Web and 3-row increasing tableaux

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Webs in Algebra, Geometry, Topology, and Combinatorics

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Standard Young tableaux

A **standard Young tableau** is a filling of a Young diagram with with n boxes positive integers such that

- entries $1, \ldots, n$ each appear exactly once and
- entries strictly increase across rows and down columns.

1	2	4	7
3	6		
5			

1	3	4
2	5	6
7		
8		

1	2	5	7
3	6	8	9
4	10	11	12

Promotion on SYT

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 2 & 5 & 6 \\ \hline 3 & 7 & 9 \\ \hline 4 & 8 & 10 \\ \hline \end{array}$$

$$\longrightarrow \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 & 8 \\ \hline 3 & 7 & 9 \\ \hline \end{array} = p(T)$$

Promotion on SYT

What are the orbit sizes of standard Young tableaux under promotion?

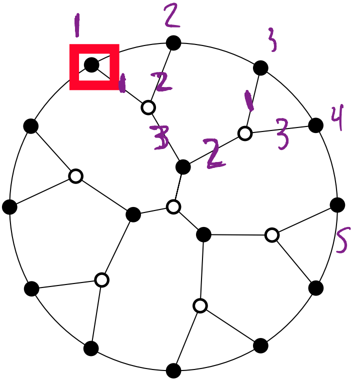
Schützenberger: For rectangular T with n boxes, $p^n(T) = T$.

Bijection between webs and 3-row rectangular SYT

(Originally due to Khovanov–Kuperberg)

- Make a proper edge coloring using the following preference:
  - • prefers 1, then 2, then 3
- Look at edge colors adjacent to boundary vertices to determine row. (Bazier-Matte–Douville–Garver–P.–Thomas–Yildirim)

1 3
2
4

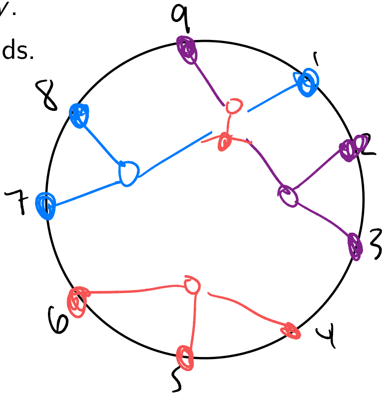


Bijection between webs and 3-row rectangular SYT

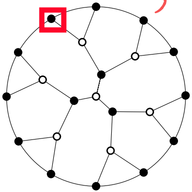
(Originally due to Khovanov–Kuperberg)

- From left to right, connect entry y with the largest entry in the row above that is $\leq y$.
- Form corresponding tripods.
- Resolve crossings. (Tymoczko)

1	2	4
3	5	7
6	8	9



1 2 3



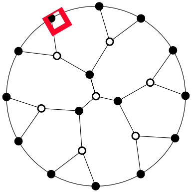
bij

1	3	5	9
2	6	7	11
4	8	10	12

$= T$

rotation

dpro



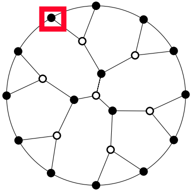
bij

1	2	4	8
3	5	6	10
7	9	11	12

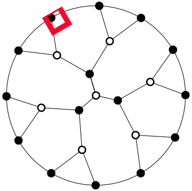
$= p(T)$

Theorem (Petersen–Pylyavskyy–Rhoades)

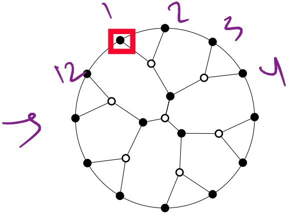
Let D be a web with cyclically labeled boundary vertices and all black boundary vertices. The standard Young tableau associated with counterclockwise rotation of D is given by promotion of the tableau associated with D itself.



1	3	5	9
2	6	7	11
4	8	10	12

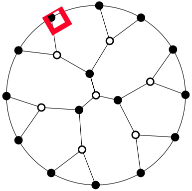


1	2	4	8
3	5	6	10
7	9	11	12



1	3	5	9
2	6	7	11
4	8	10	12

$= T$



1	2	4	8
3	5	6	10
7	9	11	12

Corollary: Let $p(T)$ denote the promotion of a rectangular, 3-row standard Young tableau with n boxes. Then $p^n(T) = T$.

Theorem (Petersen–Pylyavskyy–Rhoades)

Let D be a web with cyclically labeled boundary vertices and all black boundary vertices. The standard Young tableau associated with counterclockwise rotation of D is given by promotion of the tableau associated with D itself.

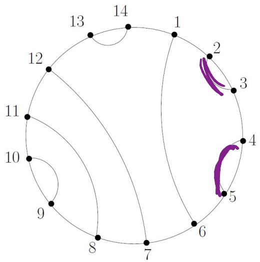
Theorem (Russell, P.)

Let D be a web with cyclically labeled boundary vertices. Web rotation corresponds to semistandard/generalized oscillating tableau promotion.

