

FACTORIZATIONS, CENTERS AND THE JUCYS-MURPHY ELEMENTS OF THE HECKE ALGEBRA

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ICERM WORKSHOP: CATEGORY THEORY,
COMBINATORICS AND MACHINE LEARNING
SEPTEMBER 18, 2025

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BIG IDEA:

* Theory of PERMUTATION FACTORIZATIONS

with connections to combinatorics, representation theory,
algebraic geometry, random matrix theory, mathematical physics

* **GOAL**: deform this theory to the **IWAHORI-HECKE ALGEBRA**

* **KEY TECHNIQUE**: work in the **HECKE ALGEBRA CENTER**

↳ Brings up many new **QUESTIONS**

OUTLINE:

I. PERMUTATION FACTORIZATIONS

II. q -ANALOGUES

III. REFRAMING FACTORIZATIONS

IV. CENTRAL QUESTIONS

I. PERMUTATION FACTORIZATIONS

QUESTION: How many ways are there to write the long cycle $\tau_n = (12 \dots n)$ as a product of $n-1$ TRANSPOSITIONS?

NOTATION: Let's call this $a(n, n-1)$

Annotations:
An arrow points from $a(n, n-1)$ to $\tau_n = (12 \dots n)$.
Another arrow points from $a(n, n-1)$ to the text " $n-1$ transpositions".

EXAMPLE: $n=3$: $\tau_3 = (123)$

$$(123) = (12)(13)$$

$$(123) = (13)(23)$$

$$(123) = (23)(12)$$

$$\left. \begin{array}{l} (123) = (12)(13) \\ (123) = (13)(23) \\ (123) = (23)(12) \end{array} \right\} a(3, 2) = 3.$$

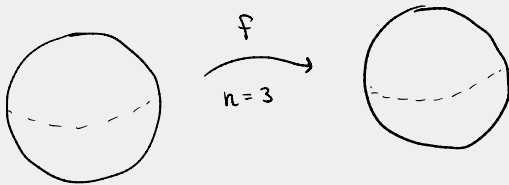
THEOREM (Hurwitz, 1891)

$$a(n, n-1) = n^{n-2}$$

Hurwitz was counting

GENUS 0 RAMIFIED COVERINGS of S^n with

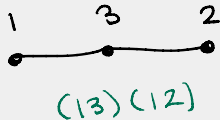
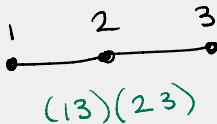
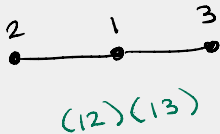
- * $n-1$ simple critical points
- * a critical point of type n



We can also relate $a(n, n-1) = n^{n-2}$ to trees...

* $n^{n-2} = \# \text{ trees on } \{1, \dots, n\}$

* Moszkowski 1989 gave a bijection



INITIATES the study of **PERMUTATION FACTORIZATIONS**

→ connections to random matrix theory,
mathematical physics,
representation theory ✓ **TODAY.**

A-B-C's of PERMUTATION FACTORIZATION

* $a(n, j) := \#$ of ways to write $\tau_n := (12 \dots n)$
as a product of j **TRANSPOSITIONS**

THEOREM (Jackson, 1988)

$$\sum_{j \geq 0} a(n, j) t^j = \frac{n^{n-2} + n^{-1}}{\prod_{k=0}^{n-1} (1 - n(\frac{n-1}{2} - k)t)}$$

In particular, $a(n, n-1) = n^{n-2}$.

A-B-C's OF PERMUTATION FACTORIZATION

* A MONOTONE FACTORIZATION:

$(i_1, j_1) (i_2, j_2) \dots (i_k, j_k)$
such that $i_x < i_e$ and $j_1 \leq j_2 \leq \dots \leq j_k$

$(1\ 2)(1\ 3)$

$(1\ 3)(2\ 3)$

✓ MONOTONE

$(1\ 3)(1\ 2)$

✗ NOT MONOTONE

* $b(n, k) = \#$ monotone factorizations of τ_n into k transpositions

THEOREM (Matsumoto - Novak, 2013)

$$b(n, n-1) = C_{n-1} = \frac{1}{n-1} \binom{2(n-1)}{n-1}$$

← CATALAN NUMBERS

and more generally

$$\sum_{j \geq n-1} b(n, j) t^j = \frac{C_{n-1} t^{n-1}}{\prod_{k=0}^{n-1} (1 - k^2 t^2)}$$

