Supercoinvariant Combinatorics

Brendon Rhoades (UCSD)

⌜joint w/ Robert Angarone, Patricia Commins, Trevor Karn,
Satoshi Murai, & Andy Wilson⌟

OUTLINE

① CLASSICAL COINVARIANT RING R_n

② DIAGONAL COINVARIANT RING DR_n

③ SUPERSPACE COINVARIANT RING SR_n

④ THE FUTURE...

CLASSICAL COINVARIANTS

$\mathfrak{S}_n \circlearrowright \mathbb{Q}[x_1, \ldots, x_n] \qquad w \cdot x_i = x_{w(i)}$

COINVARIANT IDEAL: $I_n = \left(\mathbb{Q}[\underline{x}_n]^{\mathfrak{S}_n}_{+} \right)$

COINVARIANT RING: $R_n = \mathbb{Q}[\underline{x}_n] / I_n \qquad \ulcorner \text{graded } \mathfrak{S}_n\text{-module} \lrcorner$

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COINVARIANT RING: $R_n = \mathbb{Q}[\underline{x}_n]/I_n$ ⌜graded $\mathfrak{S}_n$-module⌟

THM (CHEVALLEY) $R_n \cong \mathbb{Q}[\mathfrak{S}_n]$ as ungraded $\mathfrak{S}_n$-modules

R_n and $\mathfrak{S}_n$

CHEVALLEY: $R_n \cong \mathbb{Q}[\mathfrak{S}_n]$ as ungraded $\mathfrak{S}_n$-modules

E. ARTIN: $A_n = \{x_1^{a_1} \cdots x_n^{a_n} : a_i < i\}$ descends to a **BASIS** of R_n. $\ulcorner \Rightarrow \mathrm{Hilb}(R_n; q) = [n]!_q \lrcorner$

LUSZTIG-STANLEY: $\mathrm{grFrob}(R_n; q) = \sum\limits_{T \in SYT(n)} q^{\mathrm{maj}\, T} s_{\mathrm{shape}\, T}$.

INVERSE SYSTEMS ⌜HARMONIC SPACES⌟

* If $f = f(x_1,\dots,x_n) \in \mathbb{Q}[\underline{x}_n]$, have

$$\partial f : \mathbb{Q}[\underline{x}_n] \longrightarrow \mathbb{Q}[\underline{x}_n] \qquad \partial f = f\left(\partial/\partial x_1, \dots, \partial/\partial x_n\right)$$

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* **MODULE STRUCTURE** $\odot : \mathbb{Q}[\underline{x}_n] \times \mathbb{Q}[\underline{x}_n] \to \mathbb{Q}[\underline{x}_n]$

$$f \odot g = \partial f(g).$$

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⌜graded subspace of $\mathbb{Q}[\underline{x}_n]$⌟

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$$\mathbb{Q}[\underline{x}_n] = I \oplus I^{\perp} \rightsquigarrow \underset{\substack{\nearrow \\ \text{COSETS}}}{\mathbb{Q}[\underline{x}_n]/I} \cong \underset{\substack{\nwarrow \\ \text{NO COSETS}}}{I^{\perp}}.$$

STEINBERG'S THM

$$I_n = \left(\mathbb{Q}[\underline{x}_n]^{\mathfrak{S}_n}_+\right) = \text{defining ideal of } R_n$$

$$\delta_n = \prod_{1\le i<j\le n} (x_i - x_j)$$ VANDERMONDE

STEINBERG: $I_n^\perp$ is the SMALLEST LINEAR SUBSPACE OF $\mathbb{Q}[\underline{x}_n]$ ...

- containing δ_n,
- closed under $\partial/\partial x_i$ $i=1,\ldots,n$

OTHER TYPES

W : irreducible Weyl group, rank n

$W \curvearrowright V$ reflection rep'n $\rightsquigarrow$ $W \curvearrowright S = \mathbb{Q}[V]$

CHEVALLEY : $\cdot\ S^W = \mathbb{Q}[\underbrace{f_1, \dots, f_n}_{\text{algebraically indep}}]$

$\cdot\ R_W := S/(S^W_+) \cong \mathbb{Q}[W]$

BOREL : $H^\bullet(G/B) \cong R_W$.

STEINBERG : $(S^W_+)^\perp = \odot$-submod gen'd by $\prod_{\alpha \in \Phi^+(W)} \alpha$.

DIAGONAL COINVARIANTS

$\mathfrak{S}_n \circlearrowright \mathbb{Q}[\underline{x}_n, \underline{y}_n]$ $\quad w \cdot x_i = x_{w(i)} \quad w \cdot y_i = y_{w(i)}$

GARSIA-HAIMAN : $DI_n := \left(\mathbb{Q}[\underline{x}_n, \underline{y}_n]^{\mathfrak{S}_n}_{+}\right)$, $DR_n := \mathbb{Q}[\underline{x}_n, \underline{y}_n] / DI_n$.

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HAIMAN : $DR_n \underset{\mathfrak{S}_n}{\cong} \mathbb{Q}[\mathrm{Park}_n] \otimes \mathrm{sgn}$

$$\mathrm{Park}_3 = \left\{ \begin{array}{l} 123,\ 213,\ 132,\ 231,\ 312,\ 321, \\ 122,\ 212,\ 221, \\ 113,\ 131,\ 311, \\ 112,\ 121,\ 211, \\ 111 \end{array} \right\}$$

DIAGONAL COINVARIANTS

$$\mathfrak{S}_n \circlearrowright \mathbb{Q}[\underline{x}_n, \underline{y}_n] \qquad \omega \cdot x_i = x_{\omega(i)} \qquad \omega \cdot y_i = y_{\omega(i)}$$

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CARLSSON-OBLOMKOV : MONOMIAL BASIS of DR_n.

OPERATOR THM

HAIMAN $DI_n^{\perp}$ = SMALLEST LINEAR SUBSPACE OF $\mathbb{Q}[\underline{x}_n, \underline{y}_n]$ ...