Analogous 2-row result

1	2	4	7	8	9	13
3	5	6	10	11	12	14

$\longleftrightarrow$



Corollary: Let $p(T)$ denote the promotion of a rectangular, 2-row standard Young tableaux with n boxes. Then $p^n(T) = T$.

Increasing tableaux

An increasing tableau is a filling of a Young diagram with positive integers such that rows and columns are strictly increasing.

1	2	3
2	3	
3		

1	2	4	5
3	4		

1	3	4	6
3			
6			
7			

not packed

no 2, 5

We are sometimes interested in "packed" increasing tableaux, meaning if the largest entry is k, then everything in $[1, k]$ appears as an entry,

Increasing tableaux (Just vibes)

Increasing tableaux are a K-theoretic analogue of (semi)standard Young tableaux:

- Semistandard Young tableaux are used to study the cohomology ring of the Grassmannian.
- Increasing tableaux are used to study the K-theory ring of the Grassmannian. (Thomas–Yong)

K-promotion on increasing tableaux

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 5 & 6 \\ \hline 3 & 6 & 8 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 2 & 3 & 5 & 6 \\ \hline 3 & 4 & 6 & 9 \\ \hline 6 & 8 & 9 & \boxed{10} \\ \hline \end{array} \xrightarrow{\text{subtract } 1} \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 2 & 3 & 5 & 8 \\ \hline 5 & 7 & 8 & 9 \\ \hline \end{array} = p(T)$$

K-promotion

What is the order of K-promotion on rectangular increasing tableaux?

Theorem (P.–Pechenik 2020)

Let T be a rectangular increasing tableau with largest entry q and K-promotion orbit of size k. Then k and q share a prime divisor (except when T is minimal).

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \hline 3 & 4 & 5 & 6 \\ \hline 4 & 5 & 6 & 7 \\ \hline \end{array} = p(T)$$

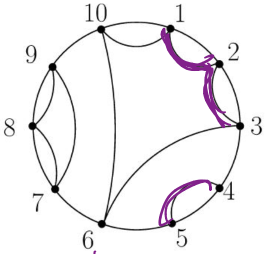
2-row rectangles

Two-row rectangles are easy (Pechenik 2014):

blocks size ≥ 2
non-crossing partition

1	2	3	4	6	7	8
2	3	5	6	8	9	10

$\longleftrightarrow$



K-promotion $\longleftrightarrow$ web rotation

For 2-row: $p^{\text{largest entry}}(T) = T$

4-row rectangles

Four-row rectangles are crazy. Tableau T has orbit size

$$675 = 25 \cdot 27$$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & 7 & 8 & 9 & 10 & 13 & 15 & 18 \\ \hline 2 & 4 & 7 & 8 & 11 & 14 & 15 & 17 & 21 & 22 \\ \hline 6 & 8 & 11 & 16 & 17 & 20 & 21 & 22 & 25 & 26 \\ \hline 8 & 10 & 12 & 17 & 19 & 22 & 23 & 24 & 26 & 27 \\ \hline \end{array}$$

3-row rectangles

Conjecture (Dilks–Pechenik–Striker 2017)

For T a three-row rectangular increasing tableau with largest entry q, we have $p^q(T) = T$.

Our motivation for this project

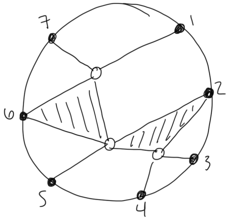
Conjecture (Dilks–Pechenik–Striker 2017)

For T a three-row rectangular increasing tableau with largest entry q, we have $p^q(T) = T$.

1	2	3
3	5	6
4	6	7

$??\longleftrightarrow??$

webs?

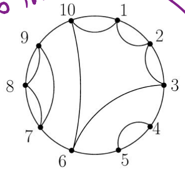


Question: Can we prove this conjecture for 3-row increasing tableaux by inventing the correct web analogue?

web invariants

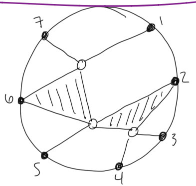
1	2	3	4	6	7	8
2	3	5	6	8	9	10

$\longleftrightarrow$



1	2	3
3	5	6
4	6	7

$??\longleftrightarrow??$



Idea: Understand the 2-row story better.

Both matchings and sl_3 webs have nice web invariants associated to them. Maybe we first need to understand what polynomials should be associated with noncrossing partitions.

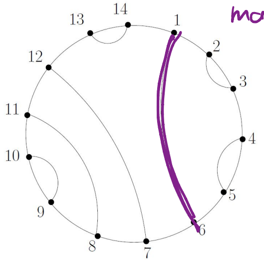
2-row SYT/noncrossing matching review

noncrossing matching

1	2	4	7	8	9	13
3	5	6	10	11	12	14

2

$\longleftrightarrow$



$$M_n = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{14} \\ y_1 & y_2 & y_3 & \cdots & y_{14} \end{pmatrix}$$

$$\begin{vmatrix} x_1 & x_6 \\ y_1 & y_6 \end{vmatrix} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \begin{vmatrix} x_4 & x_5 \\ y_4 & y_5 \end{vmatrix} \begin{vmatrix} x_7 & x_{12} \\ y_7 & y_{12} \end{vmatrix} \begin{vmatrix} x_8 & x_{11} \\ y_8 & y_{11} \end{vmatrix} \begin{vmatrix} x_9 & x_{10} \\ y_9 & y_{10} \end{vmatrix} \begin{vmatrix} x_{13} & x_{14} \\ y_{13} & y_{14} \end{vmatrix}$$

