Fibers of Maps to Totally Nonnegative Spaces

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- joint work with Jim Davis & Ezra Miller

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- slides at: https://pages.uoregon.edu/plhersh/ICERM-Sept25.pdf

Defn: A real matrix is totally nonnegative (TNN) if all its minors are nonnegative.

Thm (Whitney in type A; Lusztig for semisimple simply conn. alg groups)

The unipotent TNN matrices $\left\{\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots \end{pmatrix}\right\}$ are products of exponentiated Chevalley generators

how these look in type A

$$x_i(t) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots \ 1 \end{pmatrix} \text{ for } t \in \mathbb{R}_{\geq 0}$$

type A

$\leftarrow$ row i

column $i+1$

Today: When are products of $x_i(t)$'s equal?

Map Whose Fibers We Study:

$$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \to TNN(\underset{\text{unipotent radical}}{U_n})$$

$$(t_1,\dots,t_d) \mapsto x_{i_1}(t_1)\,x_{i_2}(t_2)\cdots x_{i_d}(t_d)$$

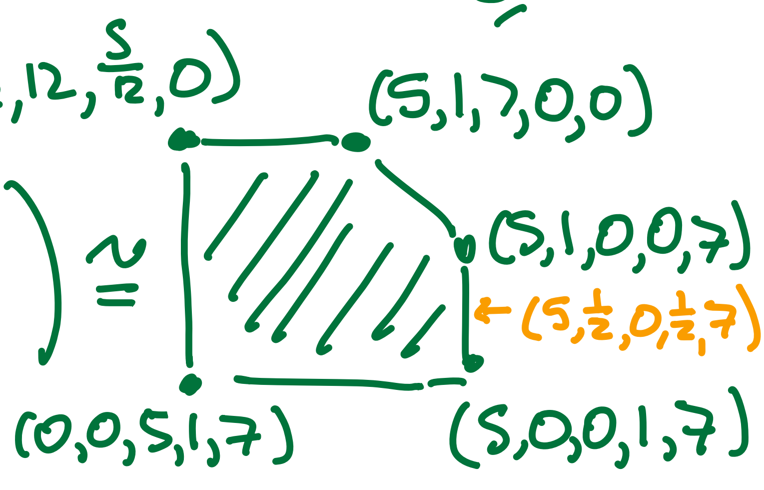
e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ & 1 & t_2 \\ & & 1 \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}}_{x_2(t_2)} \underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)}$$

Example of Fiber

$\subset \mathbb{R}^5_{\geq 0}$

$$f^{-1}_{(1,2,1,2,1)}\begin{pmatrix} 1 & 12 & 5 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \cong$$



Some Related Past Work

- Lusztig (94): Studied totally nonneg. part in reductive groups (as image of map $f_{(i_1,\ldots,i_d)}$) & connected this to canonical bases
- Fomin-Shapiro (00): Results on Bruhat stratification of $\mathrm{im}(f_{(i_1,\ldots,i_d)})$ & conjectured it is regular CW ball.
- H. (14): Proof of Fomin-Shapiro Conj.
- Gakshin-Karp-Lam (22): Totally Nonneg part of any flag variety is regular CW ball & F-S Conj via Poincaré Conjecture.
- Loosely related: Positroid varieties, cluster algebras, braid varieties...

Some Motivations:

1. fibers of $f_{(i_1 \cdots i_d)}$ encode nonneg. real relations among exponentiated Chevalley generators in Lie theory

2. "braid relations" among x_i's:

$$x_i(a)x_{i+1}(b)x_i(c) = x_{i+1}\left(\frac{bc}{a+c}\right)x_i(a+c)x_{i+1}\left(\frac{ab}{a+c}\right)$$

tropicalize to change of coords

$$(a,b,c) \mapsto \left(b+c-\min(a,c),\ \min(a,c),\ a+b-\min(a,c)\right)$$

for Lusztig's dual canonical bases.

Stratif. we use: for each $p \in TNN(U_n)$ the stratification on $\mathbb{R}^d_{\geq 0}$ based on which coords. are positive vs. 0 induces stratif. for $f^{-1}_{(i_1 \cdots i_d)}(p) \cap \mathbb{R}^d_{\geq 0}$

Baby Example of Fiber (& how we Think About It)

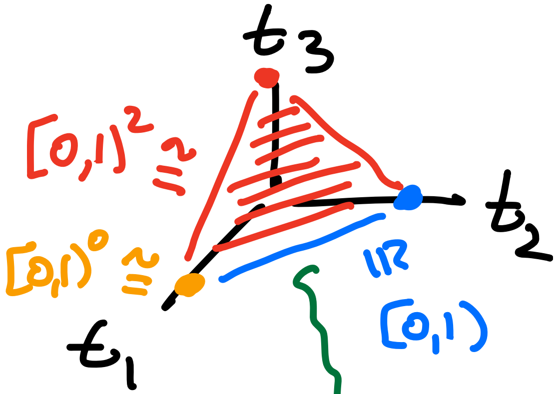
$$\overbrace{x_1(t_1)}^{\begin{pmatrix}1&t_1&\\&1&\\&&1\end{pmatrix}}\ \overbrace{x_1(t_2)}^{\begin{pmatrix}1&t_2&\\&1&\\&&1\end{pmatrix}}\ \overbrace{x_1(t_3)}^{\begin{pmatrix}1&t_3&\\&1&\\&&1\end{pmatrix}} = \overbrace{x_1(5)}^{\begin{pmatrix}1&5&\\&1&\\&&1\end{pmatrix}}$$

$$\|$$

$$x_1(t_1+t_2+t_3) = \begin{pmatrix}1&t_1+t_2+t_3&0\\&1&\\&&1\end{pmatrix}$$

$$\|$$

$$f_{(1,1,1)}(t_1,t_2,t_3)$$



Natural Description of Fiber:

$$f^{-1}_{(1,1,1)}(x_1(5)) = \left\{(t_1,t_2,t_3)\in\mathbb{R}^3_{\geq 0} \,\middle|\, \sum_{i=1}^{3} t_i = 5\right\}$$

