Rotation-invariant webs in arbitrary rank

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Based on work with Ron Cherny, Mike Cummings, Christian Gaetz, Stephan Pfannerer, Jessica Striker, & Josh Swanson

In type A, webs and relations on webs are fairly well understood.

But good web <u>bases</u> are not

You can get <u>a</u> web basis by choosing, for each dual canonical basis element, a web whose invariant has the same leading term

This is not so good!

Why is this bad?

- cannot tell if a web is a basis web without a lookup
- no strategy for expanding a web in the basis without computing invariants
- Simple algebraic operations will look complicated

<u>Claim</u>: Hourglass plabic graphs are the correct sources for good bases

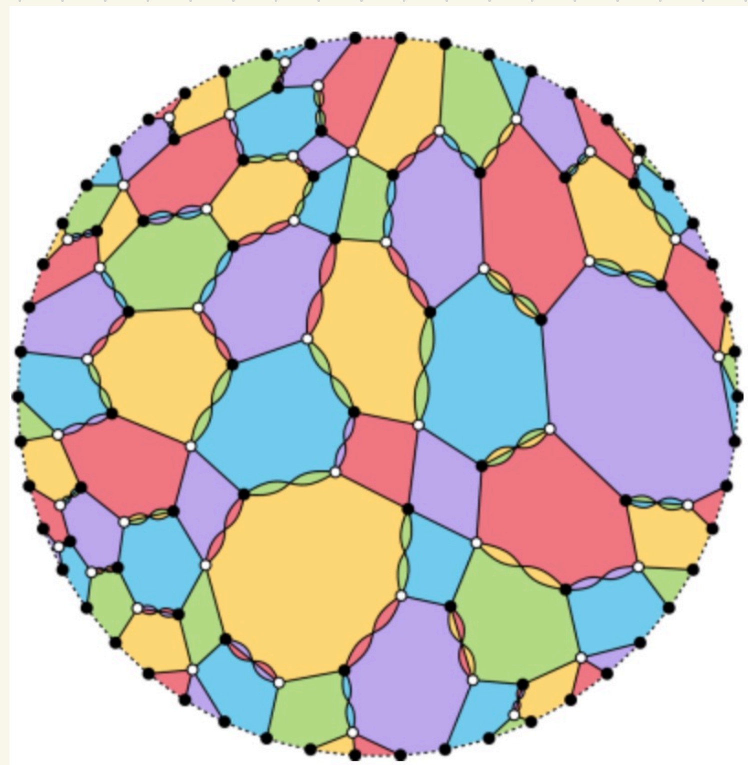
Def An m-hourglass edge is 

m-strands

Def An r-hourglass plabic graph is a properly bicolored graph embedded in a disk with internal vertices of degree r and boundary vertices of simple degree 1