2-row SYT/noncrossing matching review

$SL_2(\mathbb{C})$ acts by left multiplication. For example,

$$\begin{pmatrix} 1 & 2 \\ .5 & 2 \end{pmatrix}\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix} = \begin{pmatrix} x_1+2y_1 & x_2+2y_2 & x_3+2y_3 & \cdots & x_n+2y_n \\ .5x_1+2y_1 & .5x_2+2y_2 & .5x_3+2y_3 & \cdots & .5x_n+2y_n \end{pmatrix}$$

$$p_{13} = x_1y_3 - x_3y_1 \longrightarrow (x_1+2y_1)(.5x_3+2y_3) - (x_3+2y_3)(.5x_1+2y_1)$$
$$= \cdots = x_1y_3 - x_3y_1 = p_{13}$$

S_n acts by right multiplication and permutes the columns of M_n. For example,

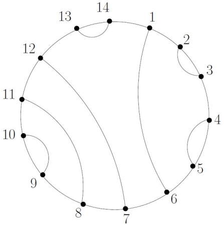
$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix} n1234\cdots(n-1) = \begin{pmatrix} x_n & x_1 & x_2 & \cdots & x_{n-1} \\ y_n & y_1 & y_2 & \cdots & y_{n-1} \end{pmatrix}$$

$$p_{13} = x_1y_3 - x_3y_1 \longrightarrow -p_{2n} = x_ny_2 - x_2y_n$$

2-row SYT/noncrossing matching review

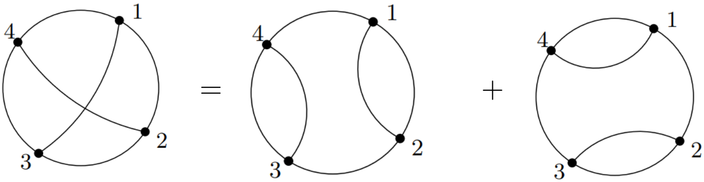
Theorem

The collection of polynomials corresponding to noncrossing matchings of n elements gives a basis for the Specht module $S^{(\frac{n}{2},\frac{n}{2})}$.



$$\begin{vmatrix} x_1 & x_6 \\ y_1 & y_6 \end{vmatrix} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \begin{vmatrix} x_4 & x_5 \\ y_4 & y_5 \end{vmatrix} \begin{vmatrix} x_7 & x_{12} \\ y_7 & y_{12} \end{vmatrix} \begin{vmatrix} x_8 & x_{11} \\ y_8 & y_{11} \end{vmatrix} \begin{vmatrix} x_9 & x_{10} \\ y_9 & y_{10} \end{vmatrix} \begin{vmatrix} x_{13} & x_{14} \\ y_{13} & y_{14} \end{vmatrix}$$

2-row SYT/noncrossing matching review

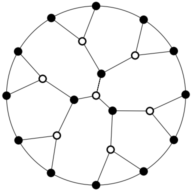


$$\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \begin{vmatrix} x_2 & x_4 \\ y_2 & y_4 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}$$

3-row SYT/$\mathfrak{sl}_3$ web review

$$M_n = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{pmatrix}.$$

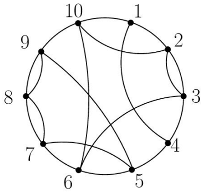
1	3	5	6
2	4	7	9
8	10	11	12



- Basis of Specht module $S^{(\frac{n}{3},\frac{n}{3},\frac{n}{3})}$.

On to 2-row increasing tableaux and noncrossing partitions (where all blocks have size ≥ 2)!

Invariants for noncrossing partitions

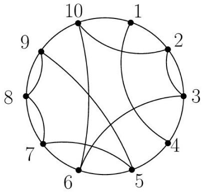


$$M = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,10} \\ x_{21} & x_{22} & \cdots & x_{2,10} \\ x_{31} & x_{32} & \cdots & x_{3,10} \\ \vdots & & & \vdots \end{pmatrix}$$

Naive guess:

$$\begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} & x_{16} & x_{1,10} \\ x_{22} & x_{23} & x_{26} & x_{2,10} \\ x_{32} & x_{33} & x_{36} & x_{3,10} \\ x_{42} & x_{43} & x_{46} & x_{4,10} \end{vmatrix} \begin{vmatrix} x_{15} & x_{17} & x_{18} & x_{19} \\ x_{25} & x_{27} & x_{28} & x_{29} \\ x_{35} & x_{37} & x_{38} & x_{39} \\ x_{45} & x_{47} & x_{48} & x_{49} \end{vmatrix}$$

