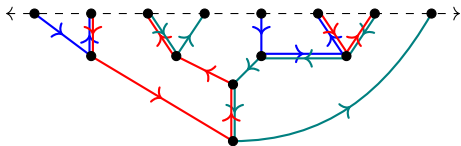


Quantum representations on webs

Julianna Tymoczko (Smith College)
with H. Russell (University of Richmond),
E. Hafken (Wake Forest), V. Lang (Michigan), O. Tashman,
L. Cowan (Smith), R. Eilfort (Smith), K. Seekamp (Smith),
and others



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This talk has three main goals:

- 1 Give context and history about the big picture.
- 2 Introduce webs and web vectors in quantum representation theory.
- 3 Provide many pictures and illustrative cases, but not technical details.

Representations of \mathfrak{sl}_n and $U(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_n)$ (vibes only)

The usual set-up of group representations considers group G acting on vector space V . For instance, take $G = SL_n(\mathbb{C})$ acting on \mathbb{C}^n .

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} = K_1 \vec{v}$$

Representations of \mathfrak{sl}_n and $U(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_n)$ (vibes only)

Upshot:

- E_1, E_2, \dots, E_{n-1} move the standard basis vectors x_1, x_2, \dots up one
- F_1, F_2, \dots, F_{n-1} move the standard basis vectors x_1, x_2, \dots down one
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The standard basis elements for the *fundamental* vector space $V_k = \bigwedge^k \mathbb{C}^n$ are:

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- for each subset $S = \{j_1 > \dots > j_k\}$
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E_i moves nonzero bits *one step right* if possible, else kills the vector:

$$\begin{array}{ccc} x_3 \wedge x_2 \xrightarrow{E_2} x_3 \wedge x_1 & \text{but} & x_3 \wedge x_2 \xrightarrow{E_3} x_2 \wedge x_2 = 0 \\ 0110 \xrightarrow{E_2} 0101 & \text{but} & 0110 \xrightarrow{E_3} \text{zero} \end{array}$$

Representations of \mathfrak{sl}_n and $U(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_n)$ (vibes only)

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 - ▶ Elementary operations of $U_q(\mathfrak{sl}_n)$ work the same as the triangular operators E_i and F_i
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- Both $U(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_n)$ act on exterior products and tensor products like before.
- We focus on tensor products of fundamental representations:

$$V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_m}$$

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This all leads to differences downstream and subtleties with duals $V_k \leftrightarrow V_k^*$.

$$\begin{array}{ll} x_1 \wedge x_3 \wedge x_2 & = x_3 \wedge x_2 \wedge x_1 & \text{in the ordinary exterior product} \\ x_1 \wedge_q x_3 \wedge_q x_2 & = (-q)^2 x_3 \wedge_q x_2 \wedge_q x_1 & \text{in the quantum exterior product} \end{array}$$

Categorification

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Something	Categorification	Map between them
nonnegative integers	vector spaces	$V \mapsto \dim(V)$
symmetric functions	S_n representations	$V_\lambda \mapsto s_\lambda$
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The point is to do this in a way that preserves more structure:

- monoid structure: add dimensions \longleftrightarrow direct sum of vector spaces
- ring structure: multiply symmetric functions \longleftrightarrow tensor product of representations

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A diagrammatic categorification has some collection of diagrams in the first column and some nice category in the second column.

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The \mathfrak{sl}_n spider category models the representation theory of \mathfrak{sl}_n and its associated quantum group using diagrams.

- The objects are representations of $U_q(\mathfrak{sl}_n)$ namely

$$V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_m}$$

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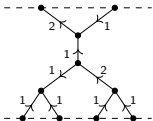
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Precise conventions for graph decorations differ but:

- Edge decorations give some information about the vector space V_j or V_j^* and interior vertices indicate some kind of compatibility.
- The web interior describes a specific homomorphism between representations.