Ex

$r=5$

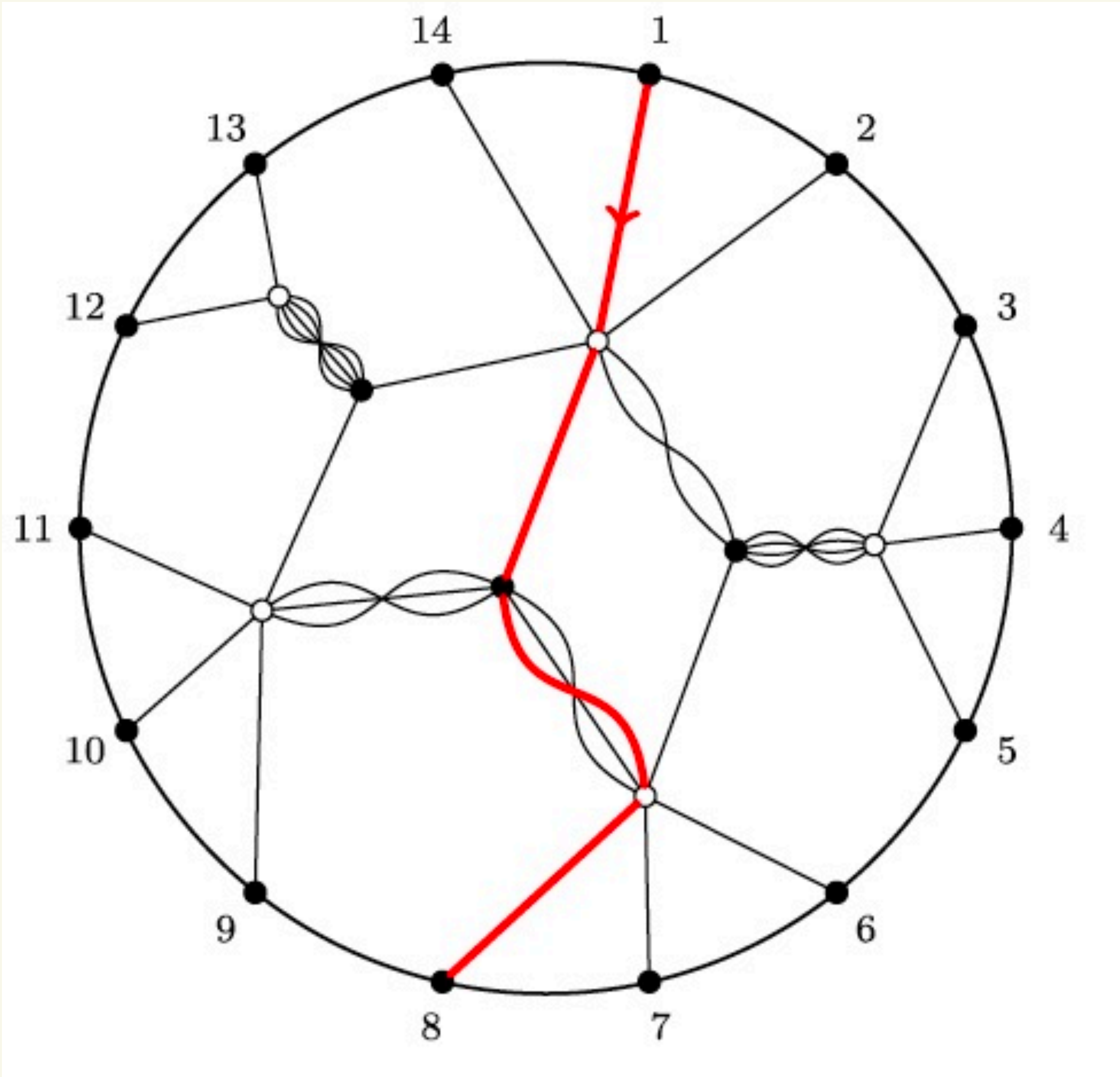


A r-HPG has r-1 trip permutations.

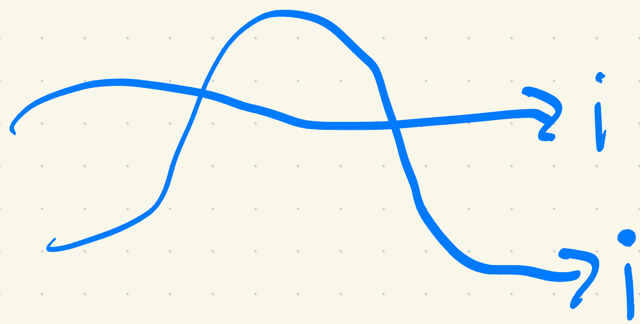
Trip i takes the ith left at each ○
and the ith right at each ●

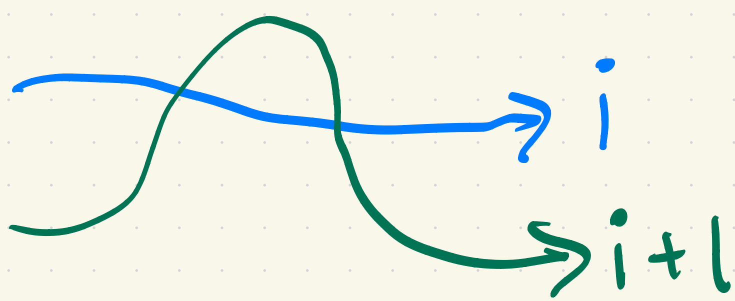
Ex
$r=7$

$\mathrm{trip}_4(1) = 8$



Def An r-HPG is <u>fully reduced</u> if it avoids the following crossings of trip strands:

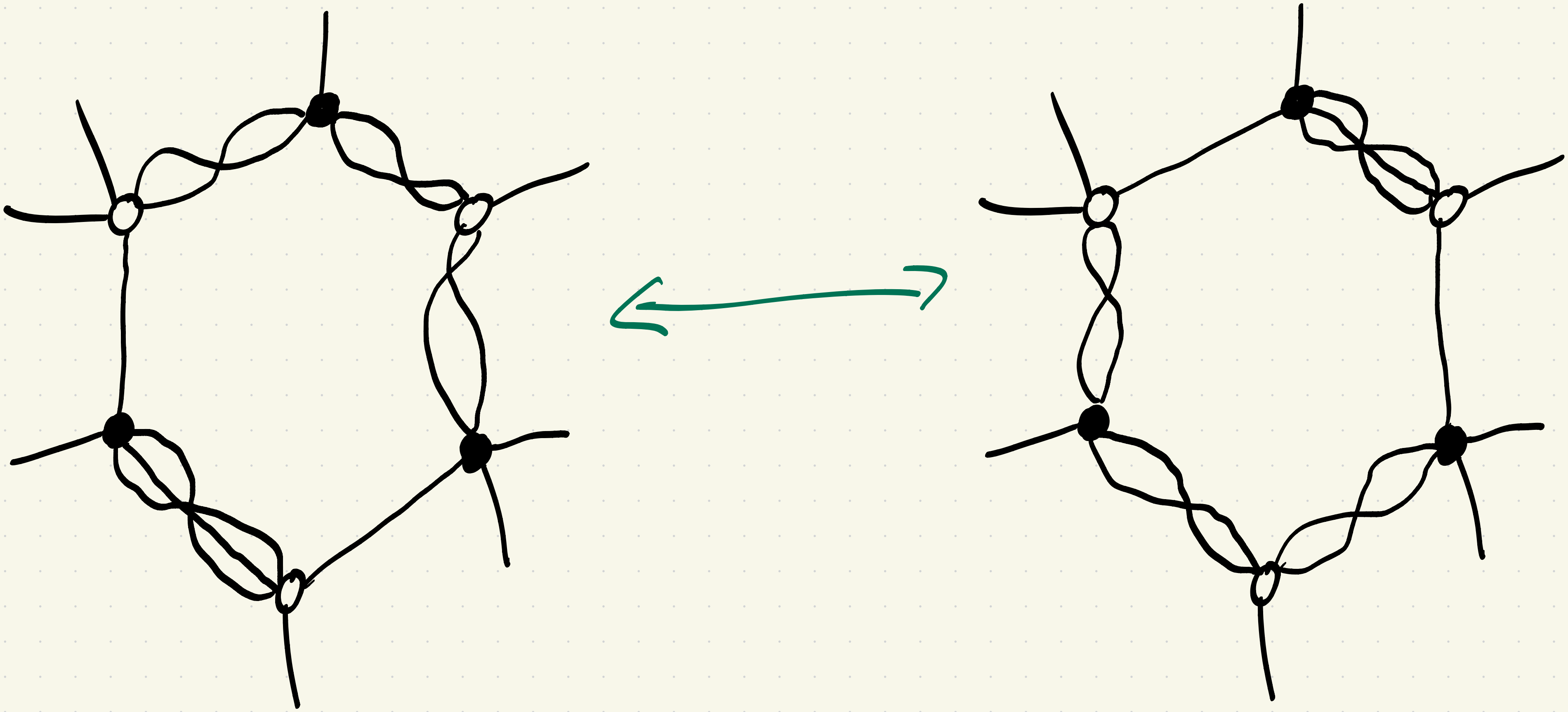




But

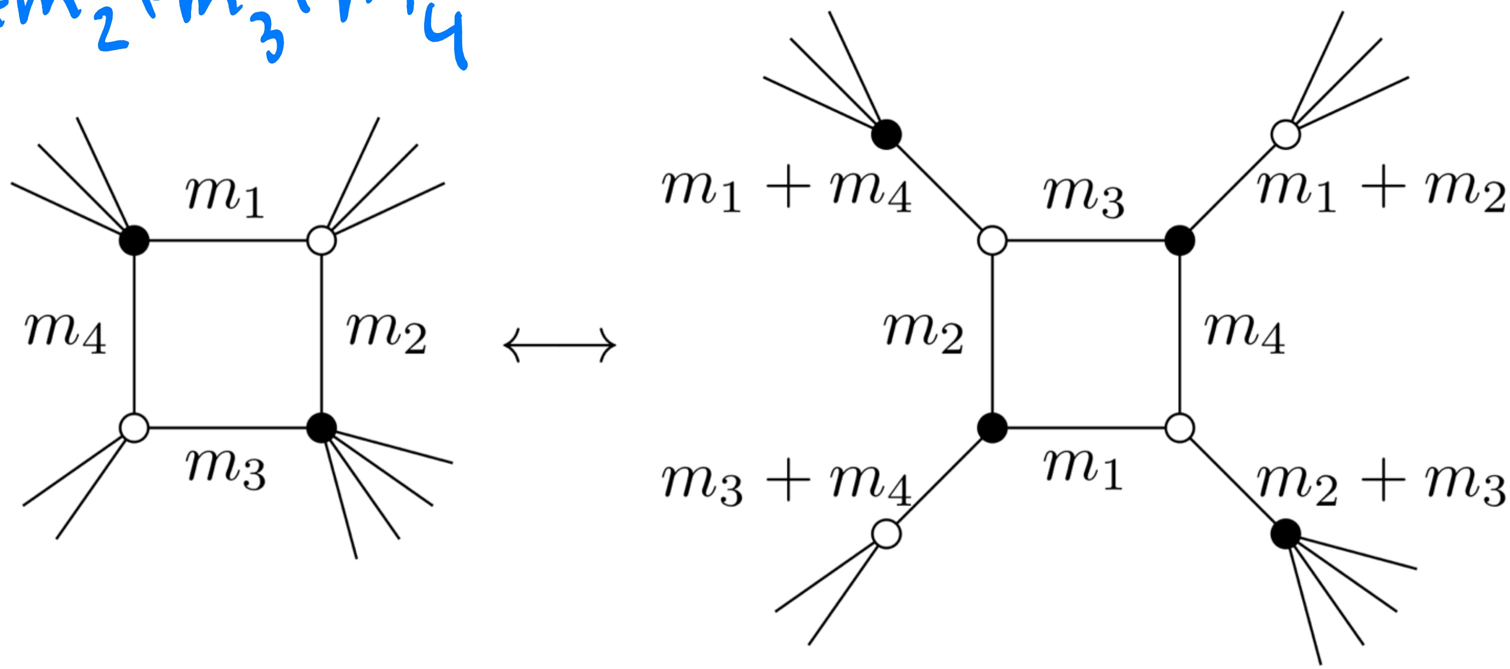
is okay!

<u>Moves</u> are local surgeries that preserve trip permutations.



There is a square move in each r

$r = m_1 + m_2 + m_3 + m_4$



This preserves trip permutations and web invariants

False but good

Conjecture

Move equivalence classes of fully reduced HPGs are a good web basis for $U_q(\mathfrak{sl}_r)$ invariants

Moreover they are in bijection with tableaux under $\mathrm{trip}_\bullet(W) = \mathrm{prom}_\bullet(T)$

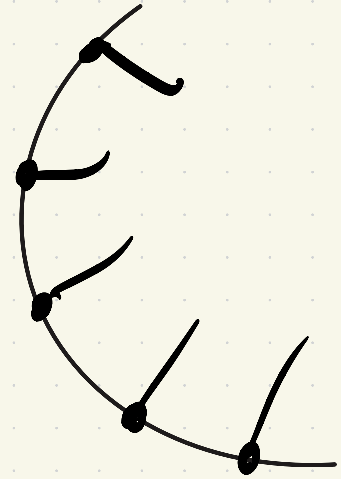
Special cases:

$r \leq 4$ (previous talk)

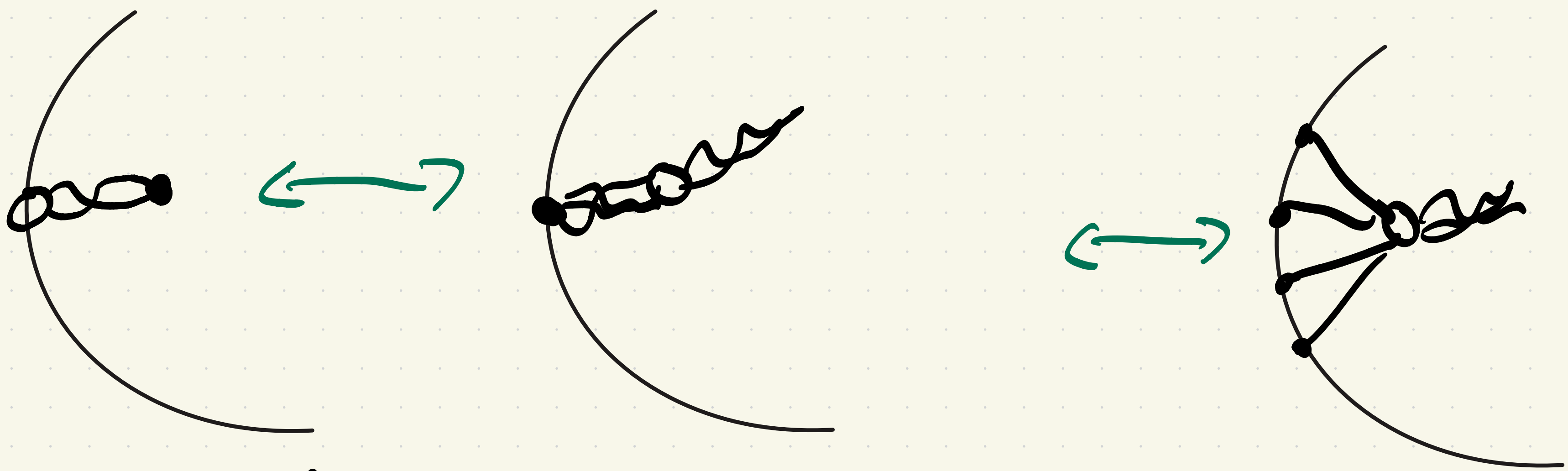
$r = 5$ (in progress w/ same people)

arbitrary r $\begin{cases} \text{2-column case} & \text{''} \\ \text{arboreal case} & \text{w/ Cherny \& Pfannerer} \end{cases}$

