Some questions around the Bruhat order

or: Counting and Classifying Bruhat Intervals (Our Favorite Pastime)
or: Euclidean Geometry Secretly Runs the Show

Nicolás Libedinsky

Joint work with F. Castillo, D. de la Fuente, D. Plaza, and G. Burrull, R. Villegas

Category Theory, Combinatorics, and Machine Learning
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First big problem: Compute the cardinalities of Bruhat intervals

$$[u, w] := \{z \in W \mid u \leq z \leq w\},$$

in particular of lower Bruhat intervals $\leq w := [id, w]$.

Some known results

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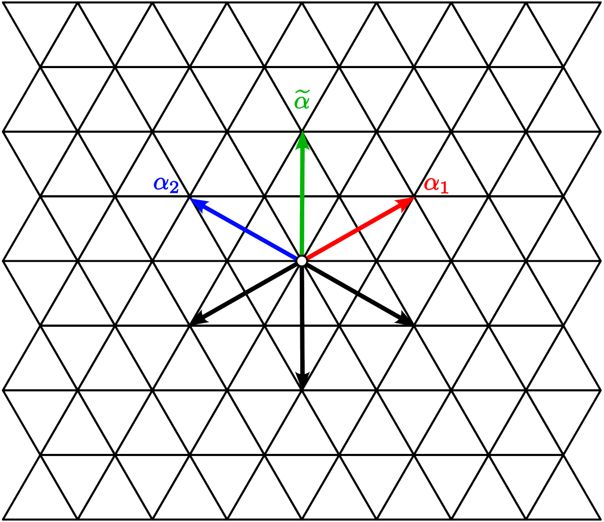
- (Oh-Yoo, 2010): Generalization of the previous result for all (finite) Weyl groups (for rationally smooth elements).
- (Libedinsky-Patimo, 2023) and (Batistelli-Bingham-Plaza, 2023): In affine type $\widetilde{A}_2$ ($\widetilde{B}_2$, respectively), explicit formula of all $| \leq w|$. Under some parametrization of the group, these are polynomial formulas.

The Lattice Formula (example in $\widetilde{A}_2$)



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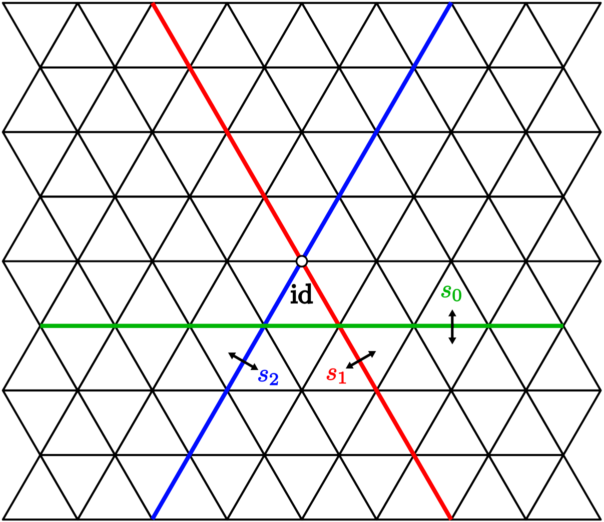

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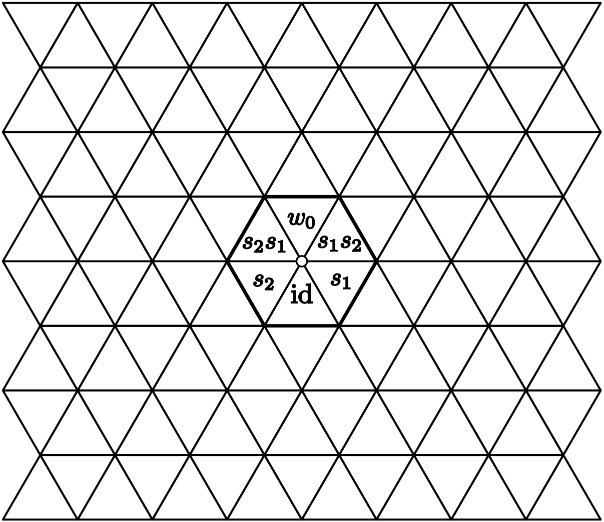