A-B-C's OF PERMUTATION FACTORIZATION

* $C(n; p_1, \dots, p_m) := \#$ factorizations of τ_n into $\pi_1, \dots, \pi_m$, with π_i having $n-p_i$ cycles

THEOREM (Jackson, 1988):

$$\sum_{0 \leq p_1, \dots, p_m \leq n-1} c(n; p_1, \dots, p_m) t_1^{n-p_1} \dots t_m^{n-p_m} = \sum_{0 \leq r_1, \dots, r_m \leq n-1} M_{(r_1, \dots, r_m)}^{(n-1)} \binom{t_1}{n-r_1} \dots \binom{t_m}{n-r_m}$$

$\swarrow$ $n-k$ cycles $\swarrow$ $k+1$ cycles

* **SPECIAL CASE:** $c(n; k, n-k-1) : \tau_n = \pi_1 \pi_2$

e.g. $n=6$, $k=3$, $n-k-1=2$, then an example contribution to $c(6; 3, 2)$ is

$$\underbrace{(1)(3)(2456)}_{\substack{3 \text{ cycles} \\ p_1 = 6-3=3}} \cdot \underbrace{(12)(34)(5)(6)}_{\substack{4 \text{ cycles} \\ p_2 = 6-4=2}} = (123456)$$

THEOREM (Bédard-Goupil, 1992):

$$C(n; k, n-k-1) = \frac{1}{n} \binom{n}{k+1} \binom{n}{k} = N_{n, k+1}$$

NARAYANA NUMBERS

SUMMARY:

A-B-C's reveal some of the most ubiquitous formulas in combinatorics!

A: $a(n, n-1) = n^{n-2}$ factorizations of τ_n into $n-1$ transpositions
LABELLED TREES, PARKING FUNCTIONS

B: $b(n, n-1) = \frac{1}{n-1} \binom{2(n-1)}{n-1} = C_{n-1}$ monotone factorizations of τ_n into $n-1$ transpositions
CATALAN NUMBERS

↪ Dyck paths, triangulations of n -gon

C: $c(n; k, n-k-1) = \frac{1}{n} \binom{n}{k+1} \binom{n}{k} = N_{n, k+1}$ factorizations of τ_n into $n-k$ cycles π_1, π_2 and $k+1$ cycles
NARAYANA NUMBERS

↪ rooted trees, monotonic lattice paths

II. q -ANALOGUES

IDEA: * Introduce a parameter q
↳ enriches and refines classical object.

* we recover classical objects at $q=1$

CLASSICAL
OBJECT

q -ANALOGUE

$$n \in \mathbb{Z}$$

$$[n]_q := \frac{1-q^n}{1-q} = \begin{cases} 1+q+q^2+\dots+q^{n-1} & n > 0 \\ 0 & n = 0 \\ -q^{-1}-q^{-2}-\dots-q^{-n} & n < 0 \end{cases}$$

e.g. $[5]_q = 1+q+q^2+q^3+q^4$

$$\mathbb{Q}[G_n]$$

the group algebra

$$H_n(q)$$

the Type A Iwahori Hecke algebra

ENUMERATIVE q -ANALOGUES (POLYNOMIALS?!) assume $n, k > 0$

* q -FACTORIAL:

$$[n]!_q := [n]_q [n-1]_q \cdots [2]_q [1]_q$$

→ counts Hilbert series for Type A coinvariant algebra

* q -BINOMIAL:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

e.g. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$

→ counts points in Grassmannian of k -planes in $\mathbb{F}_q^n$

TODAY: we will see algebraic instantiations of

q -CATALAN

AND

q -NARAYANA

$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

$$N_{n,k}(q) := \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

CLASSICAL OBJECT

DEFINE: $\mathbb{Q}[\mathbb{G}_n]$ is the algebra generated by $S_1, \dots, S_{n-1}$
 $S_i = (i, i+1)$
such that for $1 \leq i \leq n-1$:

(i) $S_i^2 = 1$

EQUIVALENTLY: $(S_i - 1)(S_i + 1) = 0$

(ii) $S_i S_j = S_j S_i$ when $|i - j| \geq 2$

(iii) $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$

LINEAR BASIS: $\{w \in \mathbb{G}_n\}$

$$w = \underbrace{S_{i_1} \dots S_{i_k}}_{\text{reduced expression}}$$

q-ANALOGUE

DEFINE: $\mathcal{H}_n(q)$ is the $\mathbb{Q}[q^{\pm 1}]$ -algebra generated by $T_1, \dots, T_{n-1}$ such that for $1 \leq i \leq n-1$:

(i) $T_i^2 = (q-1)T_i + q$

EQUIVALENTLY: $(T_i - q)(T_i + 1) = 0$

(ii) $T_i T_j = T_j T_i$ when $|i - j| \geq 2$

(iii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

LINEAR BASIS: $\{T_w : w \in \mathbb{G}_n\}$

$$T_w := T_{i_1} T_{i_2} \dots T_{i_k}$$

NOTE: we will write w in **CYCLE NOTATION** with parenthesis

EXAMPLE: MULTIPLICATION

in $\mathbb{Q}[S_n]$

$$s_2 \cdot s_1 = (23)(12) \\ = (123)$$

REDUCED
EXPRESSION

$$s_1^2 \cdot s_2 = (12)^2 \cdot (23) \\ = (23)$$

NON-REDUCED
EXPRESSION

in $\mathcal{H}_n(q)$

$$T_2 \cdot T_1 = T_{(23)} \cdot T_{(12)} \\ = T_{(123)}$$

$$T_1^2 \cdot T_2 =$$

$$\begin{aligned} & ((q-1)T_1 + q)T_2 \\ &= (q-1)T_1T_2 + qT_2 \\ &= (q-1)T_{(132)} + qT_{(23)} \end{aligned}$$

QUESTION:

Can we DEFORM the **A-B-C**'s of factorization?

---> What would it mean to define

a "FACTORIZATION THEORY" for $\mathcal{H}_n(q)$?

ISSUE: IF we limit ourselves to products of the form

$$T_w = T_{u_1} T_{u_2} \dots T_{u_k}$$

we would merely be counting reduced expressions!

↳ would lose factorizations such as $(123) = (13)(23)$

SOLUTION: Reframe the **A-B-C**'s ...

III. REFRAMING FACTORIZATIONS

CLAIM: We can reframe **A-B-C**'s in terms of the **CENTER** of the **SYMMETRIC GROUP ALGEBRA**.

$$Z_{G_n} := Z(\mathbb{Q}[G_n])$$

* closely linked to **CHARACTER THEORY** of G_n

* Concretely: basis given by **CONJUGACY CLASS SUMS**

$$\mathcal{C}_\mu := \sum_{\substack{w \text{ of cycle} \\ \text{type } \mu}} w$$

e.g. $\mathcal{C}_{(2,2)} = (12)(34) + (13)(24) + (14)(23)$

$$\mathcal{C}_{(2,1,1)} = (12) + (13) + (14) + (23) + (24) + (34)$$

$\mathbb{Z}[G_n]$ has a beautiful characterization:

INGREDIENT 1: SYMMETRIC FUNCTIONS

Λ : ring of symmetric functions over $\mathbb{Q}$ in infinitely many variables

EXAMPLE: For $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

* ELEMENTARY:
$$e_k = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k} \quad e_\lambda = e_{\lambda_1} \dots e_{\lambda_\ell}$$

e.g.
$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

* HOMOGENEOUS:
$$h_k = \sum_{i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} \dots x_{i_k} \quad h_\lambda = h_{\lambda_1} \dots h_{\lambda_\ell}$$

e.g.
$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3$$

INGREDIENT 2: JUCYS MURPHY ELEMENTS

An amazing family of elements in $\mathbb{Q}[S_n]$

$$J_1 = 0$$

$$J_2 = (12)$$

$$J_3 = (13) + (23)$$

$$J_4 = (14) + (24) + (34)$$

$\vdots$

$$J_k := (1k) + (2k) + \dots + (k-1, k) \quad \text{for } 2 \leq k \leq n$$

PROPERTIES:

- * Pairwise commute: $J_k J_\ell = J_\ell J_k$ for all $1 \leq k, \ell \leq n$
- * Simultaneous eigenspaces are **YOUNG'S SEMINORMAL FORMS**
a beautiful basis of irreducible representations of $\mathbb{Q}[S_n]$
indexed by **STANDARD YOUNG TABLEAUX**

THE CHARACTERIZATION...

THEOREM (Jucys, 1974 and Murphy, 1983)

$z \in Z_{S_n}$ if and only if $z = f(J_1, \dots, J_n)$ for $f \in \Lambda$.

