Monoidal categorification and crystals

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Categorification

Question (Categorification)

Given an algebraic object A, find a category $\mathcal{C}$ with functors F such that

$$(K(\mathcal{C}), [F]) \simeq A,$$

where $K(\mathcal{C})$ is the Grothendieck group of $\mathcal{C}$.

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Example Let A be a ring. Then the question can be written as follows:

Find a good abelian category $\mathcal{C}$ with $\otimes$ such that

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Question Can we consider a categorification for cluster algebras (algebra + cluster structure)?

Cluster algebras

Cluster algebras were introduced by Fomin-Zelevinsky ([FZ02]).

- Index set $K := K^{ex} \cup K^{fr}$,
- Exchange matrix $\widetilde{B} := (b_{i,j})_{(i,j) \in K \times K^{ex}}$ such that

$$\{\widetilde{B}\} \overset{1-1}{\longleftrightarrow} \text{quiver with no loops and no 2 cycles}$$

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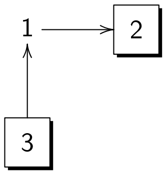
Example

Let $K = \{1,2,3\}$ with $K^{ex} = \{1\}$ and $K^{fr} = \{2,3\}$.

$$\widetilde{B} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \longleftrightarrow$$



Let A be a commutative $\mathbb{Q}$-algebra contained in a field F.

- The pair $S = (\{x_i\}_{i \in K}, \widetilde{B})$ is a seed in A if x_i's are alg. indep. and $\widetilde{B}$ is an exchange matrix.

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- For $t \in K^{ex}$, the mutation $\mu_t(\{x_i\}, \widetilde{B}) = (\{x_i'\}, \widetilde{B}')$ is given by

$$(\widetilde{B}')_{i,j} := \begin{cases} -b_{i,j} & \text{if } i = t \text{ or } j = t, \\ b_{i,j} + (-1)^{\delta(b_{i,t}<0)} \max(b_{i,t} b_{t,j}, 0) & \text{otherwise,} \end{cases}$$

$$x_i' := \begin{cases} (\prod_{b_{j,t}>0} x_j^{b_{j,t}} + \prod_{b_{j,t}<0} x_j^{-b_{j,t}})/x_t & \text{if } i = t, \\ x_i & \text{otherwise,} \end{cases}$$

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- The cluster algebra $\mathbb{A}(S)$ with the initial seed S $= \langle$all cluster variables obtained from S by applying many $\mu_t\rangle \subset F$
- A has a cluster algebra structure if $A = \mathbb{A}(S)$ for some initial seed S.

Example Let x, y, z be indeterminates.

$$A := \langle x, y, z, \frac{y+z}{x} \rangle \subset \mathbb{Q}(x, y, z) =: F$$

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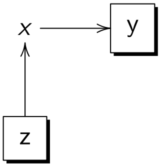
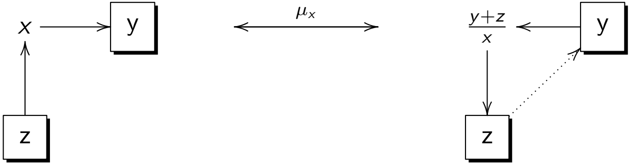
Then A has a cluster algebra structure with the initial seed:



Here, $\square$ denotes frozen variables. The mutation μ_x gives another seed.



Monoidal categorification

We assume that

- A has a cluster alg structure,
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Definition $\mathcal{S} := (\{M_i\}_{i \in K}, \widetilde{B})$ is said a monoidal seed in $\mathscr{C}$ if

(i) $\widetilde{B}$ is an exchange matrix,

(ii) any tensor product $M_{i_1} \otimes M_{i_2} \otimes \cdots \otimes M_{i_t}$ is simple,

(iii) $M_i \otimes M_j \simeq M_j \otimes M_i$ for any i, j,

Note $\mathcal{S}$ matches with a seed $(\{x_i\}_{i \in K}, \widetilde{B})$ of A.

Definition

For any $k \in K^{ex}$, a monoidal seed $\mathcal{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ admits a mutation in direction k if $\exists$ simple object M'_k such that

(i) $\exists$ exact sequence in $\mathscr{C}$

$$0 \to \bigotimes_{b_{j,k}>0} M_j^{\otimes b_{j,k}} \to M_k \otimes M'_k \to \bigotimes_{b_{j,k}<0} M_j^{\otimes -b_{j,k}} \to 0$$

(ii) the pair $\mu_k(\mathcal{S}) := (\{M_i\}_{i\neq k} \cup \{M'_k\}, \mu_k(\widetilde{B}))$ is a monoidal seed in $\mathscr{C}$.