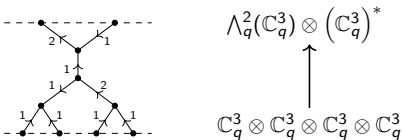
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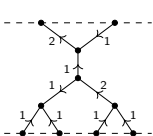
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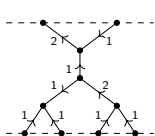
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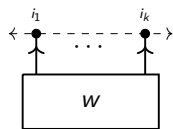
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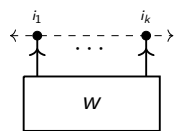
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The diagram shows a rectangular box labeled W . Above the box, there are two vertices labeled i_1 and i_k . A dashed horizontal line with arrows at both ends passes through these two vertices. Vertical lines connect i_1 and i_k to the top edge of the box W . Ellipses between i_1 and i_k indicate other vertices on the line.

$$\mathcal{U}_q(\mathfrak{sl}_n) \curvearrowright V_{i_1} \otimes V_{i_2} \cdots \otimes V_{i_k}$$
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Sometimes people refer to the $\mathcal{U}_q(\mathfrak{sl}_n)$ -invariant vector or the $U(\mathfrak{sl}_n)$ -invariant vector as a web.

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- Algebraic geometers: Springer fibers, generalized Schubert calculus (Fung 2003)

Computing with webs: web bases and web relations

The **web vector** associated to a web graph is the invariant vector lying inside some tensor product of copies of V_i or V_i^* . The vector space spanned by all \mathfrak{sl}_n web vectors is the **invariant space** for $\mathcal{U}_q(\mathfrak{sl}_n)$. A **web basis** is a set of webs that gives a basis for an invariant space.

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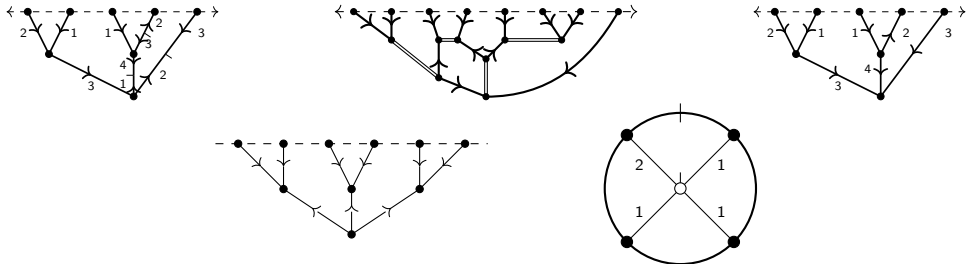
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When do web graphs represent the same web vector? **Web relations** are skein-theoretic equivalence relations that encode algebraic operations preserving an invariant vector.

\mathfrak{sl}_3 web relations:

$$\begin{aligned} \text{Loop with arrow} &= [3]_q \\ \text{Crossing of two arrows} &= -[2]_q \text{ (straight line)} \\ \text{Square web} &= \text{Two crossings} + \text{Two crossings} \end{aligned}$$

There are many different conventions for drawing webs and constructing web vectors.



Top Row: Cautis-Kamnitzer-Morrison, Kim, Fontaine,
Bottom Row: Kuperberg, Fraser-Lam-Le

Webs (incomplete history, please help add)

- $U_q(\mathfrak{sl}_2)$ webs are essentially Temperley-Lieb diagrams, or noncrossing matchings
- Kuperberg describes the general setup of spiders, and gives spiders for all rank 2 simple Lie algebras, including $U_q(\mathfrak{sl}_3)$
- Murakami, Ohtsuki, and Yamada describe how to decompose a knot diagram into a linear combination of webs
- Kim gives a definition of webs for $U_q(\mathfrak{sl}_4)$ with a set of relations he proves are necessary and conjectures are sufficient
- Morrison does the same for $U_q(\mathfrak{sl}_n)$
- Fontaine and Westbury each proposed bases for $U_q(\mathfrak{sl}_n)$ webs
- Cautis, Kamnitzer, and Morrison gave the first description of webs for $U_q(\mathfrak{sl}_n)$ that proved a complete set of relations
- Fraser, Lam, and Le use *plabic* graphs (short for planar bipartite) to describe $U(\mathfrak{sl}_n)$ invariant vectors, noting their proof of plabic web relations doesn't extend beyond $q = 1$
- Russell and T prove a complete set of relations for the web graphs Fontaine uses

Computing with webs

Good properties for a web basis:

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- *Decomposition*: Given a web vector, how do I write it in terms of basis vectors?
- *Rotation invariance*: Is the web basis preserved under operations like rotation of graphs?

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Web relations for \mathfrak{sl}_2 and \mathfrak{sl}_3 yield web bases called **reduced webs** with all “good properties.”