Invariants for noncrossing partitions



$$M = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,10} \\ x_{21} & x_{22} & \cdots & x_{2,10} \\ x_{31} & x_{32} & \cdots & x_{3,10} \\ \vdots & & & \vdots \end{pmatrix}$$

Naive guess:

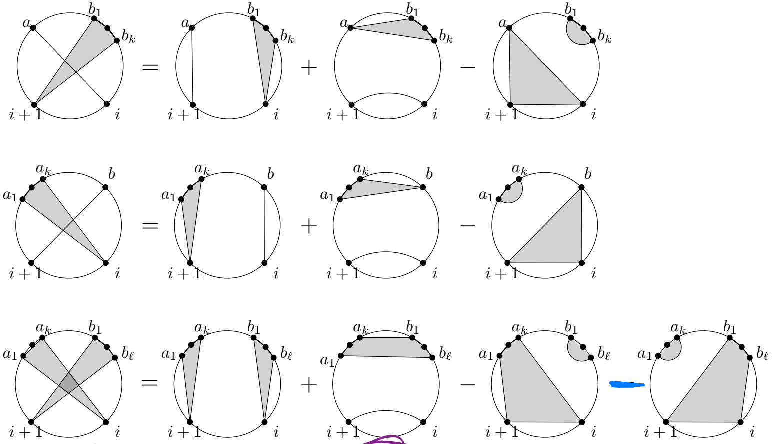
$$\begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} & x_{16} & x_{1,10} \\ x_{22} & x_{23} & x_{26} & x_{2,10} \\ x_{32} & x_{33} & x_{36} & x_{3,10} \\ x_{42} & x_{43} & x_{46} & x_{4,10} \end{vmatrix} \begin{vmatrix} x_{15} & x_{17} & x_{18} & x_{19} \\ x_{25} & x_{27} & x_{28} & x_{29} \\ x_{35} & x_{37} & x_{38} & x_{39} \\ x_{45} & x_{47} & x_{48} & x_{49} \end{vmatrix}$$

blocks
cols in minor

rows 1+2

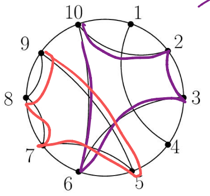
1	2	5
4	3	7
	6	8
	10	9

Want our work to agree with Brendan Rhoades's uncrossing rules up to sign.



4,4 4.4 6,2 5,3 5,3

Invariants



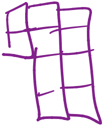
= . . . size 6

$$M = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,10} \\ x_{21} & x_{22} & \cdots & x_{2,10} \\ x_{31} & x_{32} & \cdots & x_{3,10} \\ \vdots & & & \vdots \end{pmatrix}$$

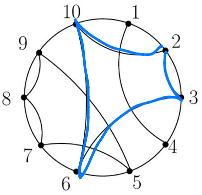
Naive guess:

only used first 4 rows of M

$$\begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} & x_{16} & x_{1,10} \\ x_{22} & x_{23} & x_{26} & x_{2,10} \\ x_{32} & x_{33} & x_{36} & x_{3,10} \\ x_{42} & x_{43} & x_{46} & x_{4,10} \end{vmatrix} \begin{vmatrix} x_{15} & x_{17} & x_{18} & x_{19} \\ x_{25} & x_{27} & x_{28} & x_{29} \\ x_{35} & x_{37} & x_{38} & x_{39} \\ x_{45} & x_{47} & x_{48} & x_{49} \end{vmatrix}$$

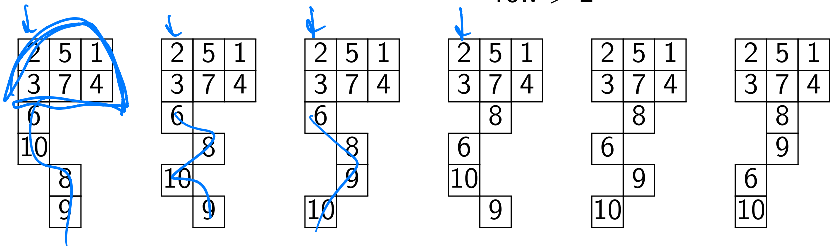


Invariants

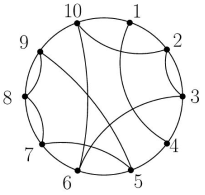


Jellyfish tableaux

- Blocks are columns
- Each column has entry in first two rows
- Exactly one entry in each row > 2



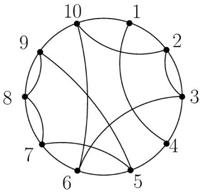
Invariants



$$M = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{16} & \cdots & x_{1,10} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{26} & \cdots & x_{2,10} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{36} & \cdots & x_{3,10} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{46} & \cdots & x_{4,10} \\ x_{51} & x_{52} & x_{53} & \cdots & x_{56} & \cdots & x_{5,10} \\ x_{61} & x_{62} & x_{63} & \cdots & x_{66} & \cdots & x_{6,10} \end{pmatrix}$$

$$T = \begin{array}{|c|c|c|} \hline 2 & 5 & 1 \\ \hline 3 & 7 & 4 \\ \hline 6 & & \\ \cline{1-1} 10 & & \\ \hline & 8 & \\ \cline{2-2} & 9 & \\ \cline{2-2} \end{array} = (-1)^{inv(T)}\, M^{cols\ 2,3,6,10}_{rows\ 1,2,3,4} \cdot M^{cols\ 5,7,8,9}_{rows\ 1,2,5,6} \cdot M^{cols\ 1,4}_{rows\ 1,2}$$