Note This exact sequence matches with mutation in A, i.e.,

$$x'_k = (\prod_{b_{j,k>0}} x_j^{b_{j,k}} + \prod_{b_{j,k<0}} x_j^{-b_{k,k}})/x_k$$

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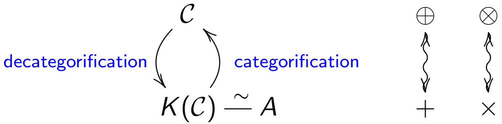
The following notion was introduced by Hernandez-Lerclerc [HL10].

Definition

The category $\mathcal{C}$ is a monoidal categorification of a cluster alg A if

(i) the category $\mathcal{C}$ categorifies A as an algebra, i.e.,



(ii) $\exists$ a monoidal seed $\mathcal{S} = (\{M_i\}_{i \in K}, \widetilde{B})$ s.t. $[\mathcal{S}] := (\{[M_i]\}_{i \in K}, \widetilde{B})$ is an initial seed of A.

(iii) the monoidal seed $\mathcal{S}$ admits successive mutations in all directions.

Remark Let $\mathcal{C}$ be a (good) category with $\otimes$, and set

$$\mathrm{G} := \{ \text{[simples] in } \mathcal{C} \} \subset A.$$

Suppose that $\mathcal{C}$ is a monoidal categorification of A.

- The structure constants for the basis G are non-negative integers.
- Every cluster monomial is contained in G.

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Example

(i) Monoidal categorification for quantum unipotent coordinate rings (Kang-Kashiwara-Kim-Oh [KKKO18]),

(ii) Monoidal categorification for Hernandez-Leclerc categories (Hernandez-Leclerc, Nakajima, Qin, Kashiwara-Kim-Oh-P. [KKOP24])

Quantum groups and quiver Hecke algebras

Let

- $A := (a_{ij})_{i,j \in I}$ (generalized Cartan matrix)
- $Q^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ (positive root lattice)

For convenience, we assume that A is of **symmetric type** in the talk.

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- (Quantum unipotent coordinate ring)

$$\mathrm{A}_q(\mathfrak{n}) := \bigoplus_{\beta\in \mathrm{Q}_-} \mathrm{A}_q(\mathfrak{n})_\beta,$$

where $\mathrm{A}_q(\mathfrak{n})_\beta := \mathrm{Hom}(U_q^+(\mathfrak{g})_\beta, \mathbb{Q}(q))$

Note As an algebra, $\mathrm{A}_q(\mathfrak{n}) \simeq U_q^-(\mathfrak{g})$.

Definition (Khovanov-Lauda [KL09], Rouquier [R08])

Let $\beta \in \mathsf{Q}^+$ with height m. The quiver Hecke algebra $R(\beta)$ (or KLR algebra) associated with A is the $\mathbb{Z}$-graded $\mathbb{C}$-algebra generated by

$$\mathrm{e}(\nu)\ (\nu \in I^\beta), \quad \mathrm{x}_k\ (1 \le k \le m), \quad \tau_t\ (1 \le t \le m-1)$$

satisfying certain defining relations.

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$\rightsquigarrow$ Convolution product $M \circ N$

$$- \circ - : R(\beta)\text{-gmod} \times R(\gamma)\text{-gmod} \longrightarrow R(\beta+\gamma)\text{-gmod}$$
$$(M, N) \longmapsto R(\beta+\gamma) \otimes_A (M \boxtimes N)$$

where $A := R(\beta) \otimes R(\gamma)$.

Theorem Categorification theorem (Khovanov-Lauda, Rouquier)

The category $R\text{-gmod}$ categotifies $\mathrm{A}_q(\mathfrak{n})$, i.e.,

$$R\text{-gmod}$$

decategorification ↓ ↑ categorification

$$K(R\text{-gmod}) \xrightarrow{\sim} \mathrm{A}_q(\mathfrak{n})$$

where the $\mathbb{Z}$-grading of R gives a $\mathbb{Z}[q, q^{-1}]$-module structure.