More Useful Description (for param. & cell decomposition)

$$\overset{\begin{pmatrix}1 & S & \\ & 1 & \\ & & 1\end{pmatrix}}{\overbrace{}}$$

$$f^{-1}_{(1,1,1)}(x_1(S))$$

$$\parallel$$

$$\left\{(t_1,t_2,t_3)\in\mathbb{R}^3_{\geq 0}\;\middle|\;\begin{array}{l} 0\leq t_1\leq 5 \\ 0\leq t_2\leq \boxed{5-t_1} \\ t_3=\boxed{5-t_1-t_2}\end{array}\right\}$$

"t_2^{max}" continuous fn of t_1

"f_3" cont fn of t_1 & t_2

Sample Stratum:

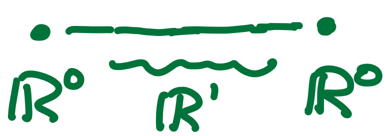
$0<t_1<S$

$0<t_2<S-t_1$

$t_3=S-t_1-t_2$

Observation: t_3 uniquely determined by t_1 & t_2 because rightmost s_1 in $s_1s_1\underline{\underline{s_1}}$ is in rightmost reduced word for $s_1=\delta(s_1,s_1,s_1)$ in nonreduced $s_1s_1s_1$.

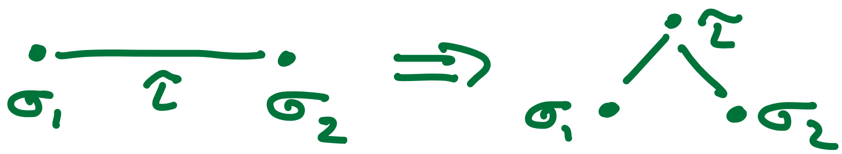
- A *cell decomposition* of topol. space X is decomp. into disjoint union of cells, namely pieces homeom. to $(0,1)^s \cong \mathbb{R}^s$ for various $s \geq 0$ e.g.



- A *cell stratification* is cell decomp. with $\sigma \cap \overline{\tau} \neq \emptyset \Rightarrow \sigma \subseteq \overline{\tau}$

- The *face poset* of a cell stratific. is partial order on cells with $\sigma \leq \tau \Longleftrightarrow \sigma \subseteq \overline{\tau}$

e.g.



Main Results for fibers

Combinatorial:

- stratification has same face poset as interior dual block complex of subword complex
- these interior dual block complexes are contractible

Topological

- each stratum is homeomorphic to $(0,1)^s$ for some $s \geq 0$.
- Parametrizations for collections of strata using $[0,1)^s$

Conjectural

- $f^{-1}_{(\iota,-id)}(p)$ is contractible regular CW complex.

How we Decide Whether Products have same Minors Positive

$$x_i(t) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = I_n + tE_{i,i+1}$$

(type A)

$$f_{(i_1,\dots,i_d)}(t_1,\dots,t_d) = x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

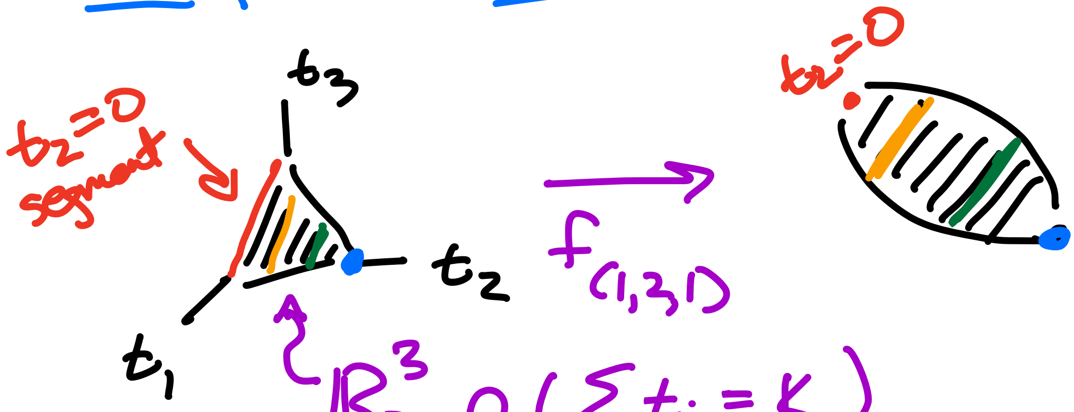
↑ reduced or nonreduced word

e.g.