- containing $\delta_n = \prod_{1 \le i < j \le n} (x_i - x_j)$

- closed under $\partial/\partial x_i$, $\partial/\partial y_i$ $(i = 1, 2, \dots, n)$

- closed under **POLARIZATION OPERATORS**

$$\sum_{i=1}^{n} \frac{\partial^{j}}{\partial x_i^{j}} y_i \qquad \ulcorner j = 1, 2, \dots, n-1 \lrcorner$$

OTHER TYPES?

irred. Weyl gp $W \circlearrowright V$ refl'n rep $\rightsquigarrow$ $W \circlearrowright \mathbb{Q}[V \oplus V^*]$

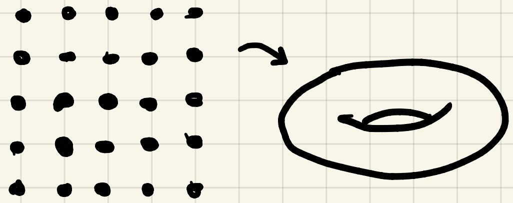
Q : ROOT LATTICE

h: COXETER #

OTHER TYPES?

$$\underset{\text{irred. Weyl gp}}{W} \circlearrowright V_{\text{refl'n rep}} \quad \rightsquigarrow \quad W \circlearrowright \mathbb{Q}[V \oplus V^*]$$

Q : ROOT LATTICE $\qquad h$: COXETER #



GORDON: There exists a W-equivariant surjection

$$DR_W := \mathbb{Q}[V \oplus V^*] \big/ \left(\mathbb{Q}[V \oplus V^*]^W_+\right) \twoheadrightarrow \underbrace{Q/(h+1)Q}_{\text{finite torus}} \otimes \det$$

OTHER TYPES?

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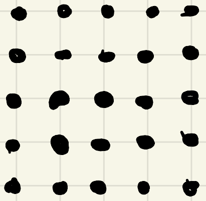

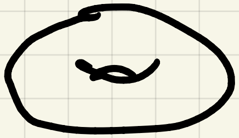

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⌜ Pf uses a DEFORMATION of $\mathbb{Q}[V \oplus V^*] * W$;

the RATIONAL CHEREDNIK ALGEBRA $\mathcal{H}_c$ ⌟

SUPERSPACE

$$\Omega_n := \mathbb{Q}[x_1 \_ x_n] \otimes \Lambda\{\theta_1, \ldots, \theta_n\}$$

$$x_i x_j = x_j x_i \qquad x_i \theta_j = \theta_j x_i \qquad \theta_i \theta_j = -\theta_j \theta_i$$

* DIFFERENTIAL FORMS ON n-space

* SUPERSYMMETRY $x_i \longleftrightarrow$ "boson"

$\theta_i \longleftrightarrow$ "fermion"

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EULER OPERATOR: $d : \Omega_n \longrightarrow \Omega_n \qquad df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \theta_i$

$$\mathfrak{S}_n \circlearrowright \Omega_n \qquad w \cdot x_i = x_{w(i)} \qquad w \cdot \theta_i = \theta_{w(i)}$$

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$$\underbrace{\text{N. BERGERON, COLMENAREJO, LI, MACHACEK, SULZGRUBER, ZABROCKI}}_{\text{“FIELDS GROUP”}}$$

$$SI_n := \left(\Omega_{n,+}^{\mathfrak{S}_n}\right) \qquad SR_n := \Omega_n / SI_n$$

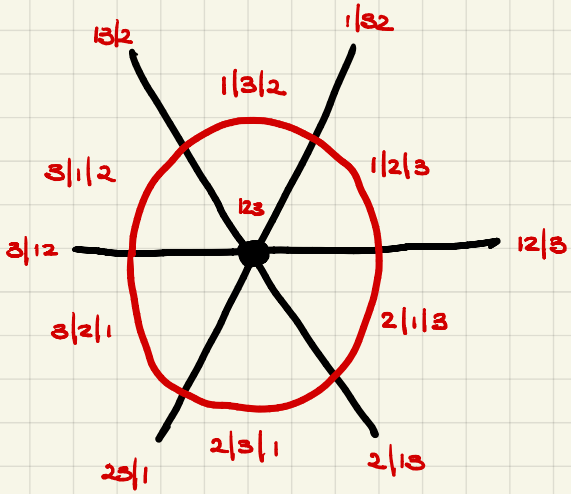
ORDERED SET PARTITIONS

$$\mathcal{OP}_n := \{\text{ordered set partitions of } [n]\}$$

$$(4 \mid 1\,3\,5 \mid 2\,6) \in \mathcal{OP}_6$$

$$\# \mathcal{OP}_n = \sum_{k \geq 0} k! \cdot \underbrace{\mathrm{Stir}(n,k)}_{\text{Stirling \# of the } 2^{\text{ND}} \text{ KIND}}$$

$\mathcal{OP}_n \longleftrightarrow$ faces in $\mathfrak{S}_n$-COXETER COMPLEX



FIELDS CONJECTURES

① $\dim SR_n = \# \mathcal{OP}_n$

② $$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=0}^{n} z^{n-k} [k]!_q \, \mathrm{Stir}_q(n,k)$$

③ $$SR_n \underset{\mathfrak{S}_n}{\cong} \mathcal{OP}_n \otimes \mathrm{sign}$$

④ $$\mathrm{grFrob}(SR_n; q, z) = \sum_{k=0}^{n} z^{n-k} \left[\sum_{T \in SYT(n)} q^{\mathrm{maj}\, T + \binom{n-k}{2} - (n-k)\,\mathrm{des}\, T} s_{\lambda(T)} \right]$$

FIELDS CONJECTURES UPDATES

① $\dim SR_n = \# \mathcal{OP}_n$ THM (R-WILSON)

② $$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=0}^{n} z^{n-k} [k]!_q \, \mathrm{Stir}_q(n,k)$$

THM [R-WILSON]

③ $$SR_n \underset{G_n}{\cong} \mathcal{OP}_n \otimes \mathrm{sign}$$

THM [MURAI-R-WILSON]

④ $$\mathrm{grFrob}(SR_n; q, z) = \sum_{k=0}^{n} z^{n-k} \left[\sum_{T \in SYT(n)} q^{\mathrm{maj}\,T + \binom{n-k}{2} - (n-k)\,\mathrm{des}\,T} s_{\lambda(T)} \right]$$

THM [MURAI-R-WILSON]

FERMIONIC CALCULUS

For $i = 1, \dots, n$ have $\partial/\partial\theta_i : \Omega_n \longrightarrow \Omega_n$

$\partial/\partial\theta_1 \, (\theta_1\theta_3\theta_4) = \theta_3\theta_4$ $\qquad$ $\partial/\partial\theta_3 \, (\theta_1\theta_3\theta_4) = -\theta_1\theta_4$

$\partial/\partial\theta_2 \, (\theta_1\theta_3\theta_4) = 0$ $\qquad$ $\partial/\partial\theta_4 \, (\theta_1\theta_3\theta_4) = \theta_1\theta_3$

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$$\partial/\partial x_i \, \partial/\partial x_j = \partial/\partial x_j \, \partial/\partial x_i \qquad \partial/\partial x_i \, \partial/\partial\theta_j = \partial/\partial\theta_j \, \partial/\partial x_i \qquad \partial/\partial\theta_i \, \partial/\partial\theta_j = -\, \partial/\partial\theta_j \, \partial/\partial\theta_i$$

$$f = f(x_1, \dots, x_n, \theta_1, \dots, \theta_n) \in \Omega_n \rightsquigarrow$$

$$\partial f = f(\partial/\partial x_1, -, \partial/\partial x_n, \partial/\partial\theta_1, -, \partial/\partial\theta_n) : \Omega_n \longrightarrow \Omega_n .$$

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$I \subseteq \Omega_n$ $\rightsquigarrow$ $I^\perp = \{ g \in \Omega_n : f \circ g = \partial f(g) = 0 \ \forall f \in I \}$

BIHOMOGENEOUS IDEAL

Ω_n OPERATOR THEOREM

"HIGHER EULER OPERATOR" $d_j : \Omega_n \longrightarrow \Omega_n$ $\quad d_j f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i^j} \theta_i$

Ω_n OPERATOR THEOREM

"HIGHER EULER OPERATOR" $d_j : \Omega_n \to \Omega_n$ $\qquad d_j f = \sum_{i=1}^{n} \frac{\partial^j f}{\partial x_i^j} \theta_i$

THM [R.-WILSON] SI_n^+ is the smallest linear subspace of Ω_n ...