- The reduced web basis for \mathfrak{sl}_2 “is” noncrossing matchings (or Temperley-Lieb diagrams).
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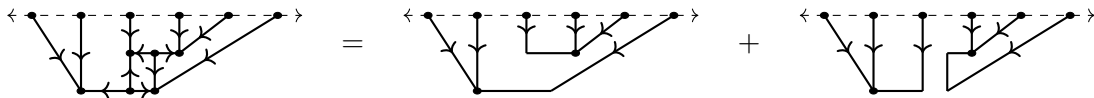
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Web relations when $n = 2$ and $n = 3$ always simplify the web graph. Edge labels can be omitted. Relations reduce the number of faces and interior vertices. Boundary depth of the web graph is a partial order preserved by web relations.



Computing with webs

- Westbury and Fontaine construct \mathfrak{sl}_n web bases that lack most of these properties.
- Gaetz-Pechenik-Pfannerer-Striker-Swanson recently constructed \mathfrak{sl}_4 web bases with these same three properties. The work is exciting but more complicated. It has been extended to two-column webs.

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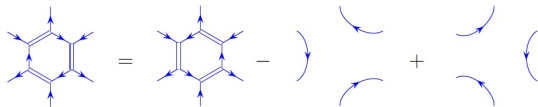
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- **Why are things harder when n gets bigger?**

Non-reducing \mathfrak{sl}_4 web relation:
(from Kim's 2003 thesis)



- Also the CKM construction of webs makes it easy to accidentally scale and hard to identify the coefficient of a web vector.

Cautis-Kamnitzer-Morrison (2014) have a complete setup for computing with webs. They give explicit formulas for web vectors and a complete set of local relations.

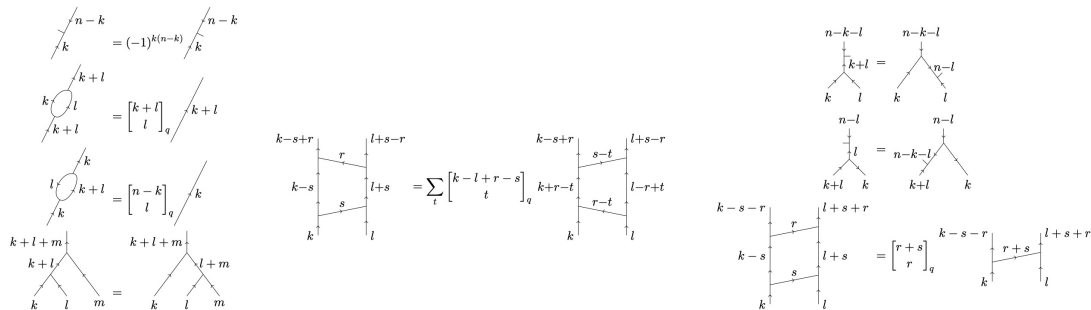


Figure: Web relations from 2014 CKM paper

Getting an invariant vector from a CKM web

To construct a web vector from the CKM web graph:

- 1 Morsify the graph.
- 2 Find all binary labelings.
- 3 Compute the local coefficient at each vertex and tag.
- 4 The web vector is the state sum over all binary labelings.

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- If you are a topologist, you don't mind Morsifying.
 - If you want a diagrammatic categorification that preserves information about duals $V \leftrightarrow V^*$ then you want something like tags.

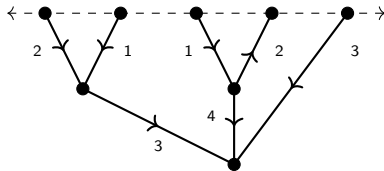
Fontaine¹ web conventions

An \mathfrak{sl}_n web is a plane graph with

- univalent boundary vertices along a top axis and
- trivalent internal vertices.
- Each edge is oriented and weighted with an element of $\{1, \dots, n-1\}$.
- At each trivalent vertex, the sum of incoming edge weights minus the sum of outgoing edge weights is divisible by n :

flow is preserved **mod** n at each vertex

An \mathfrak{sl}_5 web:



¹Insert anecdote here

Relations for Fontaine webs: strandings

Definition: Given a directed edge $u \mapsto v$ labeled k , a **(valid) stranding** of the edge is a choice of $0 < i_1 < i_2 < i_3 < \cdots < i_j < n$ with

$$i_j - i_{j-1} + i_{j-2} - i_{j-3} + \cdots + (-1)^{j-1} i_1 \in \{k, n - k\}$$

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The **strands** are denoted $\lambda_{i_1}, \dots, \lambda_{i_j}$ and drawn as colored directed paths on $u \mapsto v$ to distinguish them from each other and from the edge $u \mapsto v$.