For any $w \in \mathrm{W}$,

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- $^{\exists}$ subcategory $\mathcal{C}_w \subset R\text{-}\mathrm{gmod}$ such that

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Theorem Monoidal categorification ([Kang-Kashiwara-Kim-Oh])

- $\mathcal{C}_w$ gives a monoidal categorification of $\mathrm{A}_q(\mathfrak{n}(w))$.
- The set

$$\mathbf{G}(w) := \{ \text{ simple modules in } \mathcal{C}_w \}$$

is a $\mathbb{Z}[q^{\pm 1}]$-basis of $\mathrm{A}_q(\mathfrak{n}(w))$. Moreover

$$\{\text{every cluster monomial}\} \subset \mathbf{G}(w)$$

Example (A_2 type, $w = w_0$ case, i.e., $\mathcal{C}_{w_0} = R\text{-gmod}$)

In this case, we have

$$\mathrm{A}_q(\mathfrak{n}) \simeq U_q^-(\mathfrak{sl}_3) \simeq \mathbb{Q}(q)\langle f_1, f_2 \rangle / (\text{quantum serre relation}).$$

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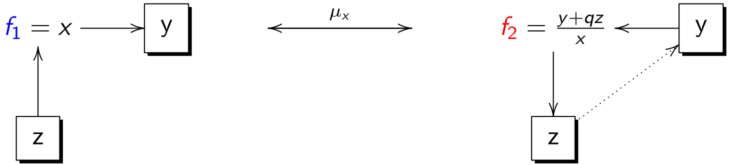
The (quantum) cluster algebra structure is given as follows:



Example (Type A_2, Cluster algebra structure on R_{A_2}-gmod)

$$\begin{array}{ccc} \mathrm{A}_q(\mathfrak{n}) & \overset{\text{categorification}}{\longleftrightarrow} & R_{A_2}\text{-gmod} \\ x, y, z, x' & \leftrightsquigarrow & L(1), L(12), L(21), L(2) \\ \text{mutation} & \leftrightsquigarrow & \text{special exact sequence} \end{array}$$

where $L(i) :=$ 1-dim'l $R(\alpha_i)$-module, $L(ij) := \mathrm{hd}(L(i) \circ L(j))$ and

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The mutation $x' = \frac{1}{x} \cdot (y + qz)$ corresponds

$$0 \to qL(21) \to L(1) \circ L(2) \to L(12) \to 0.$$

Crystals

For convenience, we assume that it is of **finite type**.

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- (Kashiwara [K91]) $B(\infty) :=$ the infinite crystal of $U_q^-(\mathfrak{g}) \simeq \mathrm{A}_q(\mathfrak{n})$

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- (Lauda-Vazirani [LV11]) There is a crystal structure on simples, i.e.,

  Categorical crystal : $\{$ simple R-modules $\} \xrightarrow{\simeq} B(\infty)$

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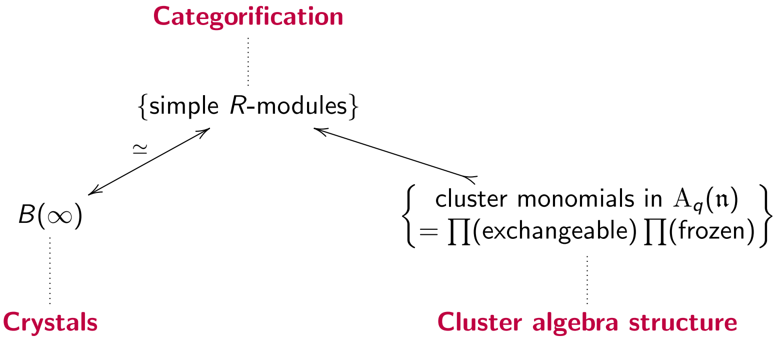
- $\mathrm{A}_q(\mathfrak{n}(w))$ has a quantum cluster algebra structure, i.e.,

  Cluster alg str : $\{ \text{ cluster monomials } \} \rightarrowtail B(\infty)$

Three structures related to $A_q(\mathfrak{n})$



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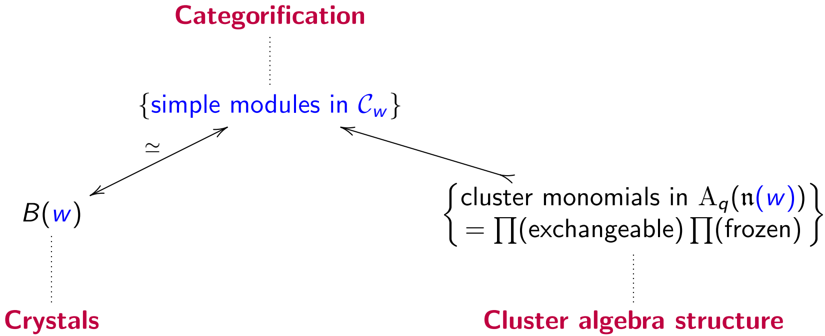

Here $B(w) := \{b \in B(\infty) \mid \mathrm{G}^{\mathrm{up}}(b) \in A_q(\mathfrak{n}(w))\}$,

$\mathrm{G}^{\mathrm{up}}(b) =$ the member of the dual canonical basis cor. to b.