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ & 1 & t_2 \\ & & 1 \end{pmatrix}$$

Example Continued



$\mathbb{R}^3_{\geq 0} \cap (\sum t_i = K)$

K: positive constant

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$\Big\downarrow t_2 = 0$

$$f_{(1,-,1)}(t_1,0,t_3) = \underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)}\underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} \Big\} x_1(t_1+t_3)$$

$$\{x_1(t) \mid t>0\} = \{x_1(t_1)x_1(t_3) \mid t_1,t_3>0\}$$

$$\Downarrow$$

$$x_i = x_i^2$$

Demazure Product δ (Equivalently Unsigned 0-Hecke Product) Governs Which Minors are Positive

- $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

  $x_i x_i \to x_i$ "modified nil move"

  $\rightsquigarrow \delta(s_i, s_i) = s_i$

- $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}(t_1')x_i(t_2')x_{i+1}(t_3')$

  for $t_1' = \dfrac{t_2t_3}{t_1+t_3}$; $t_2' = t_1+t_3$; $t_3' = \dfrac{t_1t_2}{t_1+t_3}$

  $x_i x_{i+1} x_i \to x_{i+1} x_i x_{i+1}$ "braid move"

  $\rightsquigarrow \delta(s_i, s_{i+1}, s_i) = \delta(s_{i+1}, s_i, s_{i+1})$

- $x_i(t_1)x_j(u) = x_j(u)x_i(t)$

  $\rightsquigarrow \delta(s_i, s_j) = \delta(s_j, s_i)$ for $|j-i|>1$

The Demazure product for Coxeter group W satisfies

$$\delta(s_{i_1}, s_{i_2}, \ldots, s_{i_d}) = \begin{cases} \delta(s_{i_2}, \ldots, s_{i_d}) & \text{if } \ell(u) < \ell(s_{i_1} u) \\ s_{i_1}\,\delta(s_{i_2}, \ldots, s_{i_d}) & \text{otherwise} \end{cases}$$

$u := \delta(s_{i_2}, \ldots, s_{i_d})$

e.g. $\delta(1,2,1,2,1) = ?$ $\quad \delta(1) = s_1 \Rightarrow$

$\delta(2,1) = s_2 s_1 \Rightarrow \delta(1,2,1) = s_1 s_2 s_1 \Rightarrow$

$\delta(2,1,2,1) = \underbrace{\delta(1,2,1)}_{= s_1 s_2 s_1} \Rightarrow \delta(1,2,1,2,1) = \underbrace{\delta(1,2,1)}_{= s_1 s_2 s_1}$

Fact: $f_{(i_1 \ldots i_d)}(\mathbb{R}^{Q}_{>0}) = f_{(i_1 \ldots i_d)}(\mathbb{R}^{Q'}_{>0})$

$\Longleftrightarrow \delta(Q) = \delta(Q')$

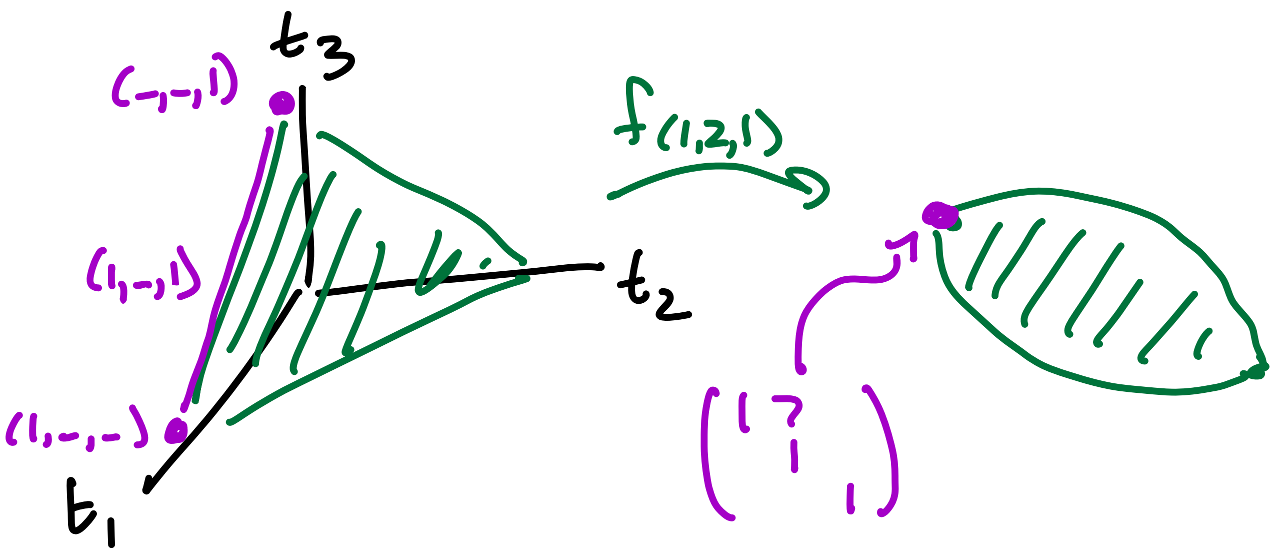
<u>Notation:</u> $U(w) = f_{(i_1 \ldots i_d)}(\mathbb{R}^Q_{>0})$

for $w = \delta(Q)$.

$$\overset{\shortparallel}{\left[B^- w B^- \cap \begin{matrix}\text{unipotent}\\ \text{subgp of } B\end{matrix} \right]_{\geq 0}}$$

e.g. $f^{-1}_{(1,2,1)} \begin{pmatrix} 1 & ? & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has nonempty strata given by subwords $(1,-,-)$, $(1,-,1)$, & $(-,-,1)$ of $(1,2,1)$ since

$\begin{pmatrix} 1 & ? & \\ & 1 & \\ & & 1 \end{pmatrix} \in U(s_1)$ for $s_1 = \delta(1,-,1) = \ldots$



Thm (Lusztig): (a) for $(i_1,\dots,i_d)$ reduced & $w=\delta(i_1,\dots,i_d)$, $f_{(i_1,\dots,i_d)}:\mathbb{R}^d_{>0}\to U(w)$ is homeomorphism.

