Surprises (!) in Specht polynomial theory

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Includes joint work with Raymond Chou,
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ICERM workshop:
Category theory, combinatorics, and machine learning

$$\begin{array}{|c|c} \hline 1 & \\ \hline 0 & \multicolumn{1}{|c|}{1} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array} \qquad x_3(x_2 - x_1)$$

Intro: S_n action on polynomial ring

Key player: symmetric group S_n of permutations of $1, \ldots, n$, acts on:

$$\mathbb{C}[x_1, \ldots, x_n]$$

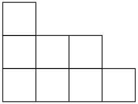
by permuting the variables.

Q: What is the decomposition into irreducibles (S_n-invariant subspaces)?

- Specht polynomials give one copy of each irreducible up to isomorphism
- What are the higher degree Specht polynomial analogs?

Specht polynomials: construction

- λ - partition/Young diagram, n boxes



Specht polynomials: construction

- λ - partition/Young diagram, n boxes
- T - fill λ with $1, \ldots, n$
- **Specht polynomial** $F_T = \prod_{i \text{ above } j}(x_i - x_j)$

3			
7	1	6	
2	8	4	5

Fact: Specht polynomials $\{F_T : \mathrm{shape}(T) = \lambda\}$ span a copy of the irreducible S_n module V_λ.

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Symmetric group action

$$\pi \cdot F_T = F_{\pi(T)}.$$

Column permutations:

$$(37) \cdot \begin{array}{|c|cccc}\cline{1-1} \mathbf{3} & & & & \\ \cline{1-3} \mathbf{7} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{6} & & \\ \hline 2 & \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \cline{1-4} \end{array} = \begin{array}{|c|cccc}\cline{1-1} \mathbf{7} & & & & \\ \cline{1-3} \mathbf{3} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{6} & & \\ \hline 2 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{8} \\ \cline{1-4} \end{array}$$

$$\begin{aligned} (37) \cdot F_T &= (37) \cdot (x_3 - x_7)(x_7 - x_2)(x_3 - x_2) \cdot \cdots \\ &= (x_7 - x_3)(x_3 - x_2)(x_7 - x_2) \cdot \cdots \\ &= -F_T \end{aligned}$$

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Column permutations:

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Row permutations:

$$(58) \cdot \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 7 & 1 & 6 \\ \cline{1-3} 2 & \mathbf{8} & 4 & \mathbf{5} \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 7 & 1 & 6 \\ \cline{1-3} 2 & 5 & 4 & 8 \\ \hline \end{array}$$

S_n action continued

- **Garnir relations:** Give *straightening algorithm*. Example:

$$F_{\begin{smallmatrix}2\\3&1\end{smallmatrix}} = -F_{\begin{smallmatrix}3\\2&1\end{smallmatrix}} = x_2 - x_3 = F_{\begin{smallmatrix}2\\1&3\end{smallmatrix}} - F_{\begin{smallmatrix}3\\1&2\end{smallmatrix}}$$

- $\mathrm{SYT}(\lambda)$ is **Standard Young Tableaux** - rows and columns sorted

$$\begin{array}{|c|c|c|c|}\cline{1-1} 7 \\ \cline{1-3} 3 & 5 & 6 \\ \hline 1 & 2 & 4 & 8 \\ \hline \end{array}$$

- **Basis** of V_λ: $\{F_T : T \in \mathrm{SYT}(\lambda)\}$

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- **Basis** of V_λ: $\{F_T : T \in \mathrm{SYT}(\lambda)\}$
- How to generalize to higher degree copies of irreducibles in $\mathbb{C}[x_1, \ldots, x_n]$?

