Webs and Tropical Grassmannians

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joint work in progress
w. N. Early

K = valued field $val : K \to \mathbb{R}$ e.g. $K = \mathbb{C}((t))$ or $\bigcup_n \mathbb{C}((t^{1/n}))$

$$M = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in Gr(k,n)(K)$$

$p_I = val(\Delta_I(M)) \in \mathbb{R}$ tropical Plücker vector

Defn The tropical Grassmannian $\mathrm{Trop}\,Gr(k,n) \subset \mathbb{R}^{\binom{n}{k}}$ is the closure of space of all such p.

[Speyer Sturmfels]

satisfies tropical Plücker relation for all $S \in \binom{[n]}{k-2}$ $\{a,b,c,d\} \subset [n] \setminus S$

$$\boxed{\min(p_{Sab} + p_{Scd},\ p_{Sac} + p_{Sbd},\ p_{Sad} + p_{Sbc}) \text{ attained twice}}$$

Positive tropical Grassmannian [Speyer-Williams]

$$\mathrm{Trop}_{>0}\, Gr(k,n) = \mathrm{val}\left(Gr(k,n)(K_{>0}) \right)$$

By

positive Dressian = positive tropical Grassmannian

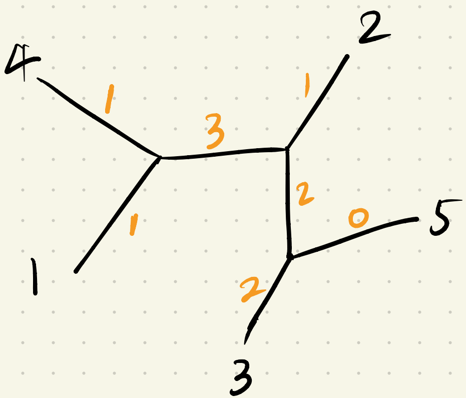
[Speyer-Williams, Arkani-Hamed - Lo - Spradlin]

$\mathrm{Trop}_{>0}\, Gr(k,n)$ is cut out by

positive tropical Plücker relation for $a<b<c<d$

$$\boxed{P_{S_{ac}} + P_{S_{bd}} = \min\left(P_{S_{ab}} + P_{S_{cd}},\, P_{S_{ad}} + P_{S_{bc}} \right)}$$

Phylogenetic trees



I	p_I
12	-5
13	-8
14	-2
15	-6
23	-5
24	-5
25	-3
34	-8
35	-2
45	-6

$S = \phi \quad \{a,b,c,d\} = \{2,3,4,5\}$

$p_{23} + p_{45} = -11$

$p_{24} + p_{35} = -7$

$p_{25} + p_{34} = -11$

min

positive
"
planar

"Tropical $Gr(2,n)$ is the space of phylogenetic trees"

Combinatorial model for higher k?

Herrmann-Jensen-Joswig-Sturmfels abstract/metric tree arrangements

Cachazo-Guevara-Umbert-Zhang planar matrices

Le, Fraser-Le higher laminations

$\vdots$

Potential applications

- understand fan structure
- duality conjectures: webs as points and functions
- tropicalize plabic graphs
- membranes L.-Postnikov vs. Keel-Tevelev
- planar basis Early
- amplitudes (CEGM, symbol alphabet)