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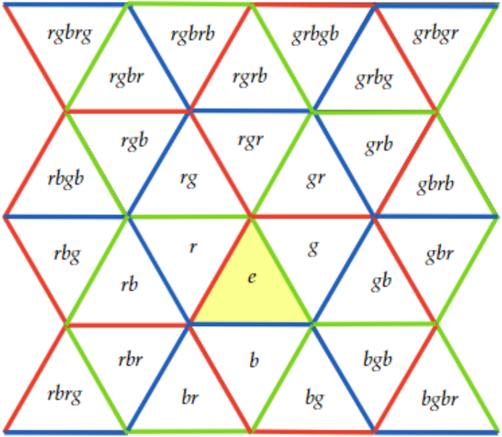

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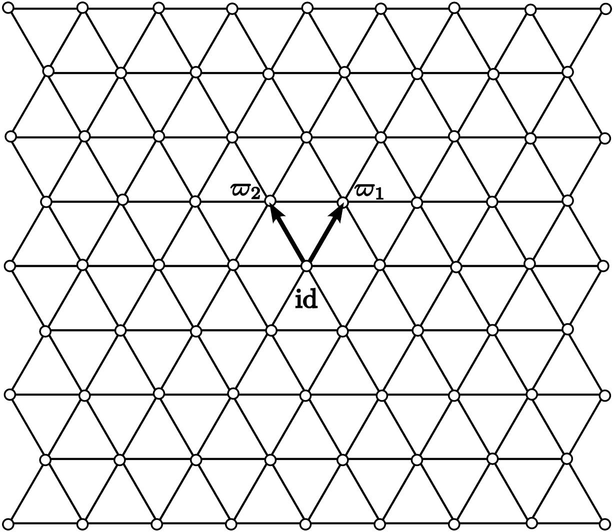

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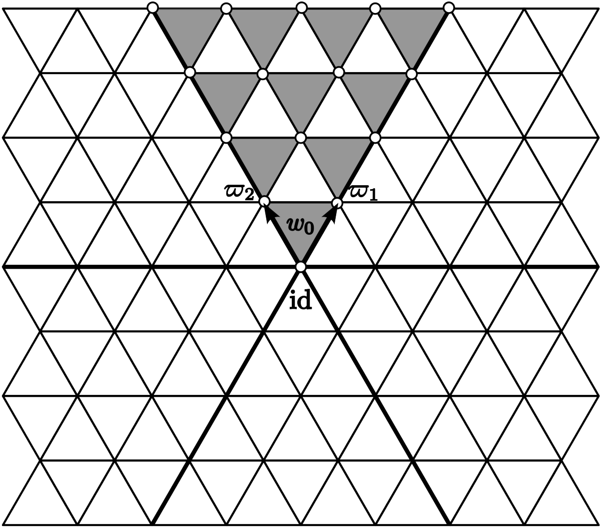

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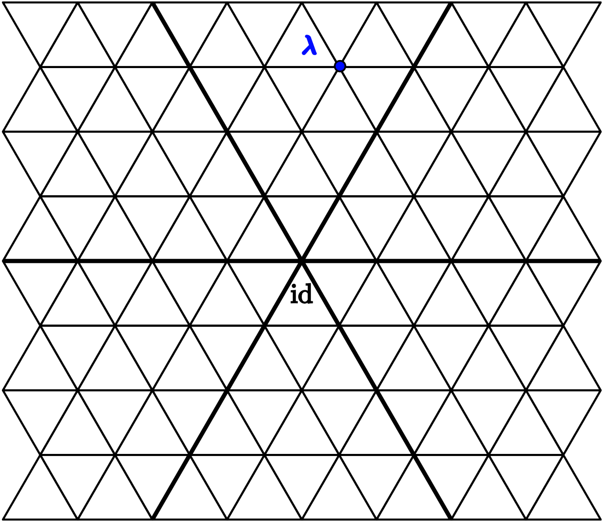