(b) $U(w)\cap U(w')=\emptyset$ for $w\neq w'$.

A Key Step in Cell Stratif. for Fibers:

Substantially generalize (a) above,

e.g. show that map

$(t_1,t_2,t_3,t_4)\mapsto x_4(1)x_2(5)x_4(t_1)x_1(3)x_2(t_2)x_1(t_3)x_2(t_4)$

$\cap$

$\mathbb{R}^4_{>0}$

rightmost red. word for

$\delta(4,2,4,1,2,1,2)=s_4s_2s_1s_2$ in $(4,2,4,1,2,1,2)$

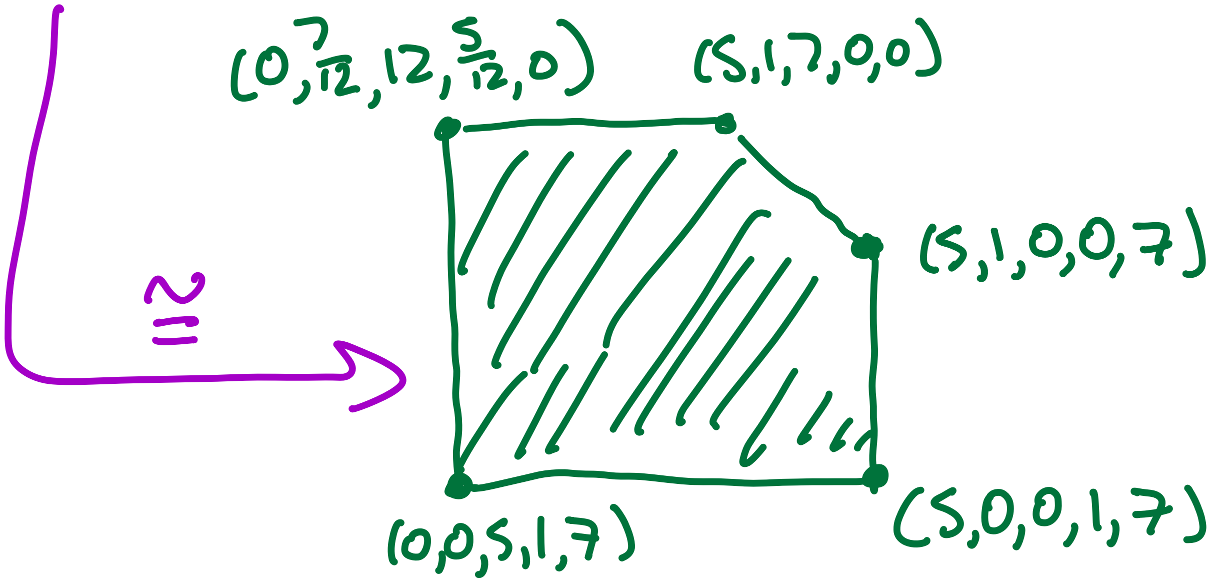
is homeomorphism from $\mathbb{R}^4_{>0}$ to its image, in particular is injective

Example of Fiber (Revisited):

$$f_{(1,2,1,2,1)}(t_1, t_2, \ldots, t_5) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix} \cdots$$

$$= \begin{pmatrix} 1 & t_1+t_3+t_5 & t_1t_2+t_1t_4+t_3t_4 \\ & 1 & t_2+t_4 \\ & & 1 \end{pmatrix}$$

$f^{-1}_{(1,2,1,2,1)}(M)$ for $M = \begin{pmatrix} 1 & 5 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 & \\ & 1 & \\ & & 1 \end{pmatrix}$

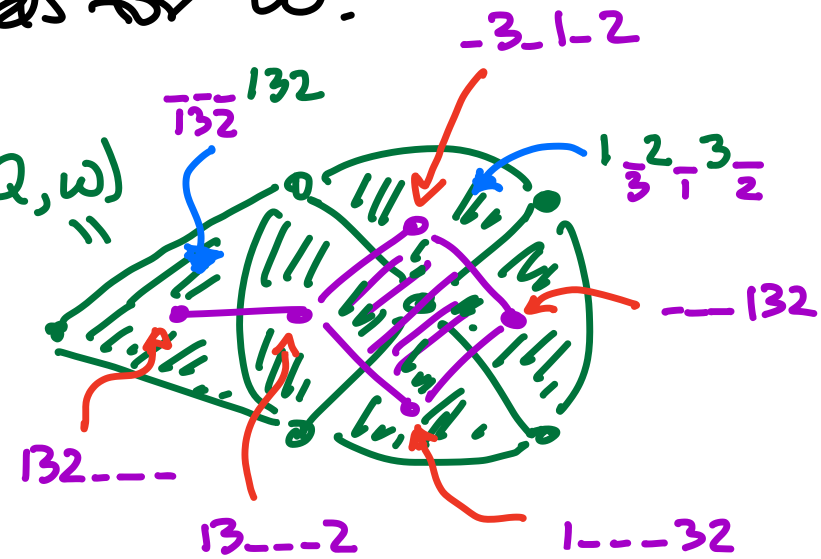


vertices $\longleftrightarrow$ reduced subwords for $s_1 s_2 s_1$ in $(1,2,1,2,1)$

