let $\Omega_{\mathcal{W}}$ be the locus where $\Delta_{\mathcal{W}}$ is nonzero.

Let $\mathcal{W}_1$, $\mathcal{W}_2$, …, $\mathcal{W}_{k(n-k)+1}$ be webs. Let $\Omega_\bullet := \bigcap \Omega_{\mathcal{W}_i}$ and let $\Delta_\bullet : \Omega_\bullet \to (\mathbb{C}^*)^{k(n-k)+1}$ be the map $(\Delta_{\mathcal{W}_1}, \Delta_{\mathcal{W}_2}, \ldots, \Delta_{\mathcal{W}_{k(n-k)+1}})$. We'll say that $\mathcal{W}_1$, $\mathcal{W}_2$, …, $\mathcal{W}_{k(n-k)+1}$ are ***web coordinates*** if $\Delta_\bullet : \Omega_\bullet \to (\mathbb{C}^*)^{k(n-k)+1}$ is an isomorphism, and we'll say that $\Omega_\bullet$ is a ***web torus***.

Conjecture: Suppose that $(v_1, v_2, \ldots, v_n)$ is connected. Then $(v_1, v_2, \ldots, v_n)$ lies in a web torus.

More generally, I think we should develop tools to compute when we would expect web invariants to form a cluster, without having to solve Fomin and Pylyavskyy's difficult conjectures:

- Given webs $\mathcal{W}_1$, $\mathcal{W}_2$, ..., $\mathcal{W}_{k(n-k)+1}$, how can we tell if $\Delta_\bullet : \Omega_\bullet \to (\mathbb{C}^*)^{k(n-k)+1}$ is an isomorphism?
- Is there a way that we could test pairwise compatability of webs? Can we compute the Poisson bracket $\{\mathcal{W}_1, \mathcal{W}_2\}$?

And we should develop ways, given vectors, to find webs which don't vanish on them. If v_1, v_2, ..., v_n is connected, then we can find a k-element subset I of $[n]$, and a spanning tree (i_1, j_1), (i_2, j_2), ..., (i_{n-1}, j_{n-1}) on $I \times ([n] \setminus I)$, such that $\Delta(I)$ and all of the $\Delta(I \setminus \{i_p\} \cup \{j_p\})$ are nonzero.

- How can we soup that up to big list of nonvanishing webs?