① containing $\delta_n = \prod_{1 \le i < j \le n} (x_i - x_j)$

② closed under $\partial / \partial x_i$ $\quad i = 1, \dots, n$

③ closed under $d_1, d_2, \dots, d_{n-1}$.

⌜CONJ. by FIELDS GP + SWANSON/WALLACH⌟

$$\begin{tikzcd}
\text{Supercommutative Algebra} \arrow[d, squiggly] \\
\text{Commutative Algebra}
\end{tikzcd}$$

BASIS TRANSFER

For $J \subseteq [n]$, define $f_J \in \mathbb{Q}[\underline{x}_n]$ by

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RECALL If $f \in \mathbb{Q}[\underline{x}_n]$, $I \subseteq \mathbb{Q}[\underline{x}_n]$ an ideal,

COLON IDEAL $(I : f) = \{ g \in \mathbb{Q}[\underline{x}_n] : fg \in I \}$

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- * $I_n \subseteq \mathbb{Q}[\underline{x}_n]$ CLASSICAL COINVARIANT IDEAL
- * $\mathcal{B}_J \subseteq \mathbb{Q}[\underline{x}_n]$ homogeneous polys $(J \subseteq [n])$

FACT (R.-WILSON)

$\mathcal{B}_J$ a basis of $\mathbb{Q}[\underline{x}_n] / (I_n : f_J)$ $\forall J \subseteq [n]$ $\Rightarrow$ $\bigsqcup_J \mathcal{B}_J \cdot \theta_J$ a basis of SR_n.

HILBERT SERIES

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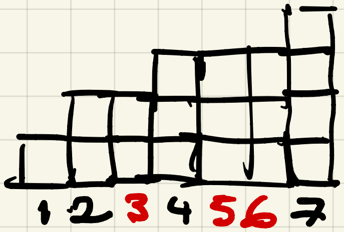
$J \subseteq [n] \rightsquigarrow st(J) = (st(J)_1, \dots, st(J)_n)$ "J-STAIRCASE"

$n = 7$ $J = \{3,5,6\}$



$st(J) = (1,2,2,3,3,3,4)$

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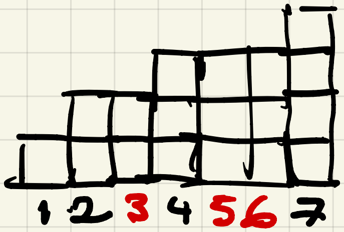

$st(J) = (1,2,2,3,3,3,4)$

THM (R-WILSON) Let $J \subseteq [n]$. We have $1 \in (I_n : f_J) \iff 1 \in J$.

If $1 \notin J$ then $(I_n : f_J) = (\underbrace{p_{J,1}, \ldots, p_{J,n}}_{\text{homog. regular sequence}})$, $\deg p_{J,i} = st(J)_i$.

So $\mathrm{Hilb}(\mathbb{Q}[\underline{x}_n]/(I_n : f_J); q) = \prod_{i=1}^{n} [st(J)_i]_q$.

ARTIN BASIS

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$\mathcal{B}_J$ a basis of $\mathbb{Q}[\underline{x}_n]/(I_n : f_J)$ $\forall\ J \subseteq [n]$ $\implies$ $\bigsqcup_J \mathcal{B}_J \cdot \theta_J$ a basis of SR_n.

$A_n(J) = \{ x_1^{a_1} \cdots x_n^{a_n} : a_i < st(J)_i \}$ "J-ARTIN MONOMIALS"

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THM (ANGARONE-COMMINS-KARN-MURAI-R)

$\mathcal{A}_n(J)$ is a basis of $\mathbb{Q}[\underline{x}_n]/(I_n : f_J)$ $\forall$ $J \subseteq [n]$ $\Longrightarrow$ $\bigsqcup_J \mathcal{A}_n(J)\, \theta_J$ is a basis of SR_n.

⌜**CONJ**: SAGAN-SWANSON⌟

THM (ANGARONE - COMMINS - KARN - MURAI - R)

$\mathcal{A}_n(J)$ is a basis of $\mathbb{Q}[\underline{x}_n]/(I_n : f_J)$ $\forall$ $J \subseteq [n]$ $\Rightarrow$ $\bigsqcup_J \mathcal{A}_n(J)\,\theta_J$ is a basis of SR_n.

Issue $\mathbb{Q}[\underline{x}_n]/(I_n : f_J)$ has inscrutable Gröbner theory!

SOLUTION HYPERPLANE ARRANGEMENTS!

HYPERPLANE ARRANGEMENTS

$\mathbb{F}$: field of char. 0 $\quad S = \mathbb{F}[x_1, \dots, x_n]$

$\mathcal{A}$: finite, linear hyperplane arr. in $\mathbb{F}^n$.

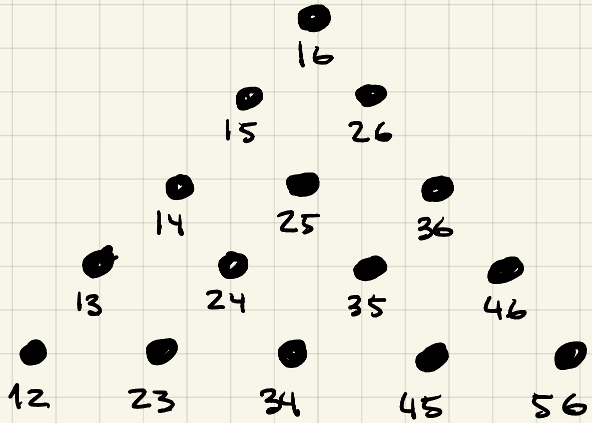
H hyperplane $\rightsquigarrow$ $\alpha_H \in S_1$ linear form ⌜$\ker \alpha_H = H$⌟

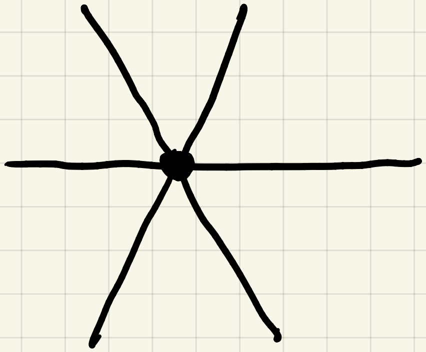
DELETION $\quad \mathcal{A} - H$

RESTRICTION $\quad \mathcal{A}|_H = \{ H' \cap H : H' \in \mathcal{A}, H' \neq H \}$

MOST IMPORTANT EXAMPLE

Φ_n BRAID ARRANGEMENT $\{x_i - x_j : 1 \leq i < j \leq n\}$





DERIVATION MODULES

$$S = \mathbb{F}[x_1, \ldots, x_n]$$

$$\mathrm{Der}(S) = \bigoplus_{i=1}^{n} S\,\partial_i = \text{VECTOR FIELDS ON } \mathbb{F}^n.$$

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DEF Let $\mathcal{A}$ be an arrangement in $\mathbb{F}^n$. The **DERIVATION MODULE** is

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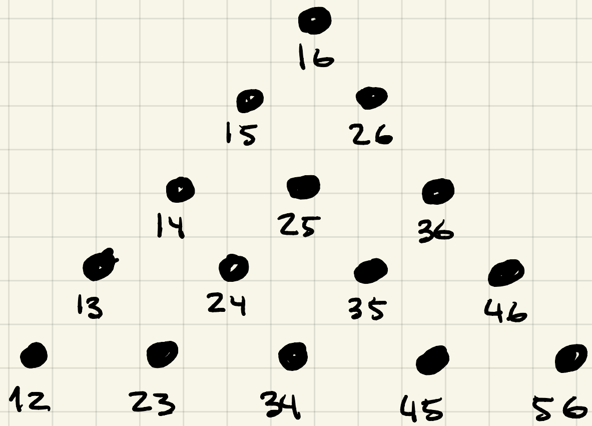
$=$ VECTOR FIELDS PARALLEL TO $\mathcal{A}$

⌜ $\mathrm{Der}(\mathcal{A})$ is an S-module. ⌟

FREE ARRANGEMENTS

Def $\mathcal{A}$ is **FREE** if $\mathrm{Der}(\mathcal{A})$ is a free S-module.



Ex BRAID ARRANGEMENT IS FREE,

S-module basis $\{\theta_1, \theta_2, \ldots, \theta_n\}$

$$\theta_j = x_1^{j-1}\partial_1 + \cdots + x_n^{j-1}\partial_n$$

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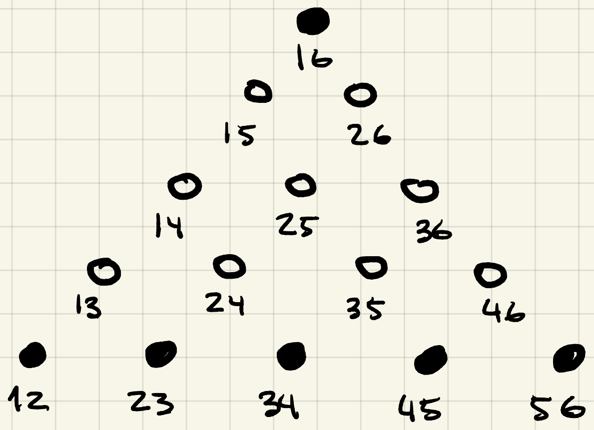

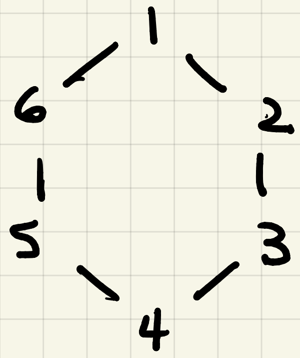

NOT FREE

STANLEY: If Γ is a graph on $[n]$,

$$\mathcal{A}_\Gamma \text{ is free} \iff \Gamma \text{ is chordal.}$$

↑ graphical arrangement

FREE ARRANGEMENTS

Def $\mathcal{A}$ is **FREE** if $\mathrm{Der}(\mathcal{A})$ is a free S-module.

TERAO'S CONJ. If $\mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}')$ and $\mathcal{A}$ is free, so is $\mathcal{A}'$.

Def If $\mathcal{A}$ is free, $\overbrace{\{\psi_1, \dots, \psi_n\}}^{\text{homog.}}$ an S-basis of $\mathrm{Der}\,\mathcal{A}$, the **exponents** of $\mathcal{A}$ are $\deg \psi_1, \_, \deg \psi_n$.

SOLOMON-TERAO ALGEBRAS

For $1 \leq i \leq n$, $c(\partial_i) \in S$ homogeneous polynomial of degree d

$$\begin{aligned} c : \operatorname{Der}(S) &\longrightarrow S \\ \partial_i &\longmapsto c(\partial_i) \end{aligned}$$

S-module homomorphism

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Def [ABE-MAENO-MURAI-NUMATA]

If $\mathcal{A}$ is an arrangement, the Solomon-Terao ideal

is $c_{\mathcal{A}} := c(\mathrm{Der}(\mathcal{A})) \subseteq S$. ⌜HOMOGENEOUS⌟

$\Longrightarrow$ $S/c_{\mathcal{A}}$ graded ring "Solomon-Terao algebra"

PROPERTIES OF ST ALGEBRAS

- Suppose $c : \operatorname{Der} S \to S$ w/ $\deg\big(c(\partial_i)\big) = d$.

- Suppose $\mathcal{A}$ is <u>free</u> where $\operatorname{Der}(\mathcal{A})$ has S-basis $\{\psi_1, \dots, \psi_n\}$, $\deg \psi_i = e_i$.

Then $c_{\mathcal{A}} = \big(c(\psi_1), \dots, c(\psi_n)\big) \subseteq S$ satisfies

① $c_{\mathcal{A}} = S$,

② $S/c_{\mathcal{A}}$ has positive Krull dimension, or

③ $S/c_{\mathcal{A}}$ is a COMPLETE INTERSECTION

w/ $\operatorname{Hilb}(S/c_{\mathcal{A}}; q) = \prod_{i=1}^{n} [d+e_i]_q$.

PROPERTIES OF ST ALGEBRAS

$c : \mathrm{Der}(S) \to S$ S-module homomorphism

FREE EXCISION Suppose $\mathcal{B} \subseteq \mathcal{A}$ are free arrangements with $c_{\mathcal{B}} \neq S$ and $\dim S/c_{\mathcal{A}} < \infty$.

If $f = \prod_{H \in \mathcal{A}-\mathcal{B}} \alpha_H$ then $c_{\mathcal{B}} = (c_{\mathcal{A}} : f)$ as ideals in S.

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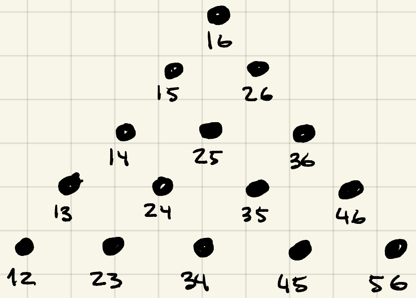
FACT If $\mathcal{A}, \mathcal{A}-H, \mathcal{A}|_H$ are ALL FREE (+ SOMETHING ELSE) have a SHORT EXACT SEQUENCE

$$0 \longrightarrow S/c_{\mathcal{A}-H} \xrightarrow{\times \alpha_H} S/c_{\mathcal{A}} \longrightarrow S/c_{\mathcal{A}|_H} \longrightarrow 0$$

BRAID ST Algebra

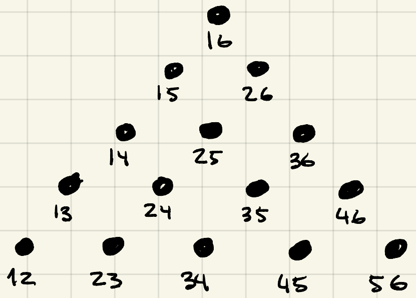


$$\Phi_n = \{x_i - x_j = 0 : 1 \leq i < j \leq n\}$$

Basis of $\mathrm{Der}(\Phi_n)$: $\left\{ \sum_{i=1}^{n} x_i^j \partial_i : j = 1, \dots, n \right\}$

$a : \mathrm{Der}(S) \to S$ $\qquad a : \partial_i \longmapsto x_i$

BRAID ST ALGEBRAS



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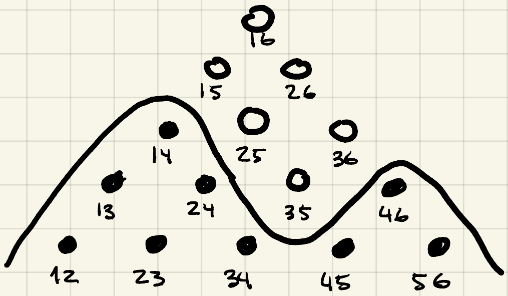
Basis of $\mathrm{Der}(\Phi_n)$: $\left\{ \sum_{i=1}^{n} x_i^j \partial_i : j = 1, \dots, n \right\}$

$\mathfrak{a} : \mathrm{Der}(S) \longrightarrow S \qquad \mathfrak{a} : \partial_i \longmapsto x_i$

$\Rightarrow \quad \mathfrak{a}_{\Phi_n} = I_n =$ COINVARIANT IDEAL

$S/\mathfrak{a}_{\Phi_n} = R_n =$ COINVARIANT RING

IDEAL ST ALGEBRAS



$\mathcal{A}_{\mathcal{L}}$ = IDEAL ARRANGEMENT

$\mathfrak{a} : \mathrm{Der}(S) \longrightarrow S \qquad \mathfrak{a} : \partial_i \longmapsto x_i$

THM (ABE-HORIGUCHI-MASUDA-MURAI-SATO)

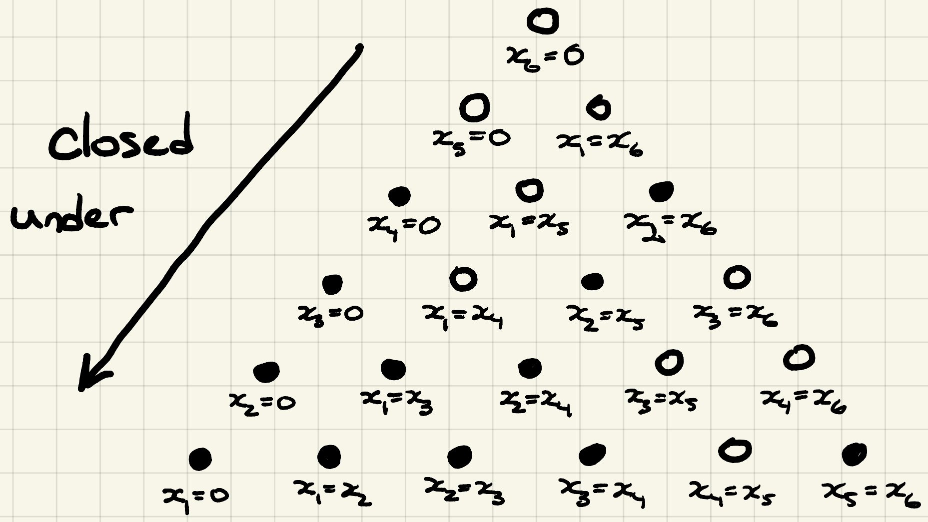
If $I = \mathfrak{a}_{\mathcal{A}_{\mathcal{L}}}$ then $H^\bullet(X_{\mathcal{L}}) = S/_I$

where

$X_{\mathcal{L}}$ = REGULAR NILPOTENT HESSENBERG VARIETY associated to $\mathcal{L}$.

SOUTHWEST ST ALGEBRAS

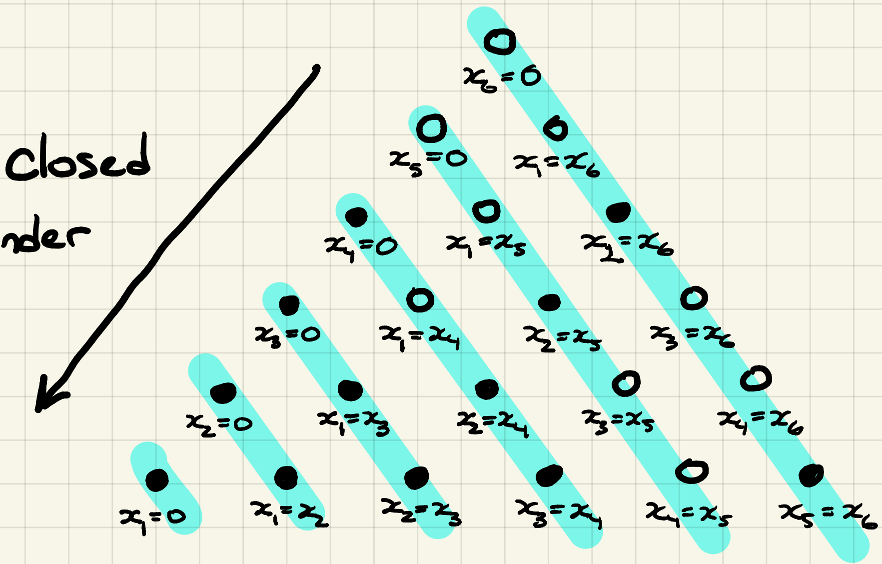
Ex:



SOUTHWEST ST ALGEBRAS

Ex:

closed under



$h(A) = (1, 2, 3, 3, 1, 2)$ "h-sequence"

SOUTHWEST ST ALGEBRAS

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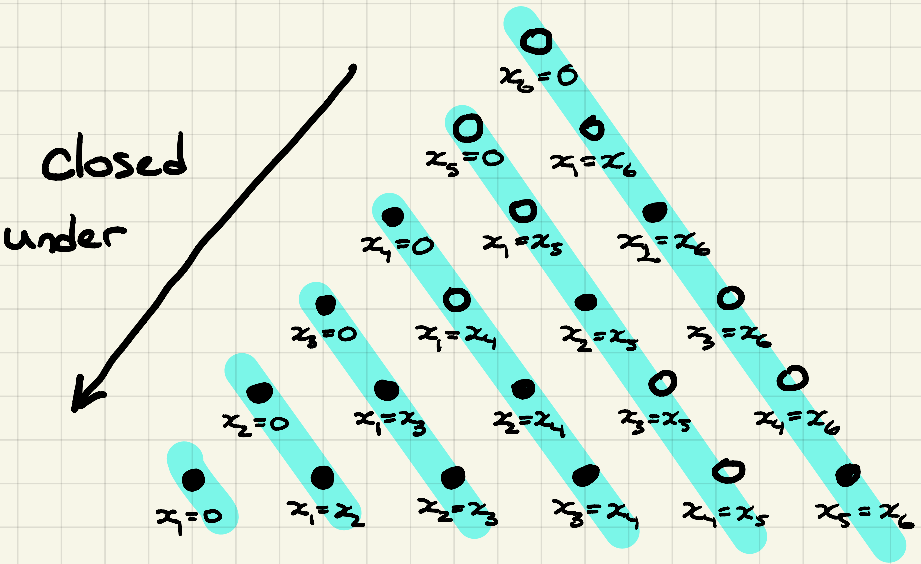

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FACT Every SW arrangement $\mathcal{A}$ is free with exponents $h(\mathcal{A})$.

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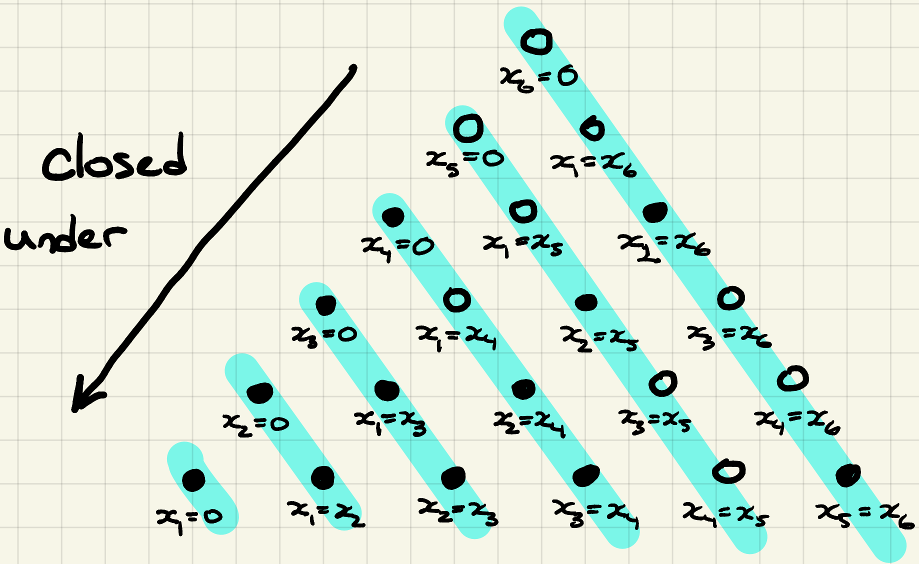


$h(\mathcal{A}) = (1, 2, 3, 3, 1, 2)$ "h-sequence"

FACT Every SW arrangement $\mathcal{A}$ is free with exponents $h(\mathcal{A})$.

FACT Let $i : \mathrm{Der}(S) \to S$ be $i : \partial_i \mapsto 1$ for $i = 1, \dots, n$.

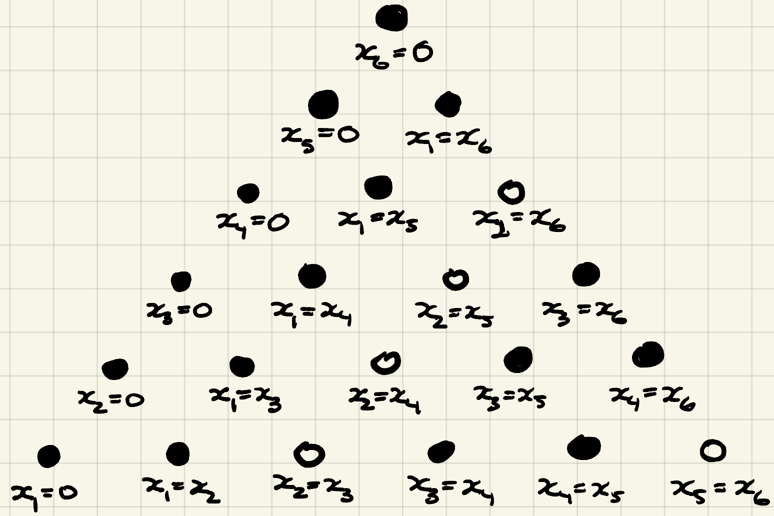
Then if $\mathcal{A}$ is SW, $S/i_{\mathcal{A}}$ has basis

$$\{ x_1^{a_1} \cdots x_n^{a_n} : a_i < h_i(\mathcal{A}) \}.$$

⌜Exact sequences of $\mathrm{Der}(\mathcal{A})$
$\Updownarrow$
Exact sequences of $S/i_{\mathcal{A}}$ ⌟

SOUTHWEST ARRANGEMENTS

If $J \subseteq \{1, -, n\}$, have a SW arrangement $\mathcal{A}_J \ldots$



$n=6, \quad J=\{2, 5\}$

$\{x_1^{a_1} \cdots x_n^{a_n} : a_i < h_i(\mathcal{A}_J)\}$ basis of $ST(\mathcal{A}_J, i)$

$\Downarrow$ ← colon ideal exact sequence

$\{x_1^{a_1} - x_n^{a_n} : a_i < st_i(J)\}$ linearly indep. in $S/(I : f_J)$.

BASIS $\Longrightarrow$ TRANSFER

THM $\bigsqcup_J \mathcal{A}_n(J)\, \Theta_J$ is a basis of SR_n.

[ACKMR]

$\mathfrak{S}_n$ - module structure ⌜w/ Murai, Wilson⌟

KEY If $\mu \vdash n$, let $\varepsilon_\mu = \sum_{w \in \mathfrak{S}_\mu} (\operatorname{sgn} w) \cdot w \in \mathbb{Q}[\mathfrak{S}_n]$

If V is an $\mathfrak{S}_n$-module,

$$\boxed{\dim(\varepsilon_\mu V) = \langle \operatorname{Frob} V, e_\mu \rangle}$$

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Upper bound on $\varepsilon_\mu SR_n$ from $\bigsqcup_J \mathcal{A}_n(J) \Theta_J$

LOWER BOUND trickier; uses $\varepsilon_\mu SI_n^\perp$.

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Upper bound on $\varepsilon_\mu SR_n$ from $\bigsqcup_J \Delta_n(J)\,\Theta_J$

Lower bound trickier; uses $\varepsilon_\mu SI_n^\perp$.

$\Longrightarrow$ [Murai-R-Wilson] ① $SR_n \cong_{\mathfrak{S}_n} \mathcal{O}P_n \otimes \operatorname{sgn}$

② $\operatorname{grFrob}(SR_n; q, z) = \sum_{k=0}^{n} z^{n-k} \Delta'_{e_{k-1}} e_n \Big|_{t \to 0}$.

GEOMETRY?

$G = SL_n\mathbb{C}$ $\quad B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \subseteq G$

$\tilde{G} = G\times B/\sim \qquad (g,b) \sim (gb_0^{-1}, b_0 b b_0^{-1}).$

$p: \tilde{G} \to G \qquad [g,b] \longmapsto gbg^{-1}$

⌜GROTHENDIECK-SPRINGER RESOLUTION⌟

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GROTHENDIECK-SPRINGER RESOLUTION

$$p^*: H^\bullet(G) \to H^\bullet(\tilde{G}) \longleftarrow R_n \otimes \Lambda\{\overbrace{\theta_1 - \theta_n}^{\overset{H^1}{\cup}}\}/(\theta_1 + - + \theta_n)$$

$J =$ ideal in $H^\bullet(\tilde{G})$ gen'd by $p^*(H^+(G))$.

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⌜GROTHENDIECK-SPRINGER RESOLUTION⌟

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J = ideal in $H^\circ(\tilde{G})$ gen'd by $p^*(H^+(G))$.

THM [MURAI-R-WILSON; REEDER]

$$SR_n \cong H^*(\tilde{G})/J.$$

FIBERS

$$\begin{array}{cc} & \begin{array}{ccc}\alpha_1 & \alpha_2 & \cdots\end{array} \\ \begin{array}{c}\alpha_1\\ \alpha_2\\ \vdots\end{array} & \left[\begin{array}{c|c|c} * & & \\ \hline & * & \\ \hline & & \ddots \end{array}\right]\end{array}$$

$p : \widetilde{G} \longrightarrow G$

$\alpha \vDash n \rightsquigarrow L_\alpha \subseteq G$ LEVI SUBGP.

$t_\alpha \in L_\alpha$ generic central element

FIBERS

$$\begin{array}{c|c|c|c} & \alpha_1 & \alpha_2 & \cdots \\ \hline \alpha_1 & * & & \\ \hline \alpha_2 & & * & \\ \hline \vdots & & & \ddots \end{array}$$

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THM (MURAI-R-WILSON) There is an UNGRADED $\mathfrak{S}_n$-mod. isomorphism

$$SR_n \underset{\mathfrak{S}_n}{\cong} \bigoplus_{\alpha \vDash n} H^{top}(p^{-1}(t_\alpha)).$$

Q GEOMETRIC PF?

MORE SETS OF VARIABLES

\# OF BOSONS $\underline{x}_n, \underline{y}_n, \ldots$

\# OF FERMIONS $\underline{\theta}_n, \underline{\xi}_n, \ldots$

	0	1	2
0	$\mathbb{Q}$ (EXCERCISE)		
1			
2			

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	2			

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	2	FDR_n ⌜JESSE KIM, JONGWON KIM, R.⌟	$R^{(1\|2)}$ ⌜LENTFER CONJ.⌟	

MORE SETS OF VARIABLES

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	1	$R^{(0 \mid 1)}$ (EXERCISE)	SR_n (THIS TALK)	$\sum_k z^{n-k} \Delta'_{e_{k-1}} e_n$ (ZABROCKI CONJ.)
	2	FDR_n [JESSE KIM, JONGWON KIM, R.]	$R^{(1 \mid 2)}$ [LENTFER CONJ.]	

MORE SETS OF VARIABLES

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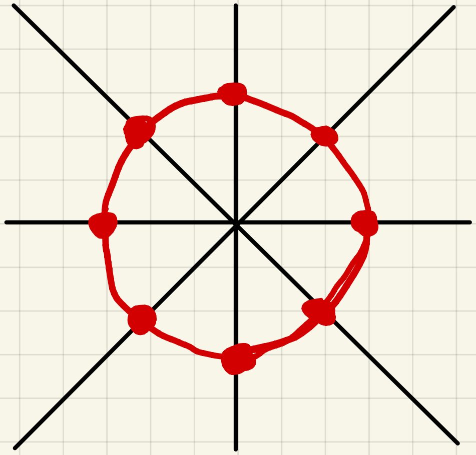
MORE SETS OF VARIABLES

\# OF BOSONS $x_n, y_n, \dots$

\# OF FERMIONS $\theta_n, \xi_n, \dots$

	0	1	2	$\infty \rightarrow$
0	$\mathbb{Q}$ (EXCERCISE)	R_n (ARTIN, CHEVALLEY, BOREL...)	DR_n (HAIMAN)	COMBIN. SUPERSYMMETRY ⌜CONJ: F. BERGERON THM: LENTFER⌟
1	$R^{(0\|1)}$ (EXERCISE)	SR_n (THIS TALK)	$\sum_k z^{n-k} \Delta'_{e_{k-1}} e_n$ (ZABROCKI CONJ.)	
2	FDR_n ⌜JESSE KIM, JONGWON KIM, R.⌟	$R^{(1\|2)}$ ⌜LENTFER CONJ.⌟	Θ-ops. ⌜D'ADDERIO-IRACI-VANDEN WYNGAERD CONJ.⌟	
$\infty \downarrow$				

OTHER TYPES?



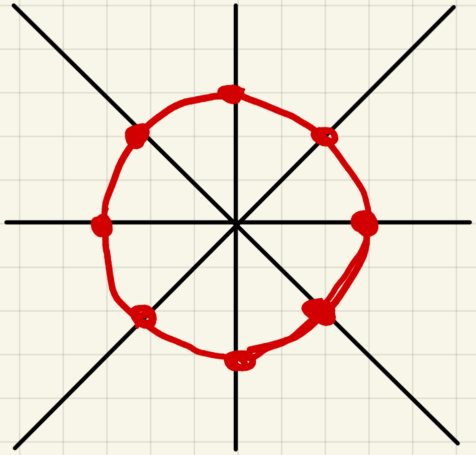
W : Weyl group

$\mathcal{P}_W$: COXETER COMPLEX

OTHER TYPES?

W : Weyl group

$\mathcal{OP}_W$: COXETER COMPLEX



CONJ [MURAI-R-WILSON] There is a W-equivariant surj'n

$$SR_W \twoheadrightarrow \mathcal{OP}_W \otimes \det .$$

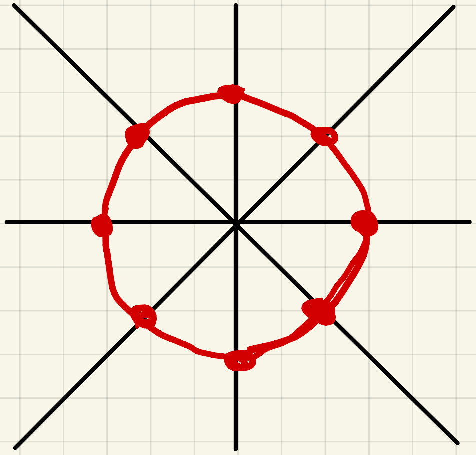
$\Omega_V / (\Omega_{V,+}^W)$

V : refl'n rep'n

OTHER TYPES?

W : Weyl group

$\mathcal{OP}_W$: COXETER COMPLEX



CONJ [MURAI-R-WILSON] There is a W-equivariant SURJ'N

$$SR_W \twoheadrightarrow \mathcal{OP}_W \otimes \det .$$

For any k, there is a W-equiv. SURJ'N

$$\begin{array}{l}\text{fermionic deg.}\\ \text{rank}\, W - k\\ \text{part of } SR_W\end{array} \twoheadrightarrow \mathcal{OP}_{\underbrace{W,k}_{k\text{-dim'l faces}}} \otimes \det$$

OTHER TYPES?

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For any k, there is a W-equiv. SURJ'N

$$\begin{array}{l}\text{fermionic deg.} \\ \text{rank } W - k \\ \text{part of } SR_W\end{array} \twoheadrightarrow \mathcal{OP}_{\underbrace{W,k}_{k\text{-dim'l faces}}} \otimes \det$$

- * ISOMORPHISM if $k = \text{rank } W$ (CHEVALLEY)
- * ISOMORPHISM if $k = 0$ (SWANSON-WALLACH)
- * ISOMORPHISM in type A (MURAI-R-WILSON)
- * DIMENSIONS MATCH in type BC (BHATTACHARYA)

Type F_4

$$\mathrm{Hilb}(SR_{F_4};1,z) = 1152 + 2304z + 1396z^2 + 244z^3 + z^4$$

reversed f-vector of $\mathcal{OP}_{F_4}$ $(1152, 2304, 1392, 240, 1)$

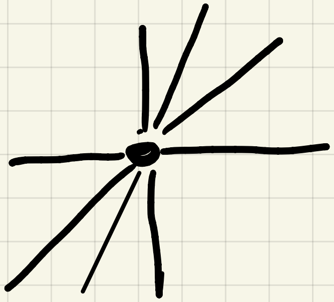
F_4 has 25 irreducible characters $\chi_1, \dots, \chi_{25}$

2304	0	4	2	2	6	2	6	2	4	8	8	12	8	12	12	12	20	12	20	24	18	18	12	24	32
2304	0	4	2	2	6	2	6	2	4	8	8	12	8	12	12	12	20	12	20	24	18	18	12	24	32
1396	0	6	1	1	6	0	6	0	0	5	5	12	5	6	8	4	16	4	16	21	10	10	3	14	18
1392	0	6	1	1	6	0	6	0	0	5	5	12	4	6	8	4	16	4	16	21	10	10	3	14	18
244	0	4	0	0	2	0	2	0	0	1	1	4	1	0	2	0	4	0	4	6	1	1	0	2	2
240	0	4	0	0	2	0	2	0	0	1	1	4	0	0	2	0	4	0	4	6	1	1	0	2	2

↑ χ_{13}, degree 4

One last conjecture

$\mathcal{A}$: arrangement in $\mathbb{R}^n$

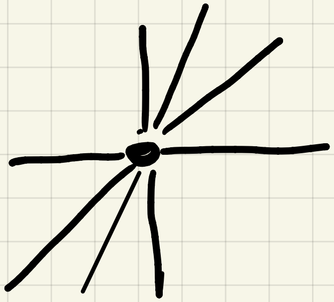


$f_d(\mathcal{A})$: # of d-dim'l faces

NO SYMMETRY ASSUMPTION!

One last conjecture

$\mathcal{A}$: arrangement in $\mathbb{R}^n$



$f_d(\mathcal{A})$: # of d-dim'l faces

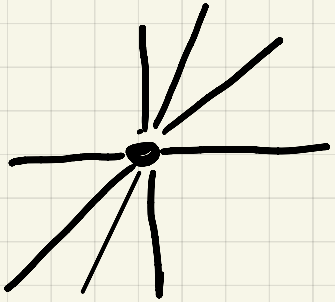
$$a : \operatorname{Der} S \longrightarrow \Omega \qquad \qquad h : \operatorname{Der} S \longrightarrow \Omega$$

$$\partial_i \longmapsto x_i \qquad \qquad \qquad \partial_i \longmapsto \theta_i$$

$$I_{\mathcal{A}} := a(\operatorname{Der} \mathcal{A}) + h(\operatorname{Der} \mathcal{A}) \subseteq \Omega$$

One last conjecture

$\mathcal{A}$: arrangement in $\mathbb{R}^n$



$f_d(\mathcal{A})$: # of d-dim'l faces

$$a : \mathrm{Der}\, S \longrightarrow \Omega \qquad\qquad b : \mathrm{Der}\, S \longrightarrow \Omega$$
$$\partial_i \longmapsto x_i \qquad\qquad\qquad \partial_i \longmapsto \theta_i$$

$$I_{\mathcal{A}} := a(\mathrm{Der}\,\mathcal{A}) + b(\mathrm{Der}\,\mathcal{A}) \subseteq \Omega$$

CONJ [MURAI-R-WILSON] For any d,

$$\dim \begin{pmatrix} \text{fermionic deg.} \\ n-d \text{ piece of} \\ \Omega / I_{\mathcal{A}} \end{pmatrix} \geq f_d(\mathcal{A})$$

Thanks for Listening!

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