Subword Complexes & their Interior Dual Block Complexes

Defn (Knutson-Miller): The subword complex $\Delta(Q,w)$ has vertices given by letters in word Q & facets the complements of subwords that are reduced words for w.

e.g. $\Delta(Q,w) =$



$Q=(1,3,2,1,3,2)$

$w=s_1 s_3 s_2$

with its interior dual block complex in purple

Map f of Face Posets Induced by Map $f_{(i_1,\dots,i_d)}$ of Topol. Spaces

$$\begin{tikzcd}[column sep=small]
 & (1,2,1) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & & & s_1s_2s_1 \arrow[dl, dash] \arrow[dr, dash] & \\
(1,2,-) \arrow[d, dash] \arrow[dr, dash] & (1,-,1) \arrow[dl, dash] \arrow[dr, dash] & (-,2,1) \arrow[dl, dash] \arrow[d, dash] & \arrow[r, "f"] & s_1s_2 \arrow[d, dash] \arrow[drr, dash] & & s_2s_1 \arrow[dll, dash] \arrow[d, dash] \\
(1,-,-) \arrow[dr, dash] & (-,2,-) \arrow[d, dash] & (-,-,1) \arrow[dl, dash] & & s_1 \arrow[dr, dash] & & s_2 \arrow[dl, dash] \\
 & (-,-,-) & & & & e &
\end{tikzcd}$$

Boolean lattice = face poset of simplex

Bruhat order = face poset of $TNN(U_n)$

"Demazure product"

$$f(i_{j_1},\dots,i_{j_\ell}) = \delta(s_{i_{j_1}},\dots,s_{i_{j_\ell}})$$

e.g. $f(1,-,1) = \delta(s_1,s_1) = s_1$

$$\begin{tikzcd}[column sep=small]
 & e \arrow[dl, dash] \arrow[dr, dash] & \\
v_1 & & v_2
\end{tikzcd} = F\left(\bullet_{v_1} \overset{}{\underset{e}{\text{—}}} \bullet_{v_2}\right)$$

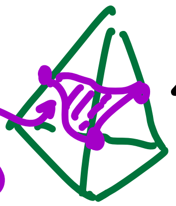
Thm (DHM): For each $u \in W$, $f^{-1}(u) := \{x \in B_n \mid f(x) = u\}$ is face poset for interior dual block complex of $\Delta((i_1, \dots, i_d), u)$.

Thm (DHM): The interior dual block complex (IDBC) of every subword complex is regular CW complex & is contractible.

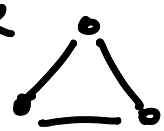
Technical Detail: If sphere, then include dual to empty face in I.D.B.C.

e.g.

$f^{-1}_{(i_1 \dots i_d)}(P)$



interior dual block complex of:

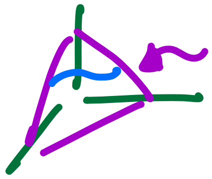


$\Delta((1,1,1,2), s_1 s_2)$

Properties of Subword Complexes We'll Use

Knutson-Miller: Each subword complex is "vertex decomposable", hence "shellable" & pure (all max faces same dim.) & homeomorphic to ball or sphere.

Corollary: They are gallery connected, i.e. max faces connected thru codimension one faces.

 $\cap\, \Delta^{d-1}_K$

Application:

$$f^{-1}_{(i_1,\dots,i_d)}(p) \cap \mathbb{R}^d_{\geq 0} = f^{-1}_{(i_1,\dots,i_d)}(p) \cap \Delta^{d-1}_K$$

for some $K>0$

coord. sum K part of $\mathbb{R}^d_{\geq 0}$

Proof:

- show change of coord's $(t_1,\dots,t_d) \mapsto (t_1',\dots,t_d')$ for braid & modified nil moves preserve sum of parameters

  e.g. $x_1(t_1)x_1(t_2) \mapsto x_1(t_1+t_2)$ $\quad K = t_1+t_2$

- use gallery-connectedness of subword complexes to show state connected by braid & modified nil-moves

Idea for Parametrizing Cell Collections

- Consider rightmost subword Q of $(i_1, \ldots, i_d)$ that is reduced word for $\delta(i_1, \ldots, i_d)$

  e.g. $(3, 1, 2, 1, 2, 1, 2)$

- Parametrize the points in those strata satisfying $t_1, t_3, t_6, t_7 > 0$

  in $f^{-1}_{(3,1,2,1,2,1,2)}(p)$, $p \in U(s_3 s_2 s_1 s_2)$

  using the set $[0,1)^3 \cong [0,A) \times [0,B) \times [0,C)$ for $A, B, C > 0$

- can do this since

$t_1, t_5, t_6, t_7 > 0 \iff t_2, t_3, t_4$ satisfy:

$$t_2 \in [0, t_2^{max})$$
$$t_3 \in [0, t_3^{max}(t_2))$$
$$t_4 \in [0, t_4^{max}(t_2, t_3))$$

continuous functions to $\mathbb{R}_{>0}$

e.g.

$$x_3(t_1)x_1(t_2)x_2(t_3)x_1(t_4)x_2(t_5)x_1(t_6)x_2(t_7)$$

choice of (nonmax.) values $\cong [0,1)$

values determined by other params

Cell Stratification: Key Lemmas (& Definitions)

Def'n: Consider word $(i_1, \dots, i_d)$.