Invariants



$$M = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{16} & \cdots & x_{1,10} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{26} & \cdots & x_{2,10} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{36} & \cdots & x_{3,10} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{46} & \cdots & x_{4,10} \\ x_{51} & x_{52} & x_{53} & \cdots & x_{56} & \cdots & x_{5,10} \\ x_{61} & x_{62} & x_{63} & \cdots & x_{66} & \cdots & x_{6,10} \end{pmatrix}$$

$$\begin{array}{|c|c|c|} \hline 2 & 5 & 1 \\ \hline 3 & 7 & 4 \\ \hline 6 & & \\ 10 & & \\ & 8 & \\ & 9 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 5 & 1 \\ \hline 3 & 7 & 4 \\ \hline 6 & & \\ & 8 & \\ 10 & & \\ & 9 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 5 & 1 \\ \hline 3 & 7 & 4 \\ \hline 6 & & \\ & 8 & \\ & 9 & \\ 10 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 5 & 1 \\ \hline 3 & 7 & 4 \\ \hline & 8 & \\ 6 & & \\ 10 & & \\ & 9 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 5 & 1 \\ \hline 3 & 7 & 4 \\ \hline & 8 & \\ 6 & & \\ & 9 & \\ 10 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 5 & 1 \\ \hline 3 & 7 & 4 \\ \hline & 8 & \\ & 9 & \\ 6 & & \\ 10 & & \\ \hline \end{array}$$

$S^{(d,d,1^\ell)}$

Theorem (Pechenik 2014, Rhoades 2017, Kim–Rhoades 2022, P.–Pechenik–Striker 2022)

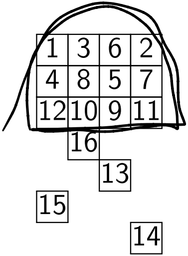
Noncrossing partitions of $2d+\ell$ *into* d *parts with no singletons give a basis for the Specht module* $S^{(d,d,1^\ell)}$*. The long cycle* $n12\cdots(n-1)$ *acts by rotation and* w_0 *by reflection.*

Generalization

In Fraser–P.–Pechenik–Striker (2025), we associate r-jellyfish tableaux to each ordered set partition with blocks sizes at least r.

blocks

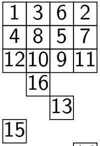
$r = 3$: $(1\ 4\ 12\ 15 \mid 3\ 8\ 10\ 16 \mid 5\ 6\ 9\ 13 \mid 2\ 7\ 11\ 14)$



Generalization: Fraser–P.–Pechenik–Striker (2025)

We associate r-jellyfish tableaux to each ordered set partition with blocks sizes at least r.

$r = 3$: $(1\ 4\ 12\ 15 \mid 3\ 8\ 10\ 16 \mid 5\ 6\ 9\ 13 \mid 2\ 7\ 11\ 14)$

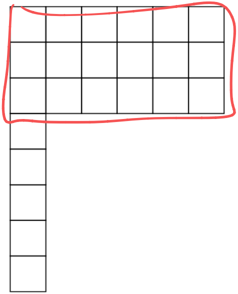


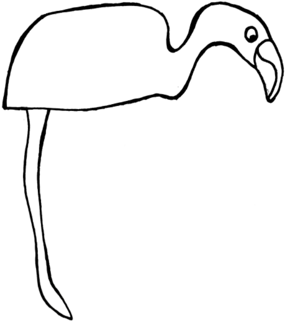
For $r > 2$,

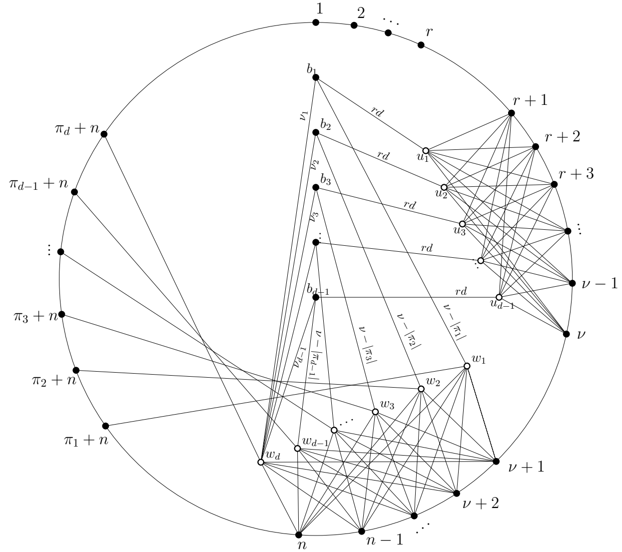
- The set of noncrossing ordered set partitions with block size at least r is linearly independent inside $S^{d^r,1^{n-dr}}$ but does not span.
- The set of all ordered set partitions with block size at least r spans $S^{d^r,1^{n-dr}}$.