- If the alternating sum is k then the strands directed *with* the edge $u \mapsto v$ are associated to *positive* coefficients (respectively against and negative).
- If the alternating sum is $n - k$ then these signs are reversed: strands directed *against* the edge are associated to *positive* coefficients (respectively with and negative).

Example:



Relations for Fontaine webs: strandings

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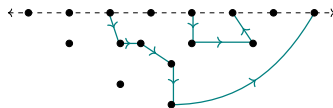
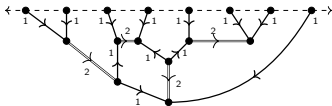
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- Each strand either starts and ends on the boundary, or forms a closed loop in the graph.
- Two strands with the same label do not intersect each other.

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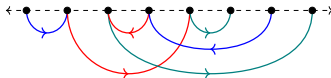
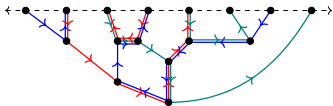
- There is a valid stranding on each edge.
- Each strand either starts and ends on the boundary, or forms a closed loop in the graph.
- Two strands with the same label do not intersect each other.

This means the set of all strands labeled λ_i form a **directed noncrossing matching** on a subset of the web's boundary vertices (possibly with closed, oriented loops in the interior of the web).



Relations for Fontaine webs: strandings

Together, strands give a **multicolored, directed noncrossing matching** that “saturates” each edge of the graph.



Strandings and web vectors: the boundary vector

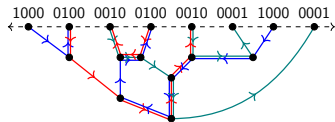
- Given a valid stranding of a web graph, label each boundary vertex v with the binary vector whose $i, i + 1$ entries are given by how λ_i runs along the edge incident to v :

$$\left\{ \begin{array}{ll} 01 & \text{if } \lambda_i \text{ is directed with the edge} \\ 10 & \text{if } \lambda_i \text{ is directed opposite the edge} \\ \text{equal} & \text{if } \lambda_i \text{ is not on the edge} \end{array} \right.$$

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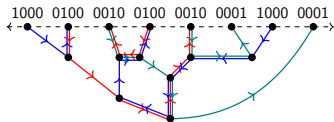


$$\mapsto x_1 \otimes x_2 \otimes x_3 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_1 \otimes x_4$$

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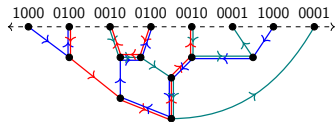
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Fact: This uniquely specifies a binary vector with the same weight as the edge and a basis vector in the desired quantum tensor space. This is the **boundary vector** of the stranding.

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- We can think of the λ_i as the binary vectors $\lambda_i = 1^i 0^{n-i}$. They form an alternate basis to $\mathbb{Z}^n / \mathbb{Z}\vec{1}$ than the standard basis (all zeroes except 1 in one entry).

Strandings and web vectors: the coefficient

Definition: An edge in a stranded web graph is an (i,j) flow edge if the edge has an odd number of strands λ_ℓ satisfying $i \leq \ell < j$.

Example: The $(i, i + 1)$ flow edges are exactly the edges with strand λ_i .

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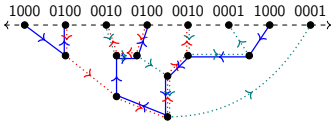
Definition: The (i,j) flows in a stranded web graph are the connected components of the subgraph formed by only considering (i,j) flow edges.

Fact: All (i,j) flows are directed paths in the web graph that either start and end at the boundary, or form a closed loop in the interior of the graph.