The letter i_ℓ is redundant in $(i_1, \dots, i_d)$ if $\delta(i_1, \dots, i_d) = \delta(i_1, \dots, \hat{i}_\ell, \dots, i_d)$ and nonredundant otherwise.

e.g. $\delta(\underbrace{1}_{\text{redundant}},2,\underbrace{1}_{\text{nonredundant}},2) = s_2 s_1 s_2 = \delta(2,1,2) \neq \delta(1,2,2)$

Note:

(1) i_1 nonredundant $\iff$

$$\delta(i_1, i_2, \dots, i_d) = s_{i_1} \delta(i_2, \dots, i_d)$$

(2) $f_{(i_1 \dots i_d)}(k_1 \dots k_d) = p \iff f_{(i_2 \dots i_d)}(k_2 \dots k_d) = x_{i_1}(-k_1) p$

Lemma: If i_1 is nonredundant in $(i_1, \dots, i_d)$, then there is unique value k_1 for t_1 s.t.

$$f_{(i_1, \dots, i_d)}(k_1, t_2, \dots, t_d) = p$$

has a solution.

This has $x_{i_1}(-k_1)p \in \mathcal{U}(\mathcal{S}(i_2, \dots, i_d))$

Lemma: If i_1 is redundant in $(i_1, \dots, i_d)$, then there exists $t_1^{max} > 0$ s.t. $f_{(i_1, \dots, i_d)}(k_1, t_2, \dots, t_d) = p$ has solution $\iff k_1 \in [0, t_1^{max}]$.

Moreover, $k_1 \in [0, t_1^{max})$ implies $x_{i_1}(-k_1)p \in \mathcal{U}(\mathcal{S}(i_2, \dots, i_d))$.

Lemma: Given i_j nonredundant in $(i_j,\dots,i_d)$ & given $k_1,\dots,k_{j-1}\geq 0$ s.t.

$$x_{i_{j-1}}(-k_{j-1})\cdots x_{i_1}(-k_1)p \in U(\delta(i_j,\dots,i_d))$$

then there is a unique value $k_j = f_j(k_1,\dots,k_{j-1}) \in \mathbb{R}_{\geq 0}$ for t_j s.t.

$$f_{(i_1,\dots,i_d)}(k_1,k_2,\dots,k_{j-1},k_j,t_{j+1},\dots,t_d) = p$$

has solution with $t_{j+1},\dots,t_d \in \mathbb{R}_{\geq 0}$.

e.g. $f^{-1}_{(1,2,1,2)}\Big(x_1\underbrace{(5)x_2(7)x_1(3)}_{\begin{pmatrix}1 & 8 & 35\\ & 1 & 7\\ & & 1\end{pmatrix}}\Big)$

has $t_1\in[0,5]$ and

- $f_2(k_1) = \frac{21}{8-k_1}$ "=" k_2
- $f_3(k_1,k_2) = 8-k_1$
- $f_4(k_1,k_2,k_3) = 7-k_2$

Lemma: Given i_ℓ redundant in $(i_\ell, \dots, i_d)$ & any $k_1, \dots, k_{\ell-1} \geq 0$ s.t.

$$x_{i_{\ell-1}}(-k_{\ell-1}) \cdots x_{i_1}(-k_1)\, p \in U(\delta(i_\ell \cdots i_d))$$

then t_ℓ takes exactly the values in $[0, K]$ for some $K > 0$.

$$K = \text{"}t_\ell^{max}(k_1, \dots, k_{\ell-1})\text{"}$$

Example: $M = \begin{pmatrix} 1 & \pi & e \\ & 1 & 14 \\ & & 1 \end{pmatrix}$ then

$$f^{-1}_{(1,2,1,2)}(M) = \{(t_1, t_2, t_3, t_4) \mid x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = M\}$$

achieves every $t_1 \in [0, \underbrace{\tfrac{e}{14}}_{\text{"}t_1^{max}\text{"}}]$

Useful Characterization of which Parameters are Uniquely Determined by Parameters to Left

Lemma: $S^c = \{j_1,\dots,j_s\} \subseteq \{1,\dots,d\}$ indexes rightmost subword of $(i_1,\dots,i_d)$ which is reduced word for $\delta(i_1,\dots,i_d)$

$$\Longleftrightarrow \quad S^c = \left\{ j \in [d] \;\middle|\; \begin{array}{l} i_j \text{ is nonredundant} \\ \text{in } (i_j,\dots,i_d) \end{array} \right\}$$

e.g. $(1,2,1,2,4,1,5,2,4,5)$ $\quad d=10$

$$1\quad 2\quad 3\quad \underset{\sim}{4}\quad 5\quad \underset{\sim}{6}\quad \underset{\sim}{7}\quad \underset{\sim}{8}\quad \underset{\sim}{9}\quad \underset{\sim}{10}$$

$$S^c$$

$S^c = \{4,6,7,8,9,10\}$ since $(2,1,5,2,4,5)$ is rightmost reduced word for $\delta(1,2,1,\dots,4,5)$