Flamingo Specht modules

- Noncrossing partitions of $2d+\ell$ into d parts with no singletons give a basis for the Specht module $S^{(d,d,1^\ell)}$.
- For $r>2$, the set of noncrossing ordered set partitions with blocks at least size r is linearly independent inside $S^{d^r,1^{n-dr}}$ but does not span






1
2
$\cdots$
r
$r+1$
$r+2$
$r+3$
$\vdots$
$\nu-1$
ν
$\nu+1$
$\nu+2$
$\cdots$
$n-1$
n
π_1+n
π_2+n
π_3+n
$\vdots$
$\pi_{d-1}+n$
π_d+n
b_1
b_2
b_3
$\vdots$
b_{d-1}
u_1
u_2
u_3
$\vdots$
u_{d-1}
w_1
w_2
w_3
$\cdots$
w_{d-1}
w_d
rd
rd
rd
rd
rd
ν_1
ν_2
ν_3
ν_{d-1}
$\nu-|\pi_1|$
$\nu-|\pi_2|$
$\nu-|\pi_3|$
$\nu-|\pi_{d-1}|$

A basis for flamingo Specht module

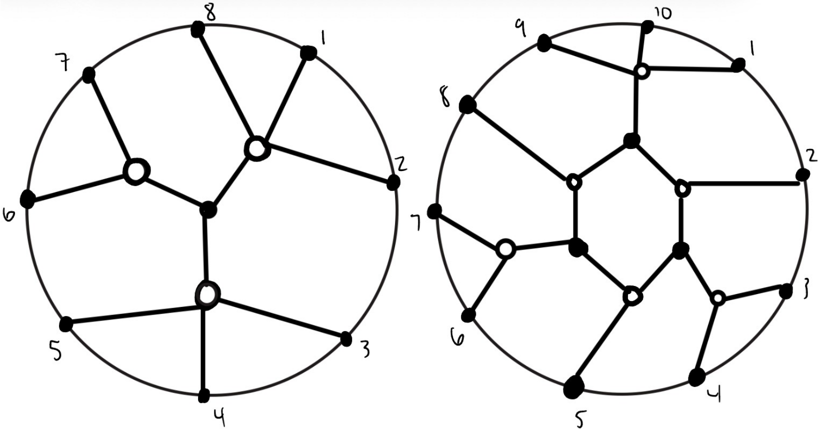
For $r > 2$,

- the set of noncrossing ordered set partitions with block size at least r is linearly independent inside $S^{d^r,1^{n-dr}}$ but does not span.
- the set of all ordered set partitions with block size at least r spans $S^{d^r,1^{n-dr}}$.

J. Kim proved that

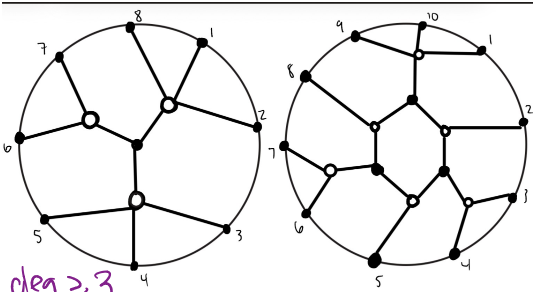
- the set of *r-weakly noncrossing partitions* is a basis of $S^{d^r,1^{n-dr}}$, but
- it is not rotation or reflection invariant.

Flamingo webs: a web basis $S^{(d,d,d,1^{n-3d})}$



$r=3$

Flamingo webs: a web basis $S^{(\underline{d,d,d},1^{n-3d})}$



white deg ≥ 3

black $=3$

Theorem (J. Kim 2024+)

The set of flamingo webs with n boundary vertices is a rotation invariant basis for $S^{(d,d,d,1^{n-3d})}$.

Idea: "trip=prom" can help us discover the correct variation of web we need to correspond to 3-row rectangular increasing tableaux.

(Hopkins–Rubey 2022,
Gaetz–Pechenik–Pfannerer–Striker–Swanson 2023, 2025)

Promotion digraphs for increasing tableaux

$$U=\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&6&9\\\hline 7&8&9&10\\\hline\end{array}\to\begin{array}{|c|c|c|c|}\hline 2&3&6&9\\\hline 4&5&9&10\\\hline 7&8&10&\mathbf{1}\\\hline\end{array}\to\begin{array}{|c|c|c|c|}\hline 3&5&6&9\\\hline 4&8&9&10\\\hline 7&10&\mathbf{1}&\mathbf{2}\\\hline\end{array}\to\begin{array}{|c|c|c|c|}\hline 4&5&6&9\\\hline 7&8&9&10\\\hline 10&\mathbf{1}&\mathbf{2}&\mathbf{3}\\\hline\end{array}$$