EXAMPLE:

$$\star e_1(J_1, \dots, J_n) = \sum_{k=1}^n J_k = \sum_{1 \leq i < j \leq n} (ij) = \mathcal{C}_{(2, 1^{n-2})}$$

$$\star e_k(J_1, \dots, J_n) = \sum_{2 \leq i_1 < \dots < i_k \leq n} J_{i_1} \dots J_{i_k} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = n-k}} \mathcal{C}_\lambda$$

eg. $k=2, n=3$

$$\begin{aligned} e_2(J_1, J_2, J_3) &= J_2 \cdot J_3 = (12) \cdot ((13) + (23)) \\ &= (123) + (132) \\ &= \mathcal{C}_{(3)} \leftarrow \ell(3) = 1 = 3 - 2 \end{aligned}$$

BACK TO THE A-B-C's...

$a(n; n-1)$ = # factorizations of τ_n into $n-1$ transpositions

Consider
$$e_1(J_1, \dots, J_n)^{n-1} = \left(\sum_{1 \leq i < j \leq n} (ij) \right)^{n-1}$$

EXAMPLE: for $n=3$,

$$\begin{aligned} e_1(J_1, J_2, J_3)^2 &= ((12) + (13) + (23))((12) + (13) + (23)) \\ &= (12)(12) + (12)(13) + (12)(23) \\ &\quad + (13)(12) + (13)(13) + (13)(23) \\ &\quad + (23)(12) + (23)(13) + (23)(23) \\ &= 3 \cdot (1)(2)(3) + 3 \cdot (132) + 3 \cdot (123) \end{aligned}$$

$\swarrow a(3,2)$

CLAIM: The coefficient of τ_n in $e_1(J_1, \dots, J_n)^{n-1}$ gives $a(n, n-1)$ i.e.:

$$[\tau_n] \cdot e_1(J_1, \dots, J_n)^{n-1} = a(n, n-1)$$

$\swarrow$ NOTATION for "coefficient of"

In fact, we can translate all **A-B-C's** in this way!

* **A: FACTORIZATIONS** of τ_n into j transpositions:

$$a(n, j) = [\tau_n] e_1 (J_1, \dots, J_n)^j$$

* **B: MONOTONE FACTORIZATIONS** of τ_n into j transpositions:

$$b(n, j) = [\tau_n] h_j (J_1, \dots, J_n)$$

e.g. $[(123)] h_2 (J_1, J_2, J_3) = [(123)] \cdot (J_2 J_3 + J_2^2 + J_3^2)$
 $= 2.$

* **C: FACTORIZATIONS** $\tau_n = \pi_1 \dots \pi_m$, $\text{cyc}(\pi_i) = n - p_i$

$$c(n; p_1, \dots, p_m) = [\tau_n] e_{p_1 \dots p_m} (J_1, \dots, J_n)$$

Computing this **ALGEBRAIC EXPRESSION**
makes sense for $H_n(q)$?

NEXT STEP:

Consider the **CENTER** of $H_n(q)$

$$Z_{H_n(q)} := Z(H_n(q))$$

CLAIM: There is a very similar characterization!

INGREDIENT 1: (again) **SYMMETRIC FUNCTIONS**

We need **INGREDIENT 2**: q -Jucys MURPHY ELEMENTS

$$J_k(q) := q^{1-k} T_{(1,k)} + q^{2-k} T_{(2,k)} + \dots + q^{-1} T_{(k-1,k)}$$

$$J_1(q) := 0$$

(For experts: these are additive Jucys-Murphy elements)

EXAMPLE: $J_3(q) = q^{-2} T_{(1,3)} + q^{-1} T_{(2,3)} \leftarrow \text{cycle notation}$
 $= q^{-2} T_1 T_2 T_1 + q^{-1} T_2 \leftarrow \text{in } \mathcal{H}_n(q) \text{ generators}$

PROPERTIES:

* Pairwise commute: $J_k(q) J_\ell(q) = J_\ell(q) J_k(q)$

* Simultaneous eigenspaces are **YOUNG'S SEMINORMAL FORMS**
 a beautiful basis of irreducible representations of $\mathcal{H}_n(q)$
 indexed by **STANDARD YOUNG TABLEAUX**

↑
 when $q \neq 0$
 or a root of unity

CHARACTERIZATION of $Z_{\mathcal{H}_n(q)}$:

THEOREM (Francis - Graham, 2006).

$z \in Z_{\mathcal{H}_n(q)}$ **IF AND ONLY IF**

$$z = f(J_1(q), \dots, J_n(q)) \text{ for } f \in \Lambda$$

↳ Resolved a conjecture of Dipper-James from 1987!