$S = \{1,2,3,5\}$ **Notation**: $S = \{j_1',\dots,j_{d-s}'\}$

Domain for Homeom. from $[0,1)^s$ to Union of Strata

Notation: $S^c = \{j_1, \ldots, j_{\ell(\omega)}\}$ s.t. $(i_{j_1}, \ldots, i_{j_{\ell(\omega)}})$ is rightmost reduced word for $\omega = \delta(i_1 \ldots i_d)$ in $(i_1, \ldots, i_d)$.

$$D^{<\max} := \{(t_1, \ldots, t_d) \mid t_j < t_j^{\max}(t_1 \ldots t_{j-1}) \text{ for all } j \in S\} \quad \stackrel{\in}{} f^{-1}_{(i_1 \ldots i_d)}(p)$$

Thm (DHM): $D^{<\max} = \bigcup \sigma$

$\cong$ $[0,1)^{d-\ell(\omega)}$

strata σ s.t. $v \subset \bar{\sigma}$ for v of support S^c

Cor: Each stratum $\cong (0,1)^s$ for some $s \geq 0$

Generalization of Lusztig Result

Lemma: Given $\mathcal{S}(i_1, \ldots, i_d) = \omega$ & $D_{k_{j_1}, \ldots, k_{j_s}} =$

$$\left\{ (t_1, \ldots, t_d) \in \mathbb{R}^d_{\geq 0} \;\middle|\; \begin{array}{l} t_{j'_\ell} > 0 \text{ for } j'_\ell \in S^c \\ t_{j_r} = k_{j_r} \text{ for } j_r \in S \end{array} \right\}$$

for any fixed $k_{j_1}, \ldots, k_{j_s} \geq 0$, then

$f_{(i_1, \ldots, i_d)}\big|_{D_{k_{j_1}, \ldots, k_{j_s}}}$ is a homeomorphism

to its image within $U(\omega)$.

e.g.

$$\underset{\in\, D_{3,7} \cong \mathbb{R}^4_{>0}}{(t_2, t_4, t_5, t_6)} \longmapsto \underset{\in\, U(s_4 s_1 s_2 s_1)}{x_1(3)\, x_4(t_2)\, x_2(7)\, x_1(t_4)\, x_2(t_5)\, x_1(t_6)}$$

Consequence: Within $D^{<\max}$, redundant parameter values determine values for nonredundant parameters

First Example, Revisited

$$x_1(t_1)\, x_1(t_2)\, x_1(t_3) = x_1(5)$$

$$\overset{\shortparallel}{f_{(1,1,1)}}(t_1, t_2, t_3)$$

Constraints for $D^{<\max}$ part of Fiber

- $0 \le t_1 < t_1^{\max} = 5$
- $0 \le t_2 < t_2^{\max}(k_1) = 5 - k_1$
- $t_3 = 5 - k_1 - k_2 =$ "$f_3(k_1, k_2)$"

$$[0,1)^2 \cong \left\{ (\underbrace{k_1}_{[0,1)}, \underbrace{k_2}_{[0,1)}, k_3) \;\middle|\; \begin{array}{l} 0 \le k_1 < 5 \\ 0 \le k_2 < 5 - k_1 \\ k_3 = 5 - k_1 - k_2 \end{array} \right\}$$

this uses continuity of $t_2^{\max}$ & f_3

Continuity Lemmas

Lemma: Given i_ℓ non-redundant in $(i_\ell, \dots, i_d)$ and $k_1, \dots, k_{\ell-1} \geq 0$ s.t.

(1) $k_i < t_i^{max}(k_1, \dots, k_{i-1}) \ \forall i \in S$ &

(2) $\exists (k_1, \dots, k_{\ell-1}, t_\ell, \dots, t_d) \in f^{-1}_{(i_1, \dots, i_d)}(p) \cap \mathbb{R}_{\geq 0}$

then f_ℓ is continuous function, letting $f_\ell(k_1, \dots, k_{\ell-1})$ be the unique value for t_ℓ in $f^{-1}_{(i_1 \dots i_d)}(p) \cap \mathbb{R}^d_{\geq 0}$ s.t. $t_i = k_i$ for $i = 1, 2, \dots, \ell - 1$.

e.g. $f^{-1}_{(1,2,1,2)}(x_1(5)x_2(7)x_1(3))$

- $t_1^{max} = 5$
- $f_2(k_1) = \dfrac{21}{8 - k_1}$
- $f_3(k_1, k_2) = 8 - k_1$
- $f_4(k_1, k_2, k_3) = 7 - k_2 = 7 - \dfrac{21}{8 - k_1}$