prom$_1$(U) moved into first row

prom$_2$(U) moved into 2nd row

$1 \to 6$, $1 \to 9$, $2 \to 5$

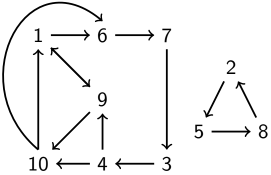
$1 \to 4$, $1 \to 10$, $2 \to 8$

Promotion digraphs for increasing tableaux

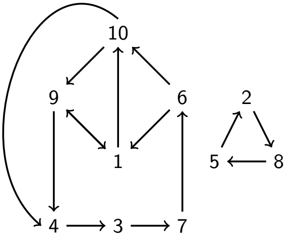
$$
\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&6&9\\\hline 7&8&9&10\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 2&3&6&9\\\hline 4&5&9&10\\\hline 7&8&10&\mathbf{1}\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 3&5&6&9\\\hline 4&8&9&10\\\hline 7&10&\mathbf{1}&\mathbf{2}\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 4&5&6&9\\\hline 7&8&9&10\\\hline 10&\mathbf{1}&\mathbf{2}&\mathbf{3}\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 5&6&9&10\\\hline 7&8&10&\mathbf{3}\\\hline 10&\mathbf{1}&\mathbf{2}&\mathbf{4}\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 6&8&9&10\\\hline 7&10&\mathbf{2}&\mathbf{3}\\\hline 10&\mathbf{1}&\mathbf{4}&\mathbf{5}\\\hline\end{array}
$$

$$
\to
\begin{array}{|c|c|c|c|}\hline 7&8&9&10\\\hline 10&\mathbf{1}&\mathbf{2}&\mathbf{3}\\\hline \mathbf{1}&\mathbf{4}&\mathbf{5}&\mathbf{6}\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 8&9&10&\mathbf{3}\\\hline 10&\mathbf{1}&\mathbf{2}&\mathbf{6}\\\hline \mathbf{1}&\mathbf{4}&\mathbf{5}&\mathbf{7}\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 9&10&\mathbf{2}&\mathbf{3}\\\hline 10&\mathbf{1}&\mathbf{5}&\mathbf{6}\\\hline \mathbf{1}&\mathbf{4}&\mathbf{7}&\mathbf{8}\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 10&\mathbf{1}&\mathbf{2}&\mathbf{3}\\\hline \mathbf{1}&\mathbf{4}&\mathbf{5}&\mathbf{6}\\\hline \mathbf{4}&\mathbf{7}&\mathbf{8}&\mathbf{9}\\\hline\end{array}
\to
\begin{array}{|c|c|c|c|}\hline \mathbf{1}&\mathbf{2}&\mathbf{3}&\mathbf{6}\\\hline \mathbf{4}&\mathbf{5}&\mathbf{6}&\mathbf{9}\\\hline \mathbf{7}&\mathbf{8}&\mathbf{9}&\mathbf{10}\\\hline\end{array}
$$

$\mathrm{prom}_1(U) =$



$\mathrm{prom}_2(U) =$



Promotion digraphs for increasing tableaux

Theorem (P.–Pechenik–Striker 2025+)

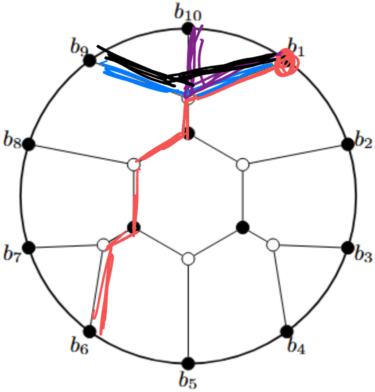
Suppose that T, T' are rectangular increasing tableaux with the same tuples of promotion digraphs. Then $T = T'$. Indeed, there is an algorithm to reconstruct the rectangular tableau from its promotion digraphs.

This does not hold for non-rectangular increasing tableaux!