EXAMPLE:

$$e_i(J_1(q), \dots, J_n(q)) = \sum_{1 \leq i < j \leq n} q^{i-j} T_{(i,j)} \in Z_{\mathcal{H}_n(q)}.$$

The q -**A**-**B**-**C**'s ...

We will consider $T_{\mathbb{Z}_n} = T_{(1,2,\dots,n)} = T_{n-1} \cdots T_1$

A: $a(n, j; q) = [T_{\mathbb{Z}_n}] e_1 (J_1(q), \dots, J_n(q))^j$

B: $b(n, j; q) = [T_{\mathbb{Z}_n}] h_j (J_1(q), \dots, J_n(q))$

C: $c(n, p_1, \dots, p_m; q) := [T_{\mathbb{Z}_n}] e_{p_1 \dots p_m} (J_1(q), \dots, J_n(q))$

EXAMPLE:

$$\begin{aligned}
 & e_1(J_1(q), J_2(q), J_3(q))^2 \\
 &= (q^{-1} T_{(12)} + q^{-2} T_{(1,3)} + q^{-1} T_{(2,3)}) \cdot (q^{-1} T_{(12)} + q^{-2} T_{(1,3)} + q^{-1} T_{(2,3)}) \\
 &= 3q^{-1} T_{(1)(2)(3)} - (2q^{-2} - 2q^{-1}) T_{(12)} - (q^{-4} + 2q^{-3} - 2q^{-2} - q^{-1}) T_{(13)} \\
 &\quad + (q^{-3} + q^{-2} + q^{-1}) T_{(1,3,2)} + (q^{-3} + q^{-2} + q^{-1}) T_{(123)}
 \end{aligned}$$

THUS...

$$e_1(J_1(q), J_2(q), J_3(q))^2 \Big|_{q=1} = 3 \cdot (1)(2)(3) + 3 \cdot (132) + 3 \cdot (123)$$

$$* [T_{(123)}] e_1(J_1(q), J_2(q), J_3(q))^2 = q^{-3} + q^{-2} + q^{-1} = q^{-3} [3]_q$$

Recall: $a(n, j) := \#$ of ways to write $\tau_n = (12 \dots n)$
 as a product of j **TRANSPOSITIONS**
 $= [Z_n] e_1(j_1 \dots j_n)^j$

THEOREM A (Bastidas, B. Guay-Pequet, Park, Morales, Salviola, 25)

$$a(n, n-1; q) = q^{-\binom{n}{2}} \cdot [n]_q^{n-2}$$

and more generally,

$$\sum_{j \geq 0} a(n, j; q) q^j = \frac{q^{-\binom{n}{2}} [n]_q^{n-2} q^{n-1}}{\prod_{j=0}^{n-1} \left(1 - \left(\frac{q^{-j} [j]_q}{1-q} \right) + \right)}$$

$$b(n, j) := \# \text{ MONOTONE FACTORIZATIONS of } \tau_n \text{ into } j \text{ transpositions} \\ = [\tau_n] h_j(\tau_1, \dots, \tau_n)$$

THEOREM B (Bastidas, B., Guay-Pequet, Park, Morales, Saliola, 25)

$$b(n, n-1; q) = q^{-\binom{n}{2}} C_{n-1}(q) = q^{-\binom{n}{2}} \frac{1}{[n-1]_q} \begin{bmatrix} 2(n-1) \\ n-1 \end{bmatrix}_q$$

↑
q-CATALAN

and more generally,

$$\sum_{j \geq 0} b(n, j; q) t^j = \frac{q^{-\binom{n}{2}} C_{n-1}(q) t^{n-1}}{\prod_{j=-(n-1)}^{n-1} (1 - [j]_q t)}$$

$$c(n; p_1, \dots, p_m) = \# \text{ FACTORIZATIONS } \tau_n = \pi_1 \dots \pi_m \text{ cyc}(\pi_i) = n - p_i \\ = [\tau_n] e_{p_1 \dots p_m}(J_1, \dots, J_n)$$

THEOREM C (Bestidas, B., Guay-Pequet, Park, Morales, Saliola, 25)

$$c(n, k, n-k-1; q) = q^{1-(k+1)(n-k)} N_{n, k+1}(q) \leftarrow q\text{-NARAYANA} \\ = q^{1-(k+1)(n-k)} \frac{1}{[n]_q} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q$$

and more generally,

$$\sum_{0 \leq p_1 \dots p_m \leq n-1} c(n; p_1, \dots, p_m; q) t_1^{n-p_1} \dots t_m^{n-p_m} \\ = [n]_q^{m-1} \sum_{0 \leq r_1, \dots, r_m \leq n-1} i^{-\sum (n-r_i)r_i} M_{(r_1, \dots, r_m)}^{n-1}(q) \binom{t_1}{n-r_1}_q \dots \binom{t_m}{n-r_m}_q$$

IN SUMMARY...