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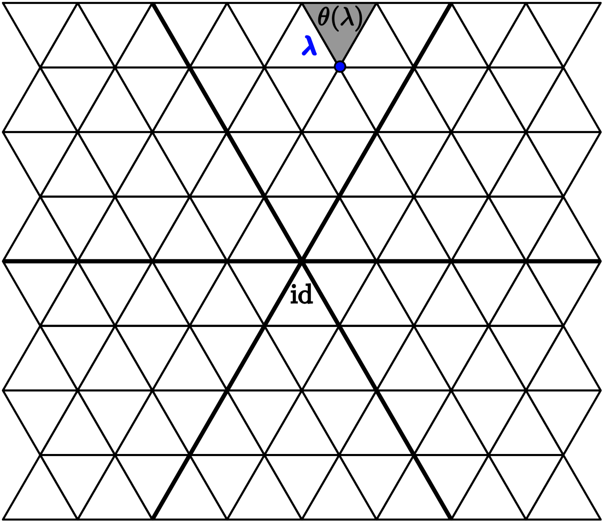

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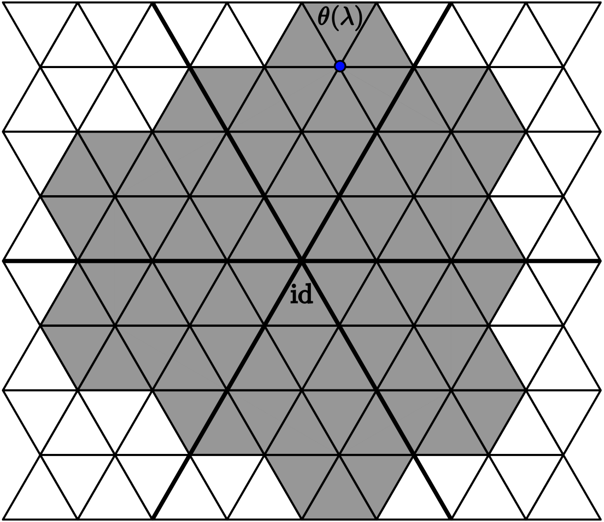

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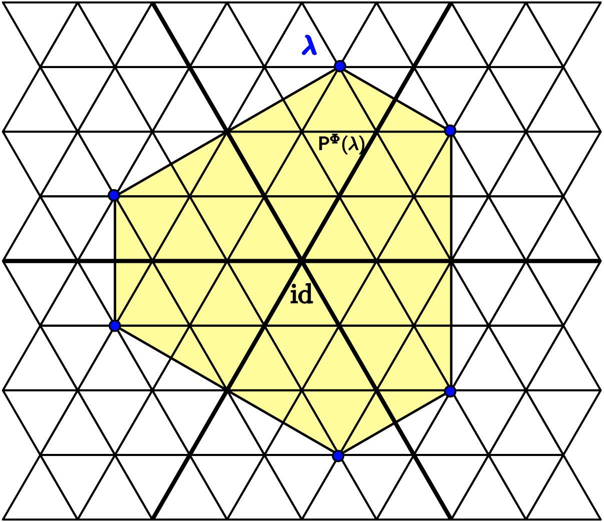

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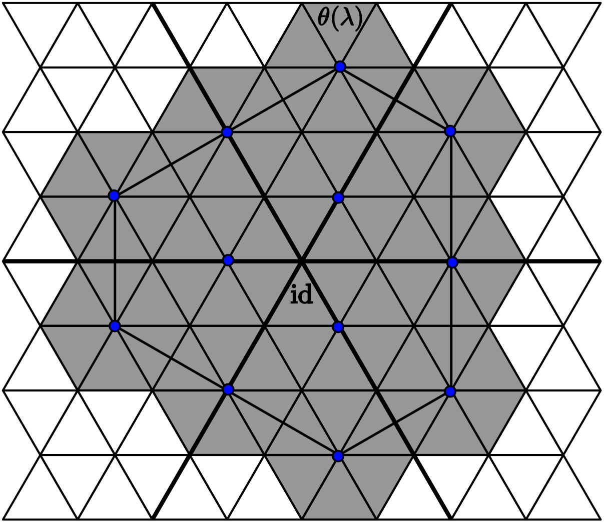