Strandings and web vectors: the coefficient

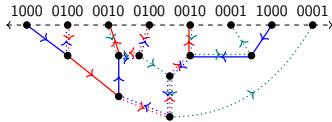
(1, 2) flows

1 clockwise,
1 counter



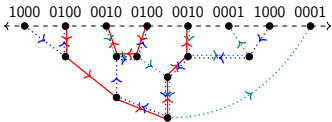
(1, 3) flows

1 clockwise,
1 counter



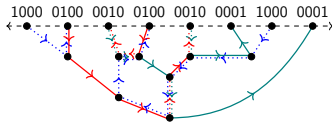
(2, 3) flows

1 clockwise
1 counter



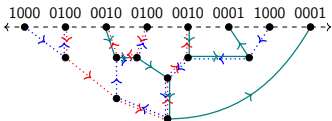
(2, 4) flows

0 clockwise
2 counter



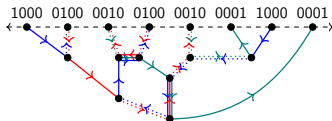
(3, 4) flows

0 clockwise
2 counter



(1, 4) flows

1 clockwise
1 counter



Strandings and web vectors

Theorem [Russell–T]: Fix a web graph G and let \mathcal{S} be the set of all valid strandings. Denote the boundary vector of the stranding $S \in \mathcal{S}$ by \vec{b}_S . Let $a(S)$ be the number of closed clockwise flows and $b(S)$ be the number of counterclockwise flows. Then the web vector corresponding to G is

$$\vec{v}_G = \sum_{S \in \mathcal{S}} (-q)^{a(S)-b(S)} \vec{b}_S$$

Previous example: $(-q)^{-8} (x_1 \otimes x_2 \otimes x_3 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_1 \otimes x_4)$

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Takeaway: Web graphs admit a global structure of strands. Web vectors can be read from web graphs by the number and direction of strands in each stranding. Relations are a natural consequence of the global structure of strands.

Some connections to other work

- **Khovanov-Kuperberg**: Use a state-sum over flows to generate \mathfrak{sl}_3 web vectors. Their flows are our $(1, 3)$ flows.

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- MOY, Morrison, and Fraser-Lam-Le also draw paths in their webs, sometimes colored paths. None of these are strands, either.

Applications

Using strandings, we prove:

- Nonvanishing/Positivity: If a web graph G has a valid stranding by S then the associated web vector w_G is nonvanishing in term x_S .
- Every web graph has a valid stranding/Base stranding: If G is a web graph, then there is a straightforward algorithm to produce a valid stranding of G by a particular S_0 .
- Basis Criteria: We provide a condition of Fontaine's for a set of webs to form a web basis, without relying on Fontaine's notion of "coherent webs."
- \mathfrak{sl}_n Web Bases: We produce a collection of web bases for \mathfrak{sl}_n webs, extending Fontaine's web basis.
- Complete Set of Relations: We prove a complete, concise set of relations.
- Geometry of Springer fibers: A family of non-reduced webs parametrizes the *entries* of top-dimensional Springer Schubert cells.
- Evacuation "is" Reflection of Web Graphs: The tableau operation of evacuation corresponds to reflecting a web graph.

Application: Webs in geometry

The flag variety is G/B

If $G = GL_n(\mathbb{C})$ and B is upper-triangular matrices then each flag is

- ... a coset gB
- ... a nested subspace $V_1 \subseteq V_2 \subseteq \dots \subseteq \mathbb{C}^n$

$$\left\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

- ... a matrix with zeros to the right and below a permutation

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Application: Webs in geometry

Each gB has a representative in exactly one of the following:

$$\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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These are the **Schubert cells** BwB/B . They are parametrized by permutation matrices w .

Application: Webs in geometry

Fix a linear operator $X : \mathbb{C}^n \rightarrow \mathbb{C}^n$

The Springer fiber of X consists of flags $gB = V_\bullet$ for which

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For example: The Springer fiber of $X = 0$ is the full flag variety.

Springer fibers

We focus on the case when X is nilpotent

- ... $X^m = 0$ for some m
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Jordan blocks partition n . The conjugacy class of X has a unique representative in Jordan canonical form with blocks arranged in nondecreasing order.

Example: If $\lambda(X)$ has 2 rows then the Springer fiber is a 2-row Springer fiber.

Theorem: (Springer's Representation)

- S_n acts naturally on the cohomology $H^*(\mathcal{S}_X)$
- The top-dimensional cohomology of $H^*(\mathcal{S}_X)$ is irreducible
- In fact $H^{top}(\mathcal{S}_X)$ is irreducible of type $\lambda(X)$
- The set $\{H^{top}(\mathcal{S}_\lambda)\}$ is precisely the collection of irreducible representations of S_n
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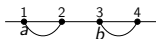
One consequence: The *number of components* of the Springer fiber $\lambda(X)$ equals the *dimension* of the irreducible representation of type $\lambda(X)$.

Concretely: If $\lambda(X)$ is a rectangle then the components of the Springer fiber are indexed by reduced webs.