Alternate definition of Specht polynomials

- **Young symmetrizer:** $C(T), R(T)$ column and row stabilizers,

$$\varepsilon_T = \sum_{\tau \in C(T)} \sum_{\sigma \in R(T)} \mathrm{sgn}(\tau)\tau\sigma$$

- **Specht polynomials:** Up to constant scalar, have

$$F_T = \varepsilon_T x_T^r$$

where $x_T^r = \prod_i x_i^{\mathrm{row}_T(i)-1}$. Example:

$$T = \begin{array}{|c|c|c|c|}\hline 4 \\ \cline{1-1} 2 & 5 & 7 \\ \cline{1-3} 1 & 3 & 6 & 8 \\ \hline\end{array} \qquad r = \begin{array}{|c|c|c|c|}\hline 2 \\ \cline{1-1} 1 & 1 & 1 \\ \cline{1-3} 0 & 0 & 0 & 0 \\ \hline\end{array} \qquad x_T^r = x_2 x_5 x_7 x_4^2$$

$$\varepsilon_T x_T^r = 144(x_4 - x_2)(x_4 - x_1)\cdots$$

Higher Specht polynomials

Recipe for constructing higher (degree) Specht polynomials:

1. S any tableau, T an SYT of the same shape

$$T = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad S = \begin{array}{|c|c}\hline 1 & \\ \hline 0 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

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$$x_2x_3 + (13)x_2x_3 - (12)x_2x_3 - (12)(13)x_2x_3 = x_3(x_2 - x_1)$$

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Lemma (G.): Garnir relations satisfied by F_T^S for any fixed S.

Cor: $\{F_T^S : T \in \mathrm{SYT}(\lambda)\}$ is a copy of V_λ for any fixed S...?

When the recipe breaks (!)

- S, T, form monomial:

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- Apply ε_T: $\quad F_T^S = \varepsilon_T x_T^S = x_1 x_2 - (12) x_1 x_2 = 0$

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Q: When is F_T^S nonzero?

Nonzero examples: Higher Specht basis for coinvariant ring

A full construction for the polynomial ring:

- Ariki, Terasoma, Yamada: constructed nonvanishing higher Specht basis of $n!$ elements for $R_n = \mathbb{C}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 + \cdots + x_{n-1}x_n$$

- Dimension $n!$, sufficient for understanding $\mathbb{C}[x_1, \ldots, x_n]$

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- **RSK:** $n!$ pairs (T, P) of SYT's of same shape, size n
- **Cocharge tableau** $S(P)$**:** label square containing 1 with 0, increment label if next letter is in a row above previous

$$T = \begin{array}{|c|cc}\hline 5 & & \\ \hline 2 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{7} \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{6} \\ \hline \end{array} \qquad P = \begin{array}{|c|cc}\hline 7 & & \\ \hline 3 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad S(P) = \begin{array}{|c|cc}\hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline 0 & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

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- **Monomial** $x_T^S = x_2x_4x_6x_7^2x_5^3$ **Polynomial** $F_T^P := \varepsilon_T\left(x_T^{S(P)}\right)$
- **Higher Specht basis:** $\{F_T^P\}$ where $T, P \in SYT(n)$ of same shape

Coinvariants in context

- Much study of R_n, many bases including: (ex for R_3):
  - Artin basis: $\{1, x_2, x_3, x_2x_3, x_3^2, x_2x_3^2\}$
  - Garsia-Stanton: $\{1, x_2, x_3, x_2x_3, x_1x_3, x_2x_3^2\}$
  - Schubert basis: $\{1, x_1, x_1 + x_2, x_1^2, x_1x_2, x_1^2x_2\}$
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- **Allen basis:** Slight modification of Procesi's:

$$\{6, 2x_2, 2x_3, x_3(x_1 + x_2), x_2(x_1 + x_3), x_2x_3^2\}$$

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- **Higher Specht (ATY)**: Respects decomposition and satisfies Garnir:

$$\{6, 2(x_2-x_1), 2(x_3-x_1), x_3(x_2-x_1), x_2(x_3-x_1), (x_3-x_2)(x_3-x_1)(x_2-x_1)\}$$

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- Many generalizations: R_μ, $R_{n,k}$, $R_{n,k,\mu}$, DR_n, DR_μ, and more