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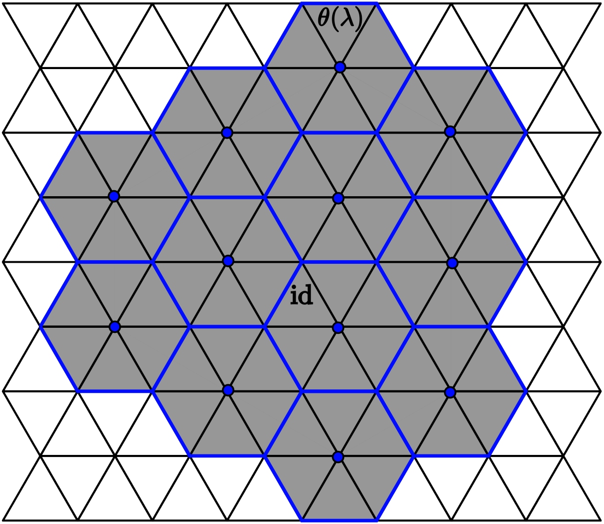

The Lattice Formula

Let Φ be an irreducible root system with finite Weyl group W_f and affine Weyl group W_a. For a dominant coweight λ, we define

$$\mathrm{P}^{\Phi}(\lambda) \coloneqq \mathrm{Conv}(W_f \cdot \lambda).$$

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$$\mathsf{P}^{\Phi}(\lambda) := \mathrm{Conv}(W_f \cdot \lambda).$$

Theorem: Lattice Formula (Castillo, de la Fuente, L., Plaza) in "On the size of Bruhat intervals"

For every dominant coweight λ, we have

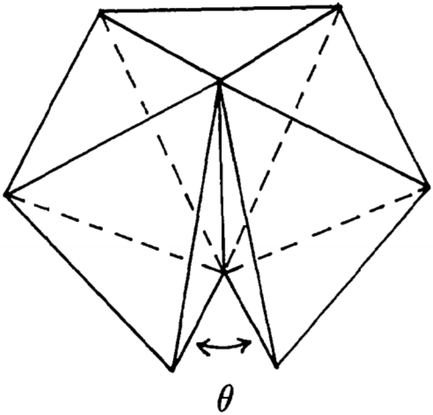
$$\mathrm{Alcoves}(\leq \theta(\lambda)) = \bigsqcup_{\mu} \mathrm{Alcoves}(W_f) + \mu,$$

where μ ranges over $\mathsf{P}^{\Phi}(\lambda) \cap (\lambda + \mathbb{Z}\Phi^{\vee})$.

In particular,

$$|\leq \theta(\lambda)| = |W_f||\mathsf{P}^{\Phi}(\lambda) \cap (\lambda + \mathbb{Z}\Phi^{\vee})|.$$

W_f in $\widetilde{A}_3$



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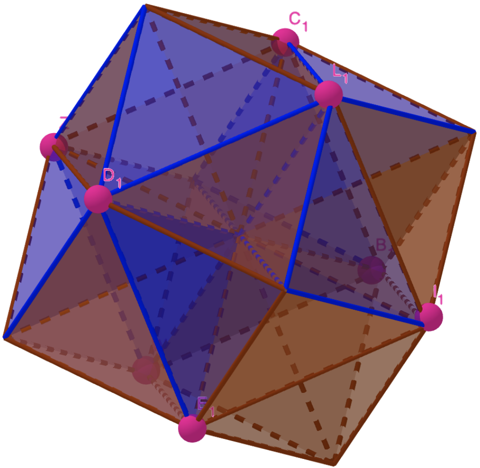

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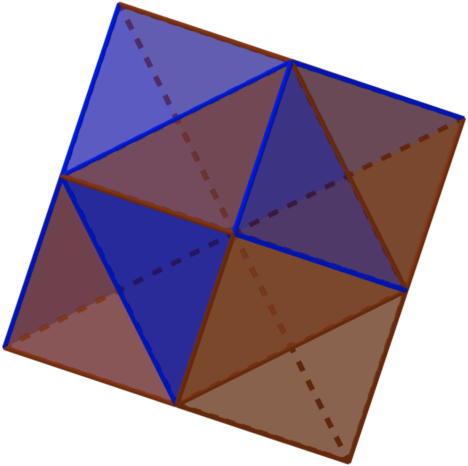