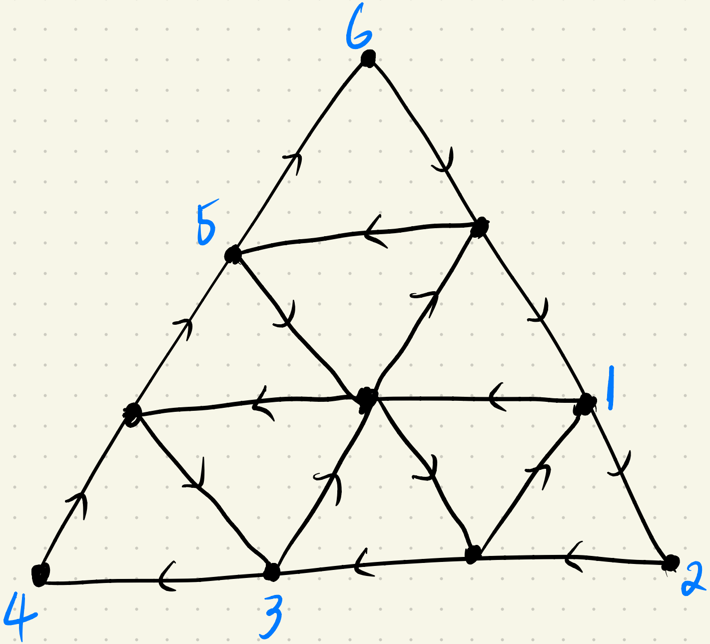
- cluster structure
- mirror symmetry

algebra is infinite and it is not clear that there even exists a cluster polytope. Using [30, 31] we have computed the vertices, found the bounding hyperplanes, and analyzed the polyhedral combinatorics of $\mathcal{C}(k,n)$ for various (k,n). Here for brevity we summarize just the f-vectors

$$\begin{aligned}
(3,6):&\quad (1,48,98,66,16,1)\,,\\
(4,7):&\quad (1,693,2163,2583,1463,392,42,1)\,,\\
(3,8):&\quad (1,13612,57768,100852,93104,48544,\\
&\qquad\qquad 14088,2072,120,1)\,,\\
(4,8):&\quad (1,90608,444930,922314,1047200,706042,\\
&\qquad\qquad 285948,66740,7984,360,1)\,.
\end{aligned}\tag{3}$$

??

Arkani-Hamed, L., Spradlin



$d(1,2) = 1$

$d(1,3) = 3$

$d(1,4) = 4$

$d(1,5) = 3$

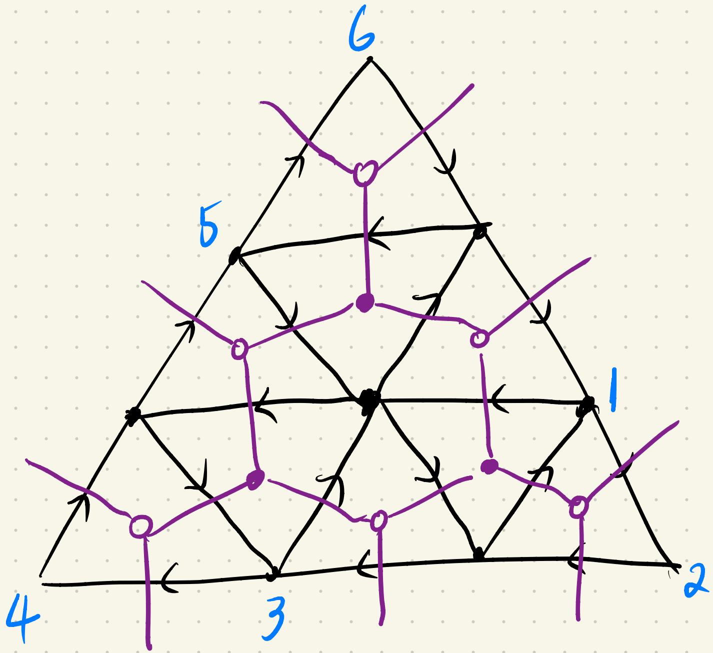
$d(1,6) = 4$

$d(2,1) = 2$

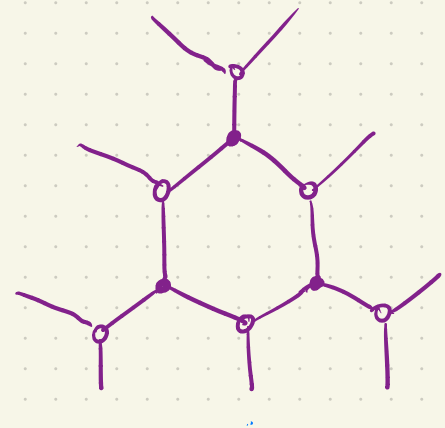
abc	d[a,b,c]
123	4
124	5
125	4
126	5
134	4
135	6
136	7
145	7
146	8
156	4
234	5
235	7
236	8