Generalization 1: to R_μ

Garsia-Procesi Modules: a generalization

- Generalization of coinvariant ring: $R_n \twoheadrightarrow R_\mu$
- $R_\mu \cong H^*(\mathcal{B}_\mu)$ where $\mathcal{B}_\mu$ is **Springer fiber** of all flags fixed by a unipotent matrix of Jordan type μ

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- Explicit presentation in terms of *Tanisaki generators* (generalize e_i's):

$$R_\mu = \mathbb{C}[x_1, \ldots, x_n]/I_\mu$$

- Bases:
  - Garsia–Procesi: a monomial basis, defined recursively
  - Carlsson–Chou: A descent (Garsia–Stanton) basis
  - G.–Rhoades: Conjectured higher Specht basis, proven for two rows

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  - Garsia–Procesi: a monomial basis, defined recursively
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  - G.–Rhoades: Conjectured higher Specht basis, proven for two rows
- **Dimension count:** need one basis element for each pair (T, P) of same shape, T standard, P semistandard of **content** μ:

$$T = \begin{array}{|c|c|c|c|}\hline 5 \\ \cline{1-1} 2 & 4 & 7 \\ \cline{1-3} 1 & 3 & 6 & 8 \\ \hline\end{array} \qquad P = \begin{array}{|c|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 & 4 \\ \cline{1-3} 1 & 1 & 1 & 2 \\ \hline\end{array} \qquad \mu = (3, 2, 2, 1)$$

Cocharge and conjectured basis

- **Cocharge** of a semistandard tableau P with partition content:
  - Search in reverse reading order for $1, 2, 3, \ldots$ and do cocharge labeling

$$P = \begin{array}{|c|c|c|c|}\cline{1-1} 3 \\ \cline{1-3} 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 2 \\ \hline \end{array} \qquad S(P) = \begin{array}{|c|c|c|c|}\cline{1-1} \\ \cline{1-3} & & \\ \hline & & & \\ \hline \end{array}$$

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  - Restart labelling and repeat on remaining letters until all letters are labeled

$$P = \begin{array}{|c|c|c|c|} \hline \mathbf{3} \\ \cline{1-1} \mathbf{2} & 3 & \mathbf{4} \\ \cline{1-3} 1 & \color{red}{1} & \mathbf{1} & 2 \\ \hline \end{array} \qquad S(P) = \begin{array}{|c|c|c|c|} \hline 2 \\ \cline{1-1} 1 & & 2 \\ \cline{1-3} & 0 & 0 & \\ \hline \end{array}$$

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$$P = \begin{array}{|c|c|c|c|} \cline{1-1} \mathbf{3} \\ \cline{1-3} \mathbf{2} & 3 & \mathbf{4} \\ \hline 1 & \color{red}{1} & \mathbf{1} & \color{red}{2} \\ \hline \end{array} \qquad S(P) = \begin{array}{|c|c|c|c|} \cline{1-1} 2 \\ \cline{1-3} 1 & & 2 \\ \hline & 0 & 0 & 0 \\ \hline \end{array}$$

Cocharge and conjectured basis

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$P =$

3			
2	3	**4**	
1	1	**1**	2

$S(P) =$

2			
1	1	2	
0	0	0	0

- Define $F_T^P = \varepsilon_T x_T^{S(P)}$
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- Define $F_T^P = \varepsilon_T x_T^{S(P)}$
- **Conjecture (G.–Rhoades '18):** F_T^P's form a basis
- Enumeratively correct in each dimension, proven for two rows
- *False in general!* Smallest known counterexample size 10. Chou, Hanada working on a correction
- **Computational time to test conjectures is a huge issue**, usually can only get up to size 8 or 9 with current methods

The issue (!)

- **Counterexample (G, Chou, 2025):**

$$T = \begin{array}{|c|cccc}\hline 4 \\ \cline{1-1} 3 \\ \hline 2 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{10} \\ \hline 1 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{9} \\ \hline \end{array} \qquad P = \begin{array}{|c|cccc}\hline 5 \\ \cline{1-1} 3 \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad S(P) = \begin{array}{|c|cccc}\hline 2 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline 0 & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

- If $\sigma \in R(T)$, term $\sigma x_T^{S(P)}$ has $x_3^2 x_4^2$, so $\tau = (3\,4)$ changes sign, cancels
- $F_T^P = 0$ as a polynomial!