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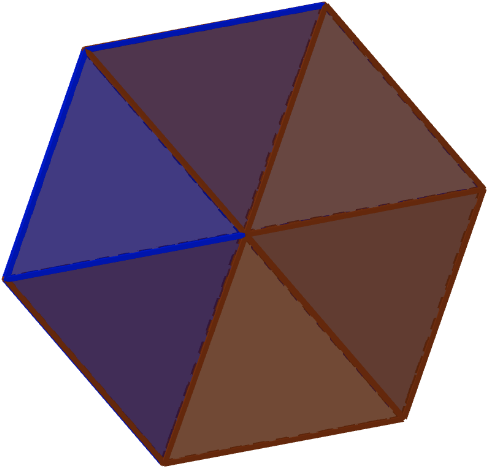

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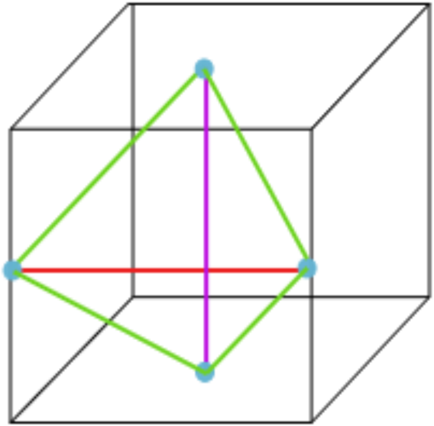

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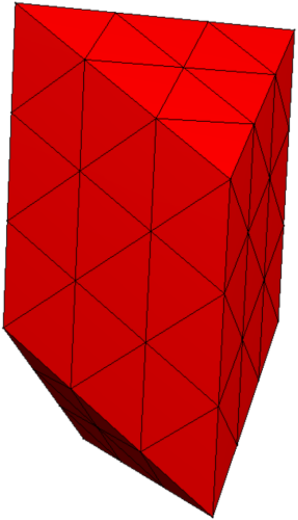

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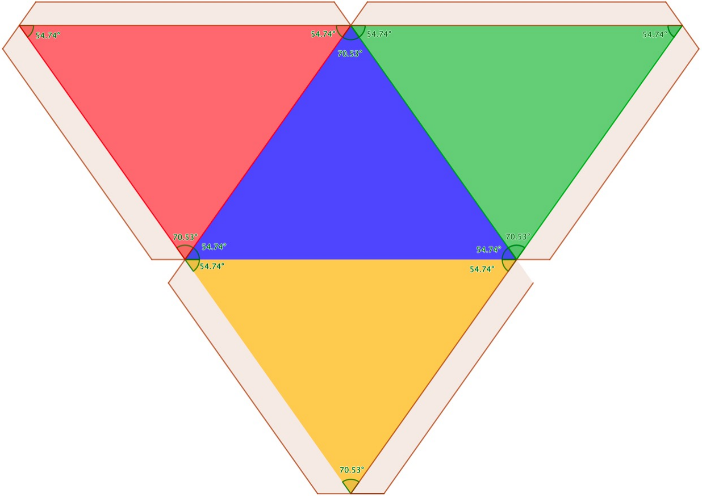
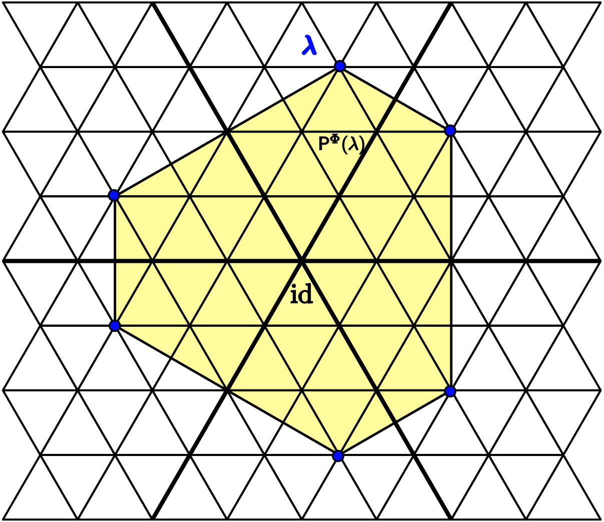