Trip digraphs for flamingo webs

left(s) at white
right(s) at black

$W =$



$\mathrm{Trip}_1(W)$

$1 \to 6$

$\to 9$

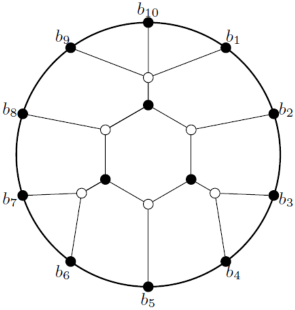
$\mathrm{Trip}_2(W)$

right at white
left at black

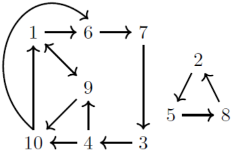
$1 \to 10$

$\to 9$

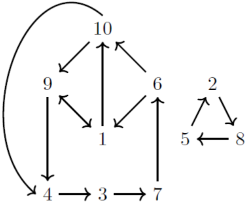
Trip digraphs for flamingo webs



$\mathsf{trip}_{1,3}(W) =$

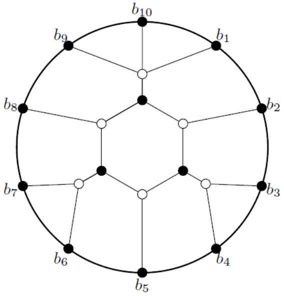


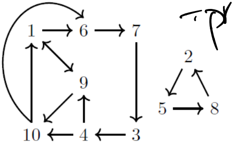
$\mathsf{trip}_{2,3}(W) =$

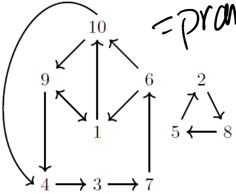


$U =$

1	2	3	6
4	5	6	9
7	8	9	10

$W =$ 

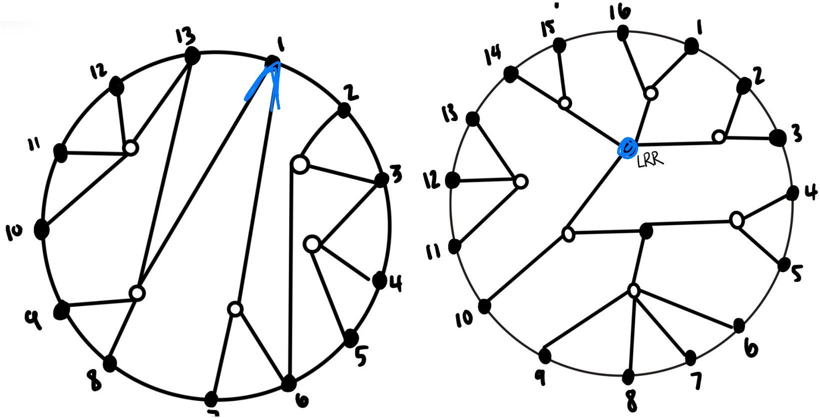
$\mathrm{trip}_{1,3}(W) =$  $= \mathrm{prom}_1(U)$

$\mathrm{trip}_{2,3}(W) =$  $= \mathrm{prom}_2(U)$

Conjecture (P.–Pechenik–Striker 2025+)

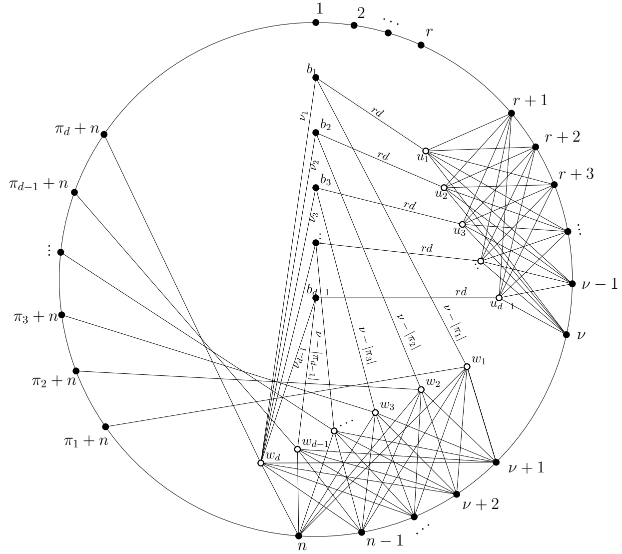
- *Each flamingo web has a distinct pair of trip digraphs.*
- *For each flamingo web W, there is a unique 3-row increasing tableau whose promotion digraphs are the same as the trip digraphs of W.*
- *This injection from flamingo webs to increasing tableaux respects K-promotion.*

Webs corresponding to 3-row increasing rectangular tableaux



Thank you!

- C. Fraser, R. Patrias, O. Pechenik, and J. Striker. *Web invariants for flamingo Specht modules,* Algebraic Combinatorics 8(1) (2025): 235–266.
- J. Kim. *Rotation invariant webs for three row flamingo Specht modules,* arXiv:2402.11994.
- J. Kim and B. Rhoades. *Set partitions, fermions, and skein relations*, Int. Math. Res. Not. IMRN (2023), 9427-9480.
- R. Patrias, O. Pechenik, and J. Striker. *Promotion digraphs*, arXiv:2508.10969.
- R. Patrias, O. Pechenik, and J. Striker. *A web basis of invariant polynomials from noncrossing partitions*, Adv. Math. 408 (2022): 108603.
- B. Rhoades. *A skein action of the symmetric group on noncrossing partitions*, J. Algebraic Combin. **45** (2017), no. 1, 81-127.


1
2
$\cdots$
r
$r+1$
$r+2$
$r+3$
$\vdots$
$\nu-1$
ν
$\nu+1$
$\nu+2$
$\cdots$
$n-1$
n
π_1+n
π_2+n
π_3+n
$\vdots$
$\pi_{d-1}+n$
π_d+n
b_1
b_2
b_3
$\vdots$
b_{d-1}
u_1
u_2
u_3
$\vdots$
u_{d-1}
rd
rd
rd
rd
rd
ν_1
ν_2
ν_3
ν_{d-1}
$\nu-|\pi_1|$
$\nu-|\pi_2|$
$\nu-|\pi_3|$
$\nu-|\pi_{d-1}|$
w_1
w_2
w_3
$\cdots$
w_{d-1}
w_d