$$d[a,b,c] = \min_x \left(d(x,a) + d(x,b) + d(x,c) \right)$$

$$P_{abc} = -\frac{1}{3} d[a,b,c]$$



non-elliptic web



cf.

Fontaine-Kamnitzer-Kuperberg

Postnikov, Shen-Sun-Weng

Le

(diskoid)

(dual quiver)

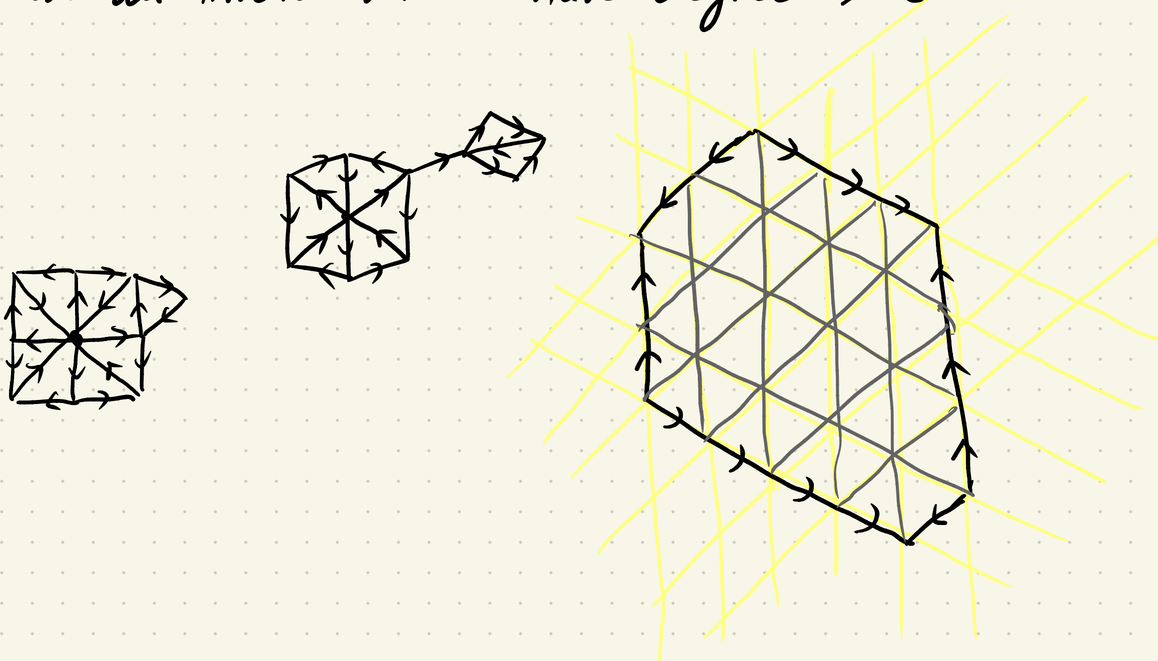
(distance function)

Definition

A CAT(0)-planar graph is a connected directed graph in the plane satisfying:

(1) all interior faces are oriented triangles

(2) all interior vertices have degree ≥ 6

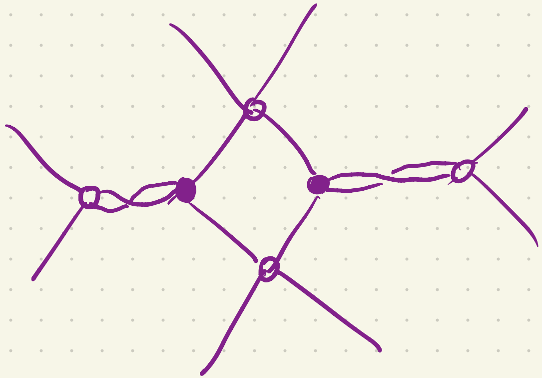


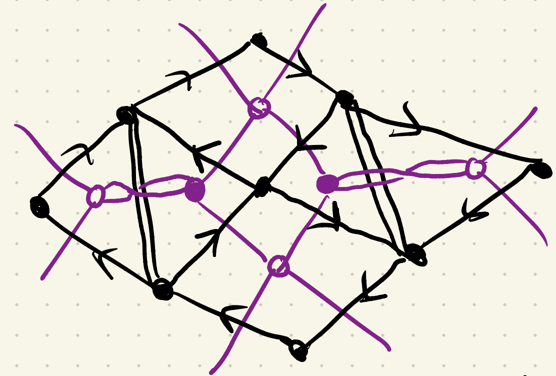
Theorem [Early - L.]

Let G be a CAT(0) planar graph, and $b_1, b_2, \ldots, b_n$ vertices.

Then
$$P_{G, i_1 i_2 i_3} = P_{i_1 i_2 i_3} = -\frac{1}{3} d[b_{i_1} b_{i_2} b_{i_3}]$$
is a tropical Plücker vector, and if
$b_1, b_2, \ldots, b_n$ are cyclically ordered on the boundary
it is a positive tropical Plücker vector.

SL(4)-webs cf. GPPSS, GSSW (pockets)

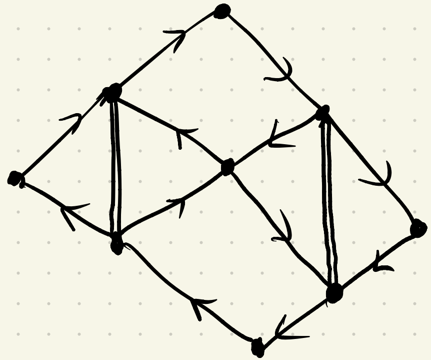




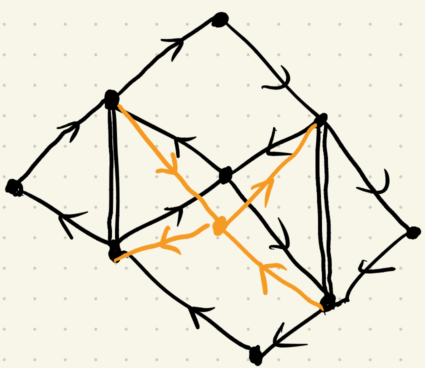
dual quiver

$d(x,y) = 1$
$d(y,x) = 3$

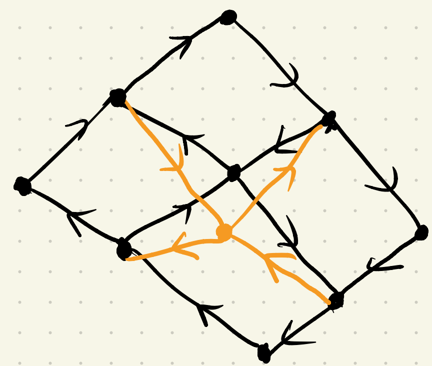
$d(x,y) = 2$
$d(y,x) = 2$



doesn't work!



$\cong$



($p_\bullet$) (positive) tropical Plücker vector

Lift each vertex $e_I = e_{i_1} + \dots + e_{i_k}$ of hypersimplex $\Delta(k,n)$ to

$$\tilde{e}_I = (e_I, p_I) \in \mathbb{R}^n \times \mathbb{R} \quad \text{giving} \quad \tilde{\Delta}(k,n) \subset \mathbb{R}^n \times \mathbb{R}$$

Projecting the lower faces of $\tilde{\Delta}(k,n)$ back to $\mathbb{R}^n$ gives a regular subdivision of $\Delta(k,n)$.