The Geometric Formula (example in $\widetilde{A}_2$)



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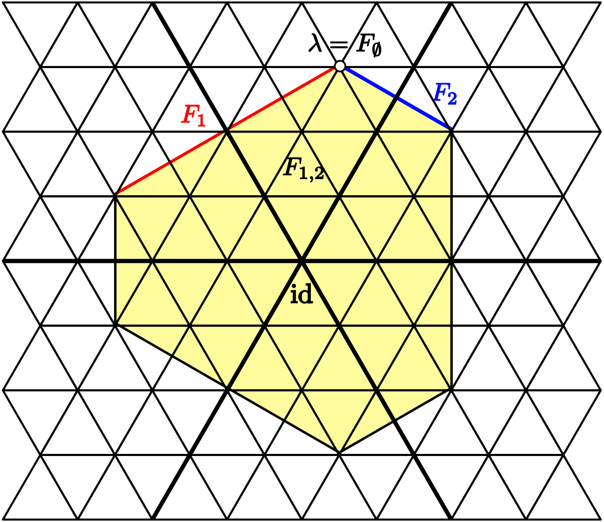

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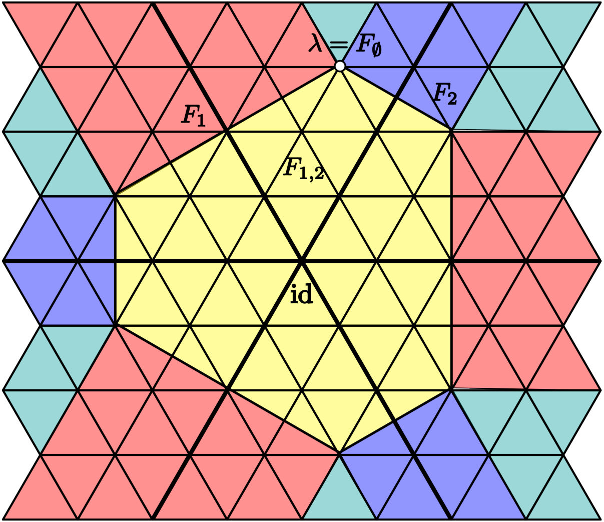

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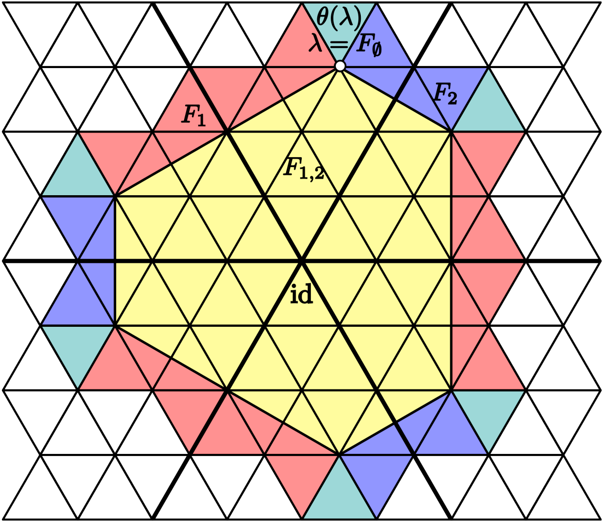
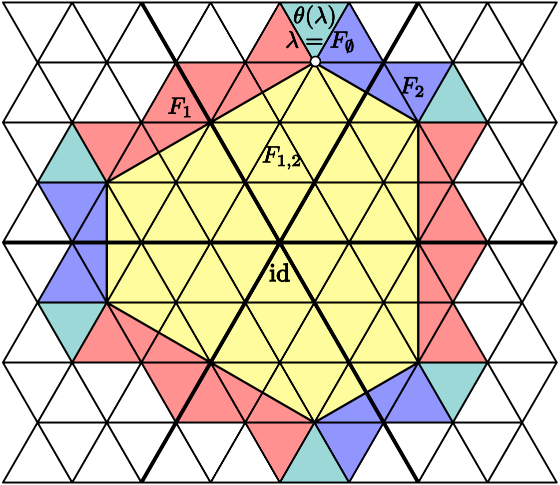