CLASSICAL:

A-B-C's for $\mathbb{G}_n$

A

$$a(n, n-1) = n^{n-2} \\ = [\tau_n] e_1 (J_1 \dots J_n)^{n-1}$$

B

$$b(n, n-1) = C_{n-1} \\ = [\tau_n] h_{n-1} (J_1 \dots J_n)$$

C

$$c(n; k, n-1-k) = N_{n, k+1} \\ = [\tau_n] e_{k, n-1-k} (J_1 \dots J_n)$$

(Bastidas, B. Guay-Piquet, Park, Morales, Salvia)

A-B-C's for $\mathcal{H}_n(q)$...

$$a(n, n-1; q) = q^{-\binom{n}{2}} [n]_q^{n-2} \\ = [\tau_n] e_1 (J_1(q), \dots, J_n(q))^{n-1}$$

$$b(n, n-1; q) = q^{-\binom{n}{2}} C_{n-1}(q) \\ = [\tau_n] h_{n-1} (J_1(q) \dots J_n(q))$$

$$c(n; k, n-1-k; q) = q^{1 - (kn)(n-k)} N_{k, n+1}(q) \\ = [\tau_n] e_{k, n-1-k} (J_1(q) \dots J_n(q))$$

PROOF TECHNIQUES:

↳ Character theory for $\mathcal{H}_n(q)$ developed by Ram
+ properties of $\check{J}_k(q)$ and associated seminormal forms

ADVANTAGE: unifies **A-B-C** proofs for $q=1\dots$

↳ we have big plans for a $\mathcal{H}_n(q)$ factorization theory

- * Conjectural q -Harer-Zagier formula
- * Bessis formula for other types?
- * Hurwitz numbers?

DISADVANTAGE: limited by what is known about $Z_{\mathcal{H}_n(q)} \dots$

IV CENTRAL QUESTIONS

Let's review what we know about $\mathbb{Z}G_n$:

(1) BASIS of CONJUGACY CLASS SUMS:

$$e_\mu = \sum_{\substack{\omega \text{ of cycle} \\ \text{type } \mu}} \omega$$

(2) CHARACTER FORMULAS for irreducible reps

$$\chi^\lambda(\mu) := \chi^\lambda(\omega) \quad \text{for } \lambda \vdash n$$

actually well-defined!

ANY ω of cycle type μ .

For $\mathcal{H}_n(q) \dots$

Analog of conjugacy class sums is called the

GECK-ROUQUIER BASIS $\{\pi_\mu\}_{\mu \vdash n}$

THEOREM (Geck-Rouquier) $\{\pi_\mu\}$ are unique central elements

$$(1) \pi_\mu|_{q=1} = \mathcal{C}_\mu.$$

(2) if w is of minimal length in its conjugacy class,

$$[Tw] \pi_\mu = \begin{cases} 1 & \text{if } w \text{ has cycle type } \mu \\ 0 & \text{if } w \text{ has cycle type } \lambda \neq \mu \end{cases}$$

EXAMPLE:

$$\begin{aligned} \pi_{(3,1)} = & (1-q^{-1}) \left((q^{-1}-q^{-2}) T_{(14)(23)} + q^{-1} T_{(1423)} + q^{-1} T_{(1324)} + 2q^{-1} T_{(14)} + T_{(133)} + T_{(24)} \right) \\ & + q^{-1} T_{(142)} + q^{-1} T_{(143)} + q^{-1} T_{(124)} + T_{(132)} + T_{(123)} + T_{(243)} + T_{(234)} \end{aligned}$$

CHARACTERS FOR $\mathcal{H}_n(q)$

* In semisimple setting, irreducibles indexed by $\lambda \vdash n$

* Ram gave character formula

$$\chi^\lambda(T_\omega) \quad \text{for } \omega \text{ of minimal length in its conjugacy class}$$

$\hookrightarrow$ otherwise compute recursively, using the $\{\rho_\mu\}$

UPSHOT: To fully understand $\chi^\lambda(T_\omega)$,

need to know expressions for ρ_μ .