Three structures related to **the localized algebra $\widetilde{\mathrm{A}}_q(\mathfrak{n}(w))$**

Categorification

$\{$simple objects in $\widetilde{\mathcal{C}}_w\}$ ··· Localized category

$\widetilde{B}(w)$ $\simeq$ $\{$simple objects in $\widetilde{\mathcal{C}}_w\}$

$\left\{ \begin{array}{l} \text{cluster monomials in } \widetilde{\mathrm{A}}_q(\mathfrak{n}(w)) \\ = \prod(\text{exchangeable}) \prod (\text{frozen})^{\pm 1} \end{array} \right\}$ → $\{$simple objects in $\widetilde{\mathcal{C}}_w\}$

Localized Crystals

Cluster algebra structure

(Localized crystal structure : (Nakashima [N22], Kashiwara-Nakashima [KN25])

Three structures related to **the localized algebra** $\widetilde{\mathrm{A}}_q(\mathfrak{n}(w))$

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Localized Crystals **Cluster algebra structure**

(Localized crystal structure : (Nakashima [N22], Kashiwara-Nakashima [KN25])

Idea Use **crystals** to study the cluster structure via categorification!!

Quantum twist automorphism

We consider

$\eta_w :=$ quantum twist automorphism of the localized alg $\widetilde{\mathrm{A}}_q(\mathfrak{n}(w))$
introduced by Kimura-Oya ([KO21]).

- η_w is a quantum analogue of the Berenstein-Fomin-Zelevinsky twist automorphisms on unipotent cells ([BFZ96]).
(※ $^\exists$ other quantum twist automorphism by Berenstein-Rupel [BR15]) .
- η_w is related to the dual functor on $\widetilde{\mathcal{C}}_w$ under monoidal categorification (Kashiwara-Kim-Oh-P. [KKOP21]).
- η_w is related to the notion of green-to-red sequence or injective-reachablity in cluster algebras.

Proposition (Kimura-Oya)

The quantum twist map η_w permutes the dual canonical basis of $\widetilde{\mathrm{A}}_q(\mathfrak{n}(w))$.

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Recent result (Jung-P.) ([JP25]; arXiv:2507.01306)

Let $U_q(\mathfrak{g})$ be a quantum group of a symmetrizable Kac-Moody type and w an arbitrary element in W.

(i) We give an crystal theoretic-description of $\mathcal{D}_w$.

(ii) Using this description, we study
- In minuscule cases (ABCD type), combinatorial description of $\mathcal{D}_w$,
- In type A, an explicit formula of the periodicity of $\mathcal{D}_w$ (up to frozens) for special w
- examples for periodicity of $\mathcal{D}_w$ (up to frozens) using computer program.

(Idea of proof for (i))

(a) Cosider two kinds of g-vectors $\mathrm{g}^{\mathrm{L}}_{\mathbf{i}}(b)$ and $\mathrm{g}^{\mathrm{R}}_{\mathbf{i}}(b)$ for $b \in \widetilde{B}(w)$

(b) ([Kashiwara-Kim], [Kashiwara-Kim-Oh-P.])

$$\mathrm{g}^{\mathrm{R}}_{\mathbf{i}}(M) + \mathrm{g}^{\mathrm{L}}_{\mathbf{i}}(\mathscr{D}_w(M)) = 0 \qquad \text{for a simple module } M \in \mathcal{C}_w.$$

where $\mathscr{D}$ is the right dual functor of $\widetilde{\mathcal{C}}_w$.

(c) Consider the following commuting diagram:

$$\begin{array}{ccccc} & & \widetilde{B}(w) & & \\ & \swarrow & & \searrow & \\ \mathrm{G}^{\mathrm{R}}_{\mathbf{i}}(w) \xleftarrow[{[KK19]}]{\mathcal{N}_{\mathbf{i}}} \widetilde{\mathsf{P}}_{\mathbf{i}}(w) & & \xrightarrow{\psi_{\mathbf{i}}} & & \widetilde{\mathsf{S}}_{\mathbf{i}}(w) \xleftarrow[{[FO21]}]{\mathcal{M}_{\mathbf{i}}} \mathrm{G}^{\mathrm{L}}_{\mathbf{i}}(w). \end{array}$$

where $\widetilde{\mathsf{P}}_{\mathbf{i}}(w)$ is the PBW parameterization and $\widetilde{\mathsf{S}}_{\mathbf{i}}(w)$ is the string parametrization for $\widetilde{B}(\infty)$.

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THANK YOU