The issue (!)

- **Counterexample (G, Chou, 2025):**

$T =$

4			
3			
2	6	8	10
1	5	7	9

$P =$

5			
3			
2	3	4	5
1	1	2	4

$S(P) =$

2			
2			
1	1	1	3
0	0	0	2

- If $\sigma \in R(T)$, term $\sigma x_T^{S(P)}$ has $x_3^2 x_4^2$, so $\tau = (3\,4)$ changes sign, cancels
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- $F_T^P = 0$ as a polynomial!
- (G., 2024): If $S(P)$ is semistandard then $F_T^P \neq 0$
- Sufficient to avoid columns with duplicate entries?

Q: When is F_T^S nonzero?

A permutative condition

Theorem (Chou, G., Hanada, literally yesterday)

We have $\varepsilon_T x_T^S = 0$ if and only if, for every row permutation of S, say S', that there is an odd column permutation of the boxes that when applied to S' fixes the content of each row.

Box labeling:

2	4	6
1	3	5

Example (!):

$$S = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 0 & 1 & 2 \\ \hline\end{array} \xrightarrow{(246)} S' = \begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline\end{array} \xrightarrow{(12)(34)(56)} \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline\end{array}$$

Have $F_T^S = 0$ for any T of shape $(3,3)$.

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Fun little corollary: Coefficients of all nonvanishing monomials in F_T^P are equal, to number of column permutations fixing the content of each row

Two more fun little corollaries

- **Two columns (plus a hook):** In this case, nonzero iff the columns can be made to each have all distinct letters using a row permutation:

5				
0				
2	3			
0	1			
2	1			
4	1	1	5	3

$\longrightarrow$

5				
0				
3	2			
1	0			
2	1			
4	5	1	1	3

- **Two rows (plus a hook):** In this case, nonzero iff chimney has no repeats and some row permutation has no odd subset of columns having the same sets of elements in first and second row

5				
1	1	1	3	2
2	1	1	3	2

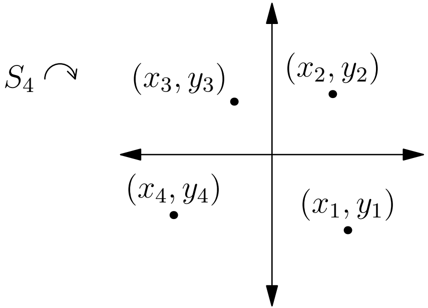
$\longrightarrow$

5				
1	2	3	1	1
2	1	1	3	2

Generalization 2: to $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$

The diagonal action

- **Diagonal action:** S_n acts on $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] = \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]$ by permuting x and y variables simultaneously



- Reduce to **diagonal coinvariant ring** $\mathrm{DR}_n = \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/I_n$, have $\dim_{\mathbb{C}} \mathrm{DR}_n = (n+1)^{n-1}$
- (Carlsson, Oblomkov) - monomial basis, not higher Specht
- **Open**: find graded decomposition into irreducibles

Recipe for two variables

1. S any tableau with **pairs** as entries, T an SYT of the same shape

$$T = \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad S = \begin{array}{|c|c} \hline 12 & \\ \hline 01 & \multicolumn{1}{c|}{10} \\ \hline \end{array}$$

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$$\begin{aligned}(I + (13) - (12) - (12)(13))y_1 x_2 y_2^2 x_3 \\ = y_1 x_2 y_2^2 x_3 + y_3 x_2 y_2^2 x_1 - y_2 x_1 y_1^2 x_3 - y_3 x_1 y_1^2 x_2\end{aligned}$$

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Lemma (G.): Garnir relations satisfied by F_T^S for any fixed S.

Cor: $\{F_T^S : T \in \mathrm{SYT}(\lambda)\}$ is a copy of V_λ for any fixed S whenever one F_T^S is nonzero (odd permutation condition still applies!)