QUESTION:

* Are there explicit formulas in Tw basis for

(1) P_μ

(2) $\chi^\lambda(P_\mu) := \sum_{w \in \mathfrak{S}_n} ([Tw] P_\mu) \chi^\lambda(Tw) ?$

Partial progress:

* **PROPOSITION** (Bastidas, B, Guay-Paquet, Park, Morales, Salasola)

$$P_{(2^k, 1^{n-2k})} = \sum_{\substack{w \text{ of} \\ \text{cycle type} \\ (2^k, 1^{n-2k})}} q^{\frac{n-\ell(w)}{2}} Tw$$

* **CONJECTURE** (Bastidas, B, Guay-Paquet, Park, Morales, Salasola)

Expression for $\chi^{(2n-k, 1^k)}(P_{2^n})$

$$\text{deforming } \chi^{(2n-k, 1^k)}(e_{2^n}) = \binom{2n-1}{k} \prod_{\substack{j \in [k, 2n-k-1] \\ j \text{ odd}}} j$$

HOW DO WE MULTIPLY THE P_n ?

Better to think in terms of **REDUCED PARTITIONS...**

↳ Take λ and remove first column to obtain $\tilde{\lambda}$

e.g. $\lambda =$  $\tilde{\lambda} = (2,1) =$ 

INTUITION: if we think in terms of cycle type, reduced partitions get "rid of" 1-cycles.

e.g. $(12) = (12)(3)(4) \dots (n) \in G_n$ has cycle type

$$\lambda = (2, 1^{n-2}) =$$
 

$$\tilde{\lambda} = (1) =$$
 

← independent of n

For $q=1$: How do we multiply the $\mathcal{C}_\mu$?

* In ZG_n : $\mathcal{C}_\lambda \cdot \mathcal{C}_\mu = \sum_{\nu \vdash n} \varphi_{\lambda, \mu}^\nu \mathcal{C}_\nu$

CONNEXION COEFFICIENTS

THEOREM (Frobenius - Higman)

The $\varphi_{\lambda, \mu}^\nu \in \mathbb{Q}[n]$ and depend only on $\tilde{\nu}, \tilde{\lambda}, \tilde{\mu}$ and n .

Moreover,

$$\varphi_{\lambda, \mu}^\nu = \begin{cases} 0 \\ \text{INDEPENDENT of } n \\ \text{??} \\ \uparrow \\ \text{HARD} \end{cases}$$

$$|\tilde{\nu}| > |\tilde{\lambda}| + |\tilde{\mu}|$$

$$|\tilde{\nu}| = |\tilde{\lambda}| + |\tilde{\mu}|$$

$$|\tilde{\nu}| < |\tilde{\lambda}| + |\tilde{\mu}|$$

"TOP CONNEXION COEFFICIENTS" for $q=1$:

$$\varphi_{\lambda, \mu}^{\nu} \quad \text{where} \quad |\tilde{\nu}| = |\tilde{\lambda}| + |\tilde{\mu}|$$

THEOREM (MacDonald)

When $\tilde{\nu} = (n-1)$ $\tilde{\mu} = (j)$ and $|\tilde{\lambda}| = n-1-j$,

$$\varphi_{\lambda, \mu}^{\nu} = \frac{j! \cdot n}{\prod_{i \geq 0} m_i(\tilde{\lambda})!}$$

where $m_0(\tilde{\lambda}) = j+1 - \ell(\tilde{\lambda})$

← counts multiplicities of parts of λ

e.g. $\lambda =$

$\mu =$

$\nu =$

$\tilde{\lambda} = (1)$

$\tilde{\mu} = (2)$

$\tilde{\nu} = (3)$

$\Rightarrow \varphi_{\lambda, \mu}^{\nu} = 4.$

$|\tilde{\lambda}| = 1 = 4-3$

$|\tilde{\mu}| = 2 = 4-2$

$|\tilde{\nu}| = 3 = 4-1$

* Goulden-Jackson compute generating function for general top connexion coefficients

How Do WE MULTIPLY THE Γ_μ ?

$$\Gamma_\lambda \Gamma_\mu = \sum_\nu \varphi_{\lambda,\mu}^\nu(q) \Gamma_\nu$$

THEOREM (Melisot)

The $\varphi_{\lambda,\mu}^\nu(q) \in \mathbb{Q}[q^\pm, n]$ and depend only on $\tilde{\nu}, \tilde{\lambda}, \tilde{\mu}, n$

THEOREM (Francis - Wang)

$$\varphi_{\lambda,\mu}^\nu(q) = \begin{cases} 0 \\ \text{INDEPENDENT of } n \end{cases}$$

??