(basically) suffices to consider the case of standard boundary



because



$$\Lambda^k(V^*) \cong \Lambda^{r-k}(V) \hookrightarrow V^{\otimes r-k}$$

special case of clasping

Plücker degree 2

$\mathrm{Inv}_{\mathfrak{sl}_r}(V^{\otimes 2r})$

Relevant tableaux are standard of shape 2^r

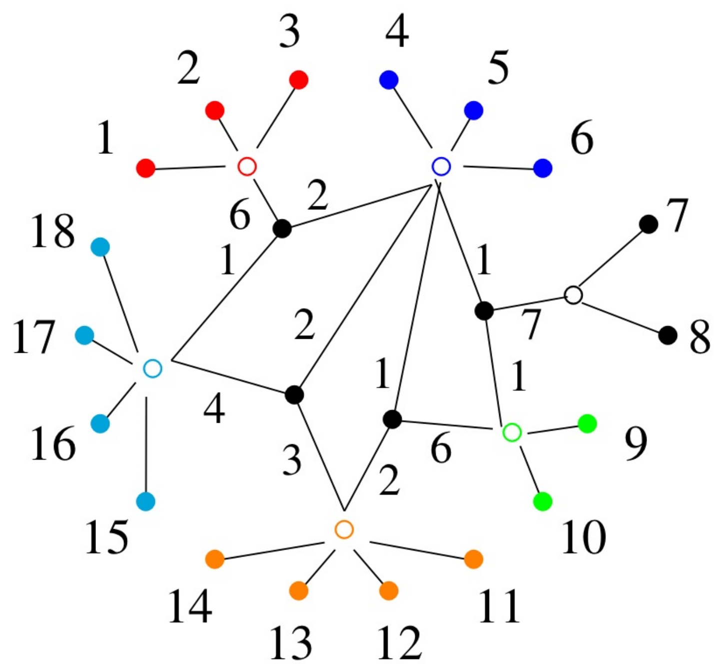
$r=8$

1	4
2	6
3	7
5	10
8	11
9	13
12	14
15	16

Chris Fraser (2023) finds a web basis element for each such tableau.

- Construction is by intricate Catalan combinatorics
- has no intrinsic characterization of the webs
- in fact, the web basis is dual canonical

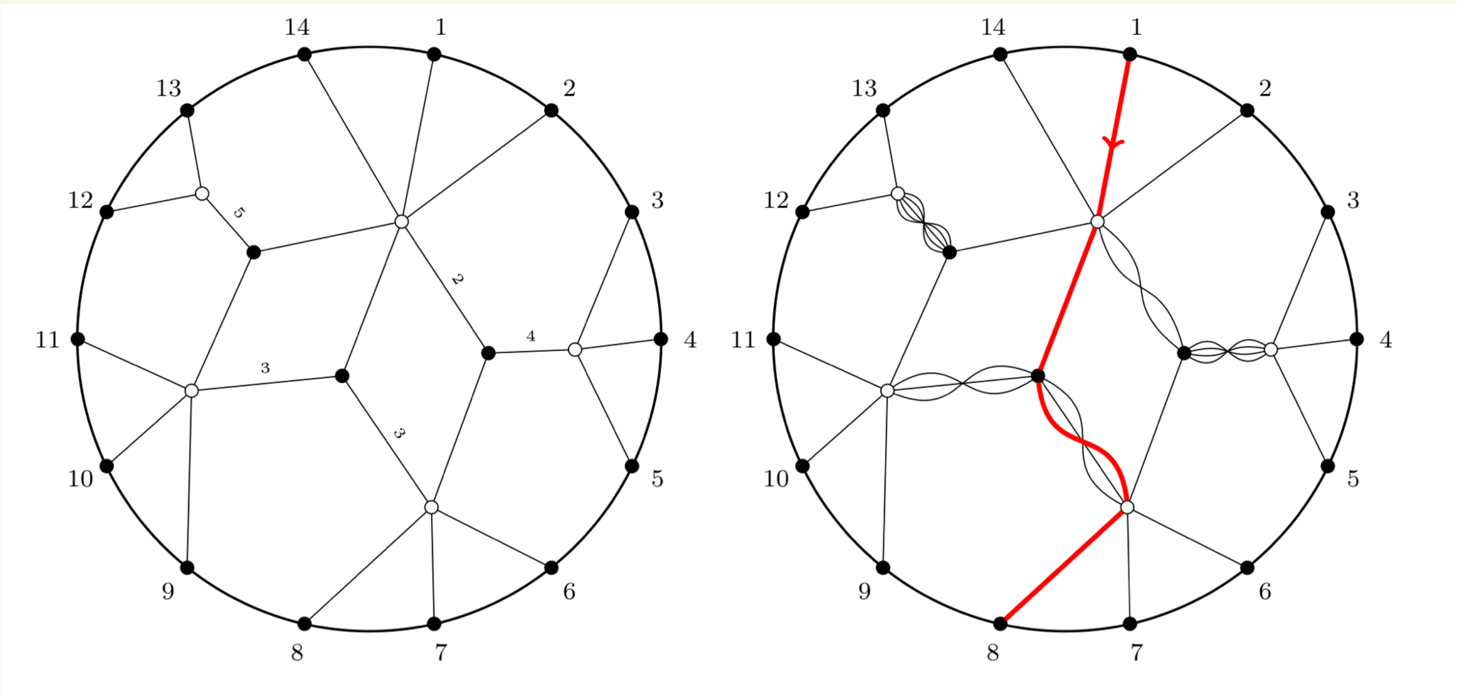
1	4
2	7
3	9
5	11
6	12
8	15
10	16
13	17
14	18