Garsia-Haiman modules

- **Garsia–Haiman:** DR_μ is quotient of DR_n,

$$\mathrm{DR}_\mu = \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/I_\mu$$

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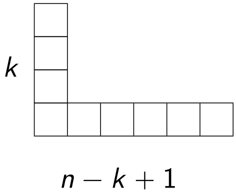
- Applications to **Macdonald polynomials**, Hilbert scheme of n points in the plane
- (G., 2024): higher Specht basis for DR_μ for μ a **hook shape**:



Hook shape Garsia–Haiman modules: construction

- μ-cocharge: reindex so that $n - k + 1$ is labeled 0:

T :

12	15				
6	9	16			
3	7	8	13	14	
1	2	4	5	10	11

P :

12	16				
6	14	15			
4	7	8	9	13	
1	2	3	5	10	11

$S(P)$:

1	3				
0	2	2			
−1	0	0	0	1	
−2	−2	−2	−1	0	0

Theorem (G.)

The set $\{F_T^P(\mu)\}$ is a higher Specht basis for DR_μ for hook shape μ.

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- Take $F_T^P(\mu) = \varepsilon_T(x_T^p y_T^n)$ where p =positive, n=negative. Here

$$x_T^p y_T^n = x_{16}^2 x_{15}^3 x_{14} x_{12} x_9^2 y_5 y_4^2 y_3 y_2^2 y_1^2$$

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Higher Specht basis for DR_3

Frobenius term	Basis polynomials	Frobenius term	Basis polynomials
$s_{(3)}$	1	$q^2 s_{(2,1)}$	$\varepsilon_{\begin{smallmatrix}3\\1&2\end{smallmatrix}} x_3 x_2$ $\varepsilon_{\begin{smallmatrix}2\\1&3\end{smallmatrix}} x_2 x_3$
$qs_{(2,1)}$	$x_2 - x_1$ $x_3 - x_1$	$t^2 s_{(2,1)}$	$\varepsilon_{\begin{smallmatrix}3\\1&2\end{smallmatrix}} y_3 y_2$ $\varepsilon_{\begin{smallmatrix}2\\1&3\end{smallmatrix}} y_2 y_3$
$ts_{(2,1)}$	$y_2 - y_1$ $y_3 - y_1$	$q^3 s_{(3)}$	$(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$
$qts_{(1,1,1)}$	$\varepsilon_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}} x_3 y_1$	$t^3 s_{(3)}$	$(y_3 - y_2)(y_3 - y_1)(y_2 - y_1)$
$qts_{(2,1)}$	$\varepsilon_{\begin{smallmatrix}3\\1&2\end{smallmatrix}} x_3 y_2$ $\varepsilon_{\begin{smallmatrix}2\\1&3\end{smallmatrix}} x_2 y_3$	$q^2 ts_{(1,1,1)}$	$\varepsilon_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}} x_3^2 y_1$
		$qt^2 s_{(1,1,1)}$	$\varepsilon_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}} x_3 y_1^2$

Future directions

Higher Specht basis for:

- DR_n? Maybe use pairs of tableaux whose words correspond to parking functions? Find guesses with machine learning??
- More DR_μ's besides hook shapes? Trying two rows, two columns (joint work with Chou, Hanada)
- Both of the above would lead to (or from) Schur expansion for Macdonalds or Shuffle polynomials

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Q: When is F_T^S nonzero *mod an ideal*? How can we use our result to search for bases? Can I just put a priority ranking on the good monomials and use FunSearch?

k	*n*	*a*	*t*	*h*					
o	*f*	—	*y*	*o*	*u*	*r*			
e	*n*	*l*	*i*	*s*	*t*	*i*	*n*	*g*	!

$\rightarrow$

t	*h*	*a*	*n*	*k*					
y	*o*	*u*	—	*f*	*o*	*r*			
l	*i*	*s*	*t*	*e*	*n*	*i*	*n*	*g*	!