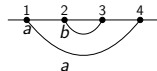
2-row Springer fibers and webs

- **Theorem: [Goldwasser, Nadeem, Sun, T]** Reduced webs parametrize the cells of 2-row Springer fibers.

$$\begin{pmatrix} a & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



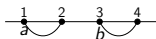
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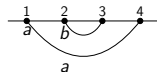
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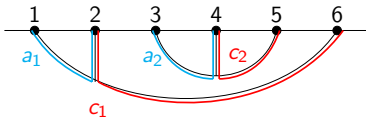
- **More Theorem:** In fact, closures of cells can be identified by cutting arcs appropriately.

3-row Springer fibers and strandings

We wanted to say the same for 3-row Springer fibers.

$$\begin{pmatrix} b_1 & c_1 & b_2 & c_2 & 1 & 0 \\ 0 & 0 & b_1 + \star & c_1 & 0 & 1 \\ \hline a_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\star = a_1 c_1 - a_2 c_1$$

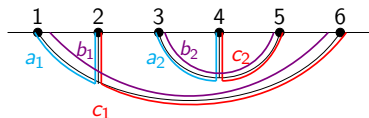


3-row Springer fibers and strandings

We realized that a purple noncrossing matching accounts for the remaining variables.

$$\begin{pmatrix} b_1 & c_1 & b_2 & c_2 & 1 & 0 \\ 0 & 0 & b_1 + \star & c_1 & 0 & 1 \\ \hline a_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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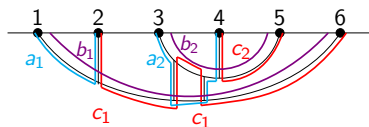
3-row Springer fibers and strandings

Emily Hafken, Veronica Lang, and Orit Tashman figured out that

- if we pull blue arcs below red arcs,
- then we can read \star from how the purple arcs cross the blue-red combos

$$\left(\begin{array}{cccccc} b_1 & c_1 & b_2 & c_2 & 1 & 0 \\ 0 & 0 & b_1 + \star & c_1 & 0 & 1 \\ \hline a_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$\star = a_1 c_1 - a_2 c_1$$



3-row Springer fibers and strandings

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- A similar result appears to hold for smaller-dimensional cells — but not all are affine.
- We believe that if this web is expressed as a linear combination of reduced webs, the reduced web corresponding to the component is the leading term.

Combinatorial connections: promotion and evacuation

- Promotion of standard Young tableaux is connected to rotation of *stranded* webs. (Peterson-Pylyavskyy-Rhoades, Cowen-Hafken-Seekamp-T)

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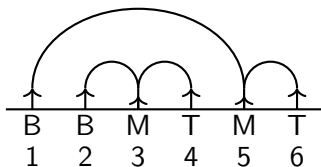
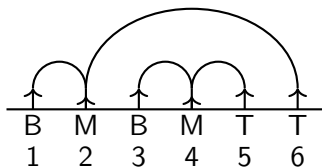
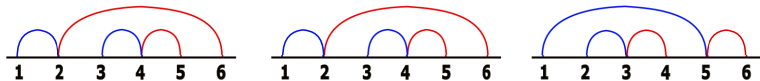
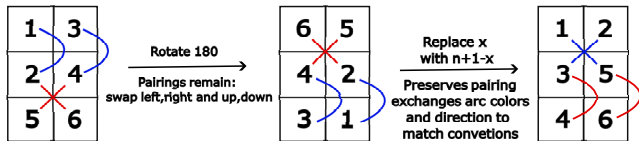
- Promotion of standard Young tableaux is connected to rotation of *stranded* webs. (Peterson-Pylyavskyy-Rhoades, Cowen-Hafken-Seekamp-T)
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Theorem [Cowen-Seekamp-Spaulling-T] If T is a rectangular SYT and w_T is its corresponding stranded web then the stranded web corresponding to the evacuation of T is the reflection of w_T .

Example of evacuation



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- Strands describe a global structure on web graphs that is more natural from a graph-theoretic point of view and more useful for computations.

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- Strands also seem to encode geometric information. $\neg_(\text{y})_/\neg$
- Good technical tools can greatly expand our understanding of a problem.

Thank you for your attention!

Preprints:

- Russell-T: arXiv:2510.12035
- Cowan-Eilfort-Seekamp-T: arXiv:2510.21989

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