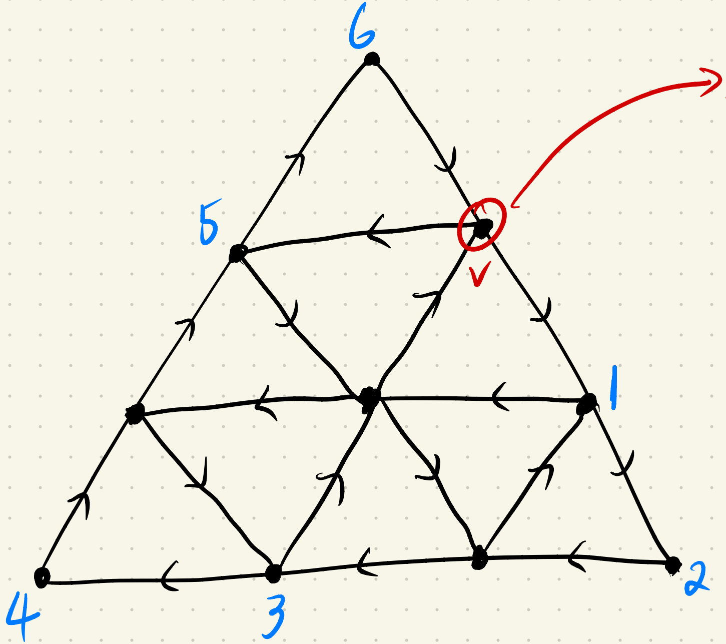
tropical Plücker vector $\rightsquigarrow$ subdivision into matroid polytopes

positive tropical Plücker vector $\rightsquigarrow$ subdivision into positroid polytopes

Q: How to read subdivision from G?

Def'n For a vertex v, define

$$M_v := \{ I \in \binom{[n]}{3} \mid v \text{ is a distance minimizer for } I = \{i_1, i_2, i_3\} \}$$



$M_v = \{125, 126, 135, 136, 145, 146, 156, 235, 236, 245, 246, 256\}$

Th'm [EL] M_v is a positroid and part of the positroid subdivision of $P_{G,\bullet}$

Not all pieces of subdivision appear! only the "positive part"

Def'n [Speyer] The tropical linear space L of $P_\bullet$ is the subset of $\mathbb{R}^n/(1,1,1,\dots)$ where

$$\min_i \left(p(\tau\setminus\tau_i) + x_i\right) \text{ is attained twice}$$

for all $\tau = (\tau_1, \tau_2, \dots, \tau_{k+1}) \in \binom{[n]}{k+1}$

Th'm [EL] We have an embedding

$$G \hookrightarrow L \text{ as a simplicial complex on } L(\mathbb{Z})$$

given by

$$v \longmapsto -\frac{1}{3}\left(d(v,b_1), d(v,b_2), \dots, d(v,b_n)\right)$$

"lineality"

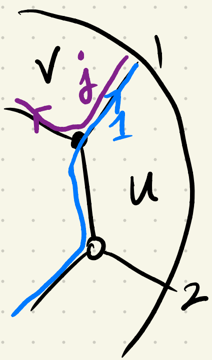
e.g. $v \longmapsto -\frac{1}{3}(1,2,4,5,1,2) \sim \frac{1}{3}(0,1,0,1,0,1) - (0,0,1,1,0,0)$

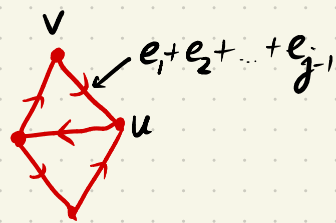
$b_1 \longmapsto -\frac{1}{3}(0,1,3,4,3,4) \sim \frac{1}{3}(0,1,0,1,0,1) - (0,0,1,1,1,1)$

This embeds $G \hookrightarrow \mathbb{R}^n/(1,1,\dots,1)$ as a membrane.
(after a standard change of coords)

<u>Defn</u> [L.-Postnikov]

A membrane is an embedding of a two-dimensional simplicial complex planar dual to a plabic graph so that all vertices are integer points and edges are in directions $e_i + e_{i+1} + \dots + e_{j-1}$





The vector $e_i + e_{i+1} + \dots + e_{j-1}$ is normal to the hyperplane separating the positroid polytopes P_{M_v}, P_{M_u}.