$$|\leq \theta(\lambda)| = \mu_{1,2}\text{Area}(F_{1,2}) + \mu_1\text{Length}(F_1) + \mu_2\text{Length}(F_2) + \mu_\emptyset\text{Card}(F_\emptyset)$$



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\end{aligned}
$$

If $\lambda = (a, b)$ in the fundamental weight basis, then:

- $V_{1,2}(a, b) = \frac{\sqrt{3}}{2}(a^2 + 4ab + b^2)$, $\quad \mu_{1,2} = 2\sqrt{3}$
- $V_1(a, b) = a\sqrt{2}$, $\quad \mu_1 = \frac{9}{2}\sqrt{2}$
- $V_2(a, b) = b\sqrt{2}$, $\quad \mu_2 = \frac{9}{2}\sqrt{2}$
- $V_\emptyset(a, b) = 1$, $\quad \mu_\emptyset = 6$

Thus,

$$|\leq \theta(a, b)| = 3a^2 + 3b^2 + 12ab + 9a + 9b + 6$$

The Geometric Formula

Let Φ be any irreducible root system and let S_f be the set of simple reflections of the finite Weyl group W_f.

Theorem: Geometric Formula (Castillo, de la Fuente, L., Plaza)

There exist unique real numbers μ_J^Φ such that for any dominant coweight λ,

$$|\leq \theta(\lambda)| = \sum_{J \subset S_f} \mu_J^\Phi V_J^\Phi(\lambda).$$

Important: The coefficients μ_J^Φ **do not** depend on λ.

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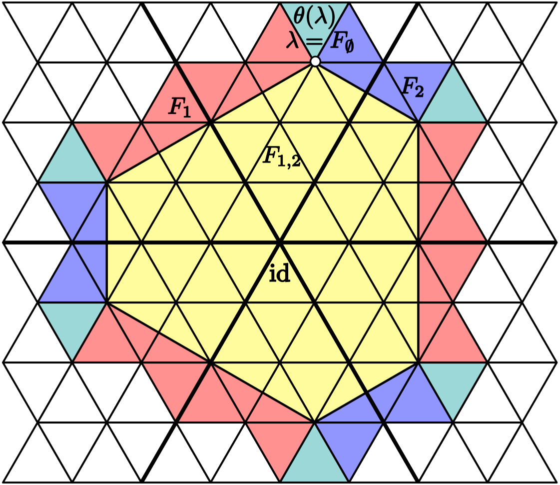
Important: The coefficients μ_J^Φ **do not** depend on λ.
If $\lambda = (m_1, \ldots, m_n)$ in the fundamental coweight basis, then the volumes $V_J^\Phi(\lambda)$ are polynomials in $m_1, \ldots, m_n$. ($n =$rank of Φ)
Thus, the Geometric Formula implies that $|\leq \theta(\lambda)|$ is also a polynomial of degree n in $m_1, \ldots, m_n$.

Treacherous $\widetilde{A}_2$

$$|\leq \theta(\lambda)| = \mu_{1,2}\mathrm{Area}(F_{1,2}) + \mu_1\mathrm{Length}(F_1) + \mu_2\mathrm{Length}(F_2) + \mu_\emptyset\mathrm{Card}(F_\emptyset)$$



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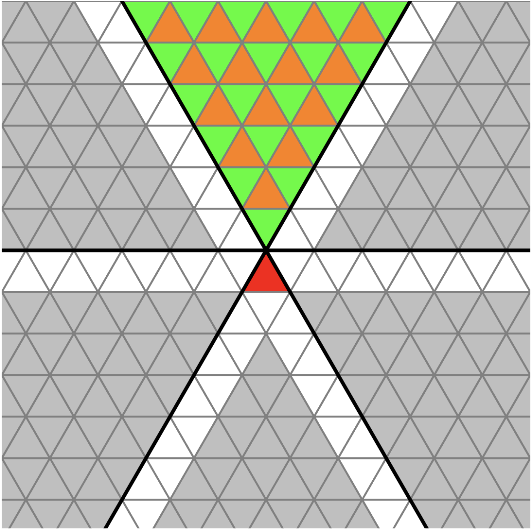
- BOAT: Bruhat order and alcovic tilings.

Paper BOAT



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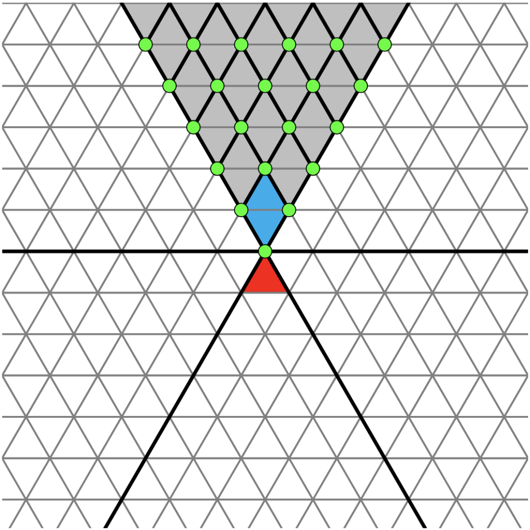

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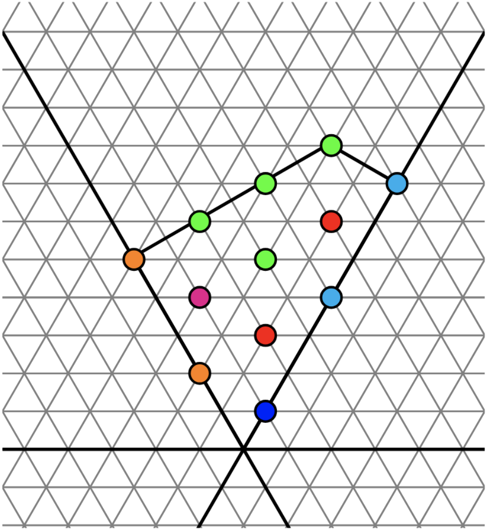

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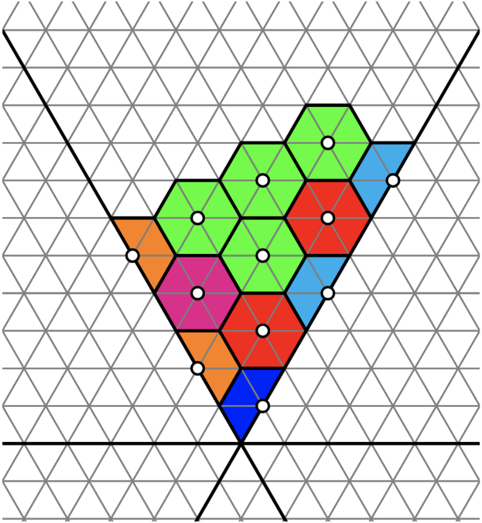

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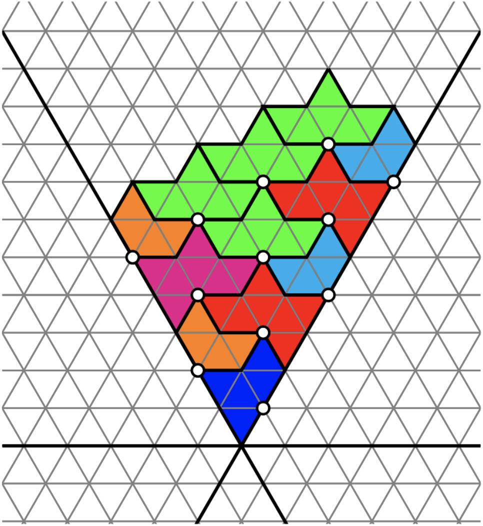

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