Thm (Gaetz-P-Pfannerer-Striker-Swanson 2025)

- Fraser's webs, interpreted as HPGs, are exactly the fully reduced HPGs with appropriate boundary
- The correspondence with tableaux is

$$\mathrm{trip}_\bullet(W) = \mathrm{prom}_\bullet(T)$$



All internal faces of 2-column webs are squares.

Only $\mathfrak{sl}_r$-square moves apply.

The web invariant is constant on the move class.

Springer (1976) gives a resolution of the variety of nilpotents in $\mathfrak{sl}_r$, fibre $X_\lambda \subseteq SL_r/B$ is a **Springer fibre**

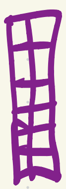
partition

Springer fibres are far from irreducible:

components $\longleftrightarrow$ $SYT(\lambda)$

X_λ is almost never smooth, but components can be smooth (not well understood)

__Thm__ (Perrin-Smirnov 2012)

If $\lambda = 2^k =$ , then all components of X_λ are normal and CM, but not all smooth

__Thm__ (Fresse-Melnikov 2011)

Combinatorial characterization of smoothness

Let T be a standard tableau of shape $Y(u)$. To T, we associate the involution $\sigma_T \in \mathbf{S}_n^2(k)$ by the following procedure. Let $a_1 < \ldots < a_{n-k}$ (resp. $j_1 < \ldots < j_k$) be the entries in the first (resp. second) column of T. Put $\sigma_T = (i_1, j_1) \cdots (i_k, j_k)$ where $i_1 = j_1 - 1$ and $i_p = \max\{a \in \{a_1, \ldots, a_{n-k}\} \setminus \{i_1, \ldots, i_{p-1}\} : a < j_p\}$ for $p = 2, \ldots, k$.

For $i = 1, \ldots, n$, let $c_T(i) \in \{1, 2\}$ be the index of the column of T containing i. Write $\tau^*(T) = \{i \in \{1, \ldots, n-1\} \ : \ c_T(i) < c_T(i+1)\}$. Let $|\tau^*(T)|$ be the cardinality of $\tau^*(T)$.

Example Let $T = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 5 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}$. Then $\sigma_T = (3, 4)(5, 6)(2, 7)$. Thus $2, 3, 4, 5, 6, 7$ are the end points of σ_T, and $1, 8$ are the fixed points. We have $\tau^*(T) = \{3, 5\}$.

Our first criterion gives an explicit description of tableaux T for which the component $\mathcal{K}^T$ is singular.

Theorem 1.2 *Suppose that $Y(u)$ has two columns. Let T be a standard tableau of shape $Y(u)$. Let $\mathcal{K}^T \subset \mathcal{F}_u$ be the irreducible component associated to T.*

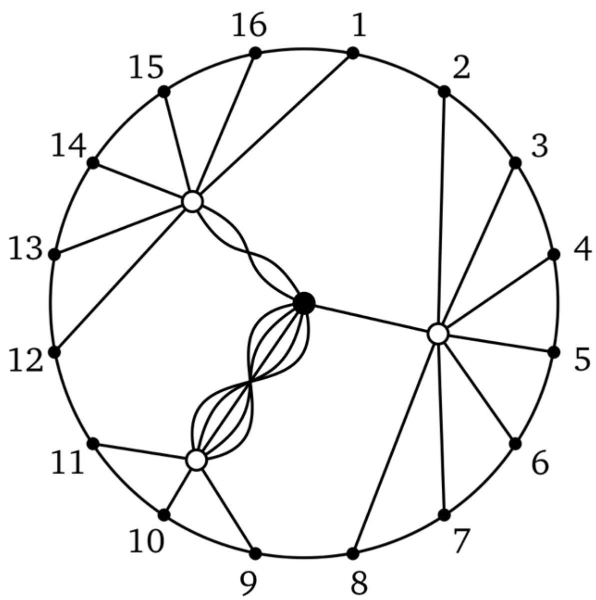
(a) *If $|\tau^*(T)| = 1$, then $\mathcal{K}^T$ is smooth.*
(b) *If $|\tau^*(T)| = 2$, then $\mathcal{K}^T$ is smooth if and only if at least one of $\{1, n\}$ is an end point of σ_T.*
(c) *If $|\tau^*(T)| = 3$, then $\mathcal{K}^T$ is smooth if and only if both 1 and n are end points of σ_T and $(1, n) \notin \sigma_T$.*
(d) *If $|\tau^*(T)| \geq 4$, then $\mathcal{K}^T$ is singular.*