The above construction is compatible with Keel–Tevelev's homeomorphism

$$L \xrightarrow{\sim} M \subset PGL_k\text{-building}$$

tropical linear space   membrane

LP membrane $\approx$ KT membrane

Connects to higher laminations   Le, Fraser–Le

diskoids   FKK, Akhmejanov

Suggests definition of $L_{>0}$:

:= tropical convex hull of the tropical cocircuits $p(\sigma *)$ $\sigma = \{i, i+1, \ldots, i+k-2\}$

Yu–Yuster: L = tropical convex hull of all $p(\sigma *)$ $\sigma \in \binom{[n]}{k-1}$

Planar basis

Defn $d(e_J, e_I)$:= distance using steps $e_i - e_{i+1}$

Define $h_\circ^J$ by $h_I^J := \frac{1}{n} d(e_J, e_I)$.

Th'm [Early] (1) The $h_\circ^J$ are positive tropical Plücker vectors

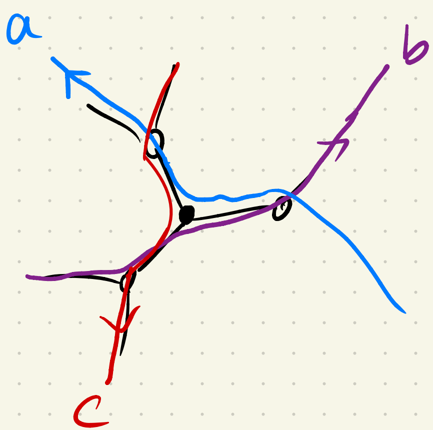
(2) $\{h_\circ^J \mid J \in \binom{[n]}{k}\}$ is a basis for $\mathbb{R}^{\binom{n}{k}}$

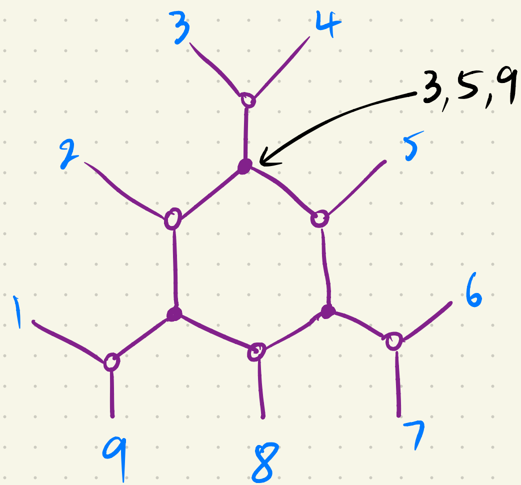
(3) $h_\circ^J$ is a ray of $\mathrm{Trop}_{>0}\, Gr(k,n)$ giving a multisplit subdivision

Th'm [EL]

$G^\vee$ = plabic graph

$$P_G = -\sum_{v\ \mathrm{white}(G^\vee)} h_{a_v b_v c_v} + \sum_{u \in \mathrm{black}(G^\vee)} h_{a_u b_u c_u}$$





Application: Ren-Spradlin-Volovich symbol alphabet recursion

Future:

Higher k

$\mathrm{Trop}_{>0}\, Gr(k,n) \subset \mathrm{Trop}\, Gr(k,n) \subset Dr(k,n)$

Intersection pairing: evaluate web as function
on
web as point