Idea: $x_{i_1}(k_1)\cdots x_{i_{\ell-1}}(k_{\ell-1})\,x_{i_\ell}(t_\ell)\cdots x_{i_d}(t_d)=p$

$$\Updownarrow$$

$$x_{i_\ell}(t_\ell)\cdots x_{i_d}(t_d)=x_{i_{\ell-1}}(-k_{\ell-1})\cdots x_{i_1}(-k_1)\,p$$

so $\{(t_\ell,\dots,t_d)\mid(k_1,\dots,k_{\ell-1},t_\ell,\dots,t_d)\in f^{-1}_{(i_1,\dots,i_d)}(p)\}$

$$\|$$

$$f^{-1}_{(i_\ell,\dots,i_d)}\big(\underbrace{x_{\ell-1}(-k_{\ell-1})\dots x_1(-k_1)\,p}_{\text{continuous function of } k_1,\dots,k_{\ell-1} \text{ for fixed } p}\big)$$

so f_ℓ is coordinate fn composed with $f^{-1}_{(i_\ell,Q)}$ where (i_ℓ,Q) is reduced subword of $(i_\ell,\dots,i_d)$

Lemma: If i_ℓ is redundant in $(i_\ell, \dots, i_d)$, then t_ℓ^{max} is continuous fn of $k_1, \dots, k_{\ell-1}$

Proof Idea:

$$\underbrace{t_\ell^{max}(k_1, \dots, k_{\ell-1})}_{\substack{\text{for word} \\ (i_1,\dots,i_d)\,\{ \\ f^{-1}_{(i_1-i_d)}(p)}} = \underbrace{f_\ell(k_1, \dots, k_{\ell-1})}_{\substack{\text{for subword } (i_1 - i_\ell, Q) \\ \{ f^{-1}_{(i_1-i_\ell, Q)}(p) \text{ where} \\ (i_\ell, Q) \text{ is reduced word}}}$$

so f_ℓ cont. $\Rightarrow$ t_ℓ^{max} continuous

Example: $f_{(1,1,1,2)}(t_1, t_2, t_3, t_4) = x_1(7)x_2(5)$

$$t_2^{max}(k_1) = 7 - k_1 \quad (\text{for } f_{(1,1,1,2)})$$

$$x_1(7)x_2(5) = p = f_{(1,1,-,2)}(t_1, t_2, -, t_4) = x_1(7)x_2(5)$$

$$\| $$

$f_2(k_1)$ $\left(\begin{matrix}\text{for} \\ f_{(1,1,2)}\end{matrix}\right)$ $(1,1,2)$ nonreduced

$(1,-,2)$ reduced $\quad Q = (2)$

Remark: We focus on $D^{<\max}$ where $t_\ell < k_\ell^{\max}$ for all $\ell \in S$ because $t_\ell = k_\ell^{\max}$ part much less well behaved.

Summary: $(0,1)^S \xrightarrow{\cong} \bigcup_{v \le \bar{\sigma} \le \bar{\tau}} \sigma$ which restricts to $(0,1)^S \xrightarrow{\cong} \tau$, showing each stratum is open ball.

An Open Qu: Given word Q, a subword $(i_{j_1}, \dots, i_{j_\ell})$ that is reduced word for $\delta(Q)$ and constants $\{k_j \ge 0 \mid j \notin \{j_1, \dots, j_\ell\}\}$, is $(t_{j_1}, \dots, t_{j_\ell}) \mapsto f_Q(u_1, \dots, u_d)$ for $u_j = \begin{cases} t_j & \text{if } j \in \{j_1, \dots, j_\ell\} \\ k_j & \text{otherwise} \end{cases}$ homeom to image?

e.g. $(t_1, t_6) \mapsto x_1(t_1) x_2(3) x_1(2) x_2(7) x_1(\pi) x_2(t_6)$

Equations w/ Unique Solution in $\mathbb{R}_{>0}$ by Lusztig Injectivity Result

$$f_{(1,2)} : (t_1, t_6) \longmapsto \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_6 \\ & & 1 \end{pmatrix} = \underbrace{M}_{\substack{\text{constant} \\ \text{matrix} \\ \text{in } U(s_1 s_2)}}$$

e.g. $t_1 = M_{12}$; $t_1 t_6 = M_{13}$

$t_6 = M_{23}$

Modified Equations for

$$(t_1, t_6) \longmapsto \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & 3 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & 7 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \pi & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_6 \\ & & 1 \end{pmatrix}$$

e.g. $\underbrace{x_1(t_1)}_{s_1}\ \underbrace{x_2(3)}_{s_2}\ \underbrace{x_1(2)}_{s_1}\ \underbrace{x_2(7)}_{s_2}\ \underbrace{x_1(\pi)}_{s_1}\ \underbrace{x_2(t_6)}_{s_2}$ ---

$$t_1 + 2 + \pi = \underbrace{M_{12}}_{\text{"}s_1\text{" minor}} \qquad 3 + 7 + t_6 = \underbrace{M_{23}}_{\substack{\text{"}s_2\text{"} \\ \text{minor}}}$$

$$t_1 \cdot 3 + t_1 \cdot 7 + t_1 t_6 + 2 \cdot 7 + 2 \cdot t_6 + \pi t_6 = \underbrace{M_{13}}_{\substack{\text{"}s_1 s_2\text{"} \\ \text{minor}}}$$

Conjecture (Davis–H.–Miller):

$f^{-1}_{(i_1,\dots,i_d)}(p) \cap \mathbb{R}^d_{\geq 0}$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\dots,i_d),w)$ where $p \in U(w)$.

Thanks!