↑
HARD

$$|\tilde{\nu}| > |\tilde{\lambda}| + |\tilde{\mu}|$$

$$|\tilde{\nu}| = |\tilde{\lambda}| + |\tilde{\mu}|$$

$$|\tilde{\nu}| < |\tilde{\lambda}| + |\tilde{\mu}|$$

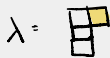
QUESTION:

"TOP CONNEXION COEFFICIENTS"

* What are

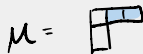
$$\varphi_{\lambda, \mu}^{\nu}(q) \quad \text{for} \quad |\tilde{\nu}| = |\tilde{\lambda}| + |\tilde{\mu}|?$$

EXAMPLE:



$$\tilde{\lambda} = (1)$$

$$|\tilde{\lambda}| = 1 = 4 - 3$$



$$\tilde{\mu} = (2)$$

$$|\tilde{\mu}| = 2 = 4 - 2$$



$$\tilde{\nu} = (3)$$

$$|\tilde{\nu}| = 3 = 4 - 1$$

MacDonald's formula says $\varphi_{\lambda, \mu}^{\nu} = 4$

$$\text{data says: } \varphi_{\lambda, \mu}^{\nu}(q) = 1 + q^{-2} [3]_{q^2}$$

$$\text{Data suggests for } \begin{matrix} \tilde{\nu} = (n-1) \\ \tilde{\mu} = (n-2) \end{matrix} \quad \varphi_{\lambda, \mu}^{\nu}(q) = 1 + q^{-(|\tilde{\mu}|)} [|\tilde{\nu}|]_{q^2}$$

IN SUMMARY... What we know:

CLASSICAL:

A-B-C's for G_n

A

$$a(n, n-1) = n^{n-2} \\ = [T_n] e_1 (J_1 \dots J_n)^{n-1}$$

B

$$b(n, n-1) = C_{n-1} \\ = [T_n] h_{n-1} (J_1 \dots J_n)$$

C

$$c(n; k, n-1-k) = N_{n, k+1} \\ = [T_n] e_{k, n-1-k} (J_1 \dots J_n)$$

(Bustidas, B. Guay-Piquet, Park, Morales, Salvia)

A-B-C's for $H_n(q)$...

$$a(n, n-1; q) = q^{-\binom{n}{2}} [n]_q^{n-2} \\ = [T_n] e_1 (J_1(q), \dots, J_n(q))^{n-1}$$

$$b(n, n-1; q) = q^{-\binom{n}{2}} C_{n-1}(q) \\ = [T_n] h_{n-1} (J_1(q) \dots J_n(q))$$

$$c(n; k, n-1-k; q) = q^{1 - (kn)(n-k)} N_{k, n+1}(q) \\ = [T_n] e_{k, n-1-k} (J_1(q) \dots J_n(q))$$

IN SUMMARY... what we would like to know...

EXPRESSIONS (or even efficient algorithms) for

* **GECK-ROUQUIER BASIS** Π_μ

* **CHARACTERS** for this basis $\chi^\lambda(\Pi_\mu)$

* **TOP CONNEXION COEFFICIENTS** $\varphi_{\lambda, \mu}^\nu(q)$

THANK
YOU!

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THEOREM (Bastidas, B. Guay-Pequet, Park, Morales, Saliola, 25)

$$+ [T_{\tau_n}] h_\lambda (J_1(q) \dots J_n(q)) = \frac{q^{-\binom{n}{2}}}{[n]_q} \prod_{i=1}^{\ell(\lambda)} \begin{bmatrix} n + \lambda_i - 1 \\ \lambda_i \end{bmatrix}_q$$

$$+ [T_{\tau_n}] p_\lambda (J_1(q) \dots J_n(q)) = \frac{q^{-\binom{n}{2}}}{[n]_q} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{n \lambda_i}}{1 - q^{\lambda_i}}$$

$$+ [T_{\tau_n}] s_\lambda (J_1(q) \dots J_n(q)) = \frac{q^{b(\lambda) - \binom{n}{2}}}{[n]_q} \prod_{(i,j) \in \lambda} \frac{1 - q^{n+j-i}}{1 - q^{h(i,j)}}$$

where $b(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1) \lambda_i$

and $h(i,j)$ = hook length of box (i,j) .