Thm (Cummings $2025+\epsilon$)

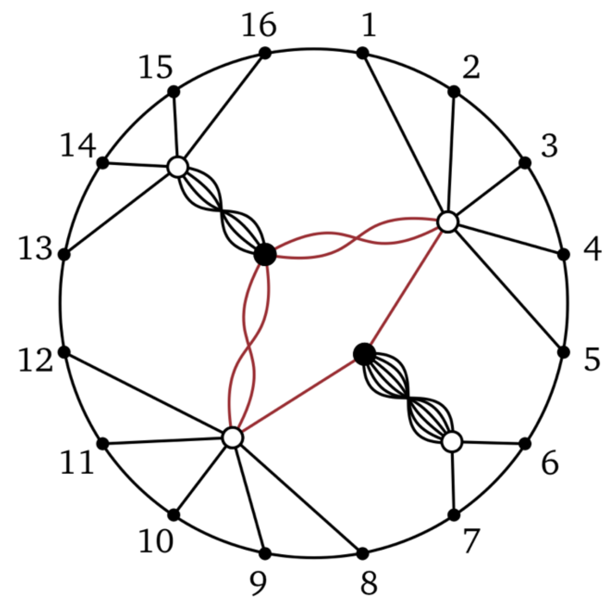
For $T \in SYT(r \times 2)$, the T-component of the Springer fibre is

smooth $\longleftrightarrow$ the web is a forest

1	2
3	9
4	10
5	12
6	13
7	14
8	15
11	16



1	6
2	8
3	9
4	10
5	13
7	14
11	15
12	16



Thm cont (Cummings 2025 + G)

Smooth components X_T, X_U

have the same Poincaré polynomial $\longleftrightarrow$ T, U lie in the same promotion-evacuation orbit

$\longleftrightarrow$ webs lie in the same dihedral orbit

Exact enumeration of smooth components, correcting Mansour (2025)

Equinumerous with a pattern avoidance class of permutations

Arboreal webs

We say an HPG is arboreal if the underlying graph is a forest

Thm (Cherny – P – Pfannerer 2025 + 2ε)

Arboreal HPGs are fully reduced and correspond to "arboreal" tableaux under

$$\mathrm{trip}_\bullet(W) = \mathrm{prom}_\bullet(T)$$

Gives a good basis of the subspace of "arboreal" invariants

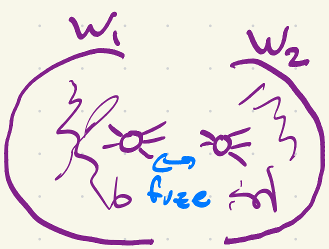
This comes from a more general "gluing" operation...

Thm (Cherny – P – Pfannerer 2025 + 2ε)

If W_1, W_2 are fully reduced r-HPGs with

$$\mathrm{trip}_\bullet(W_i) = \mathrm{prom}_\bullet(T_i),$$ then

is fully reduced with

$\mathrm{trip}_\bullet\Big($ 

$\Big) = \mathrm{prom}_\bullet\Big($

1	2	4
3	5	7
6	8	11
9	10	14
12	13	15

free

1	4
2	6
3	7
5	9
8	10

$\Big)$

So we can handle also
- few cycles
- isolated cycles
- 2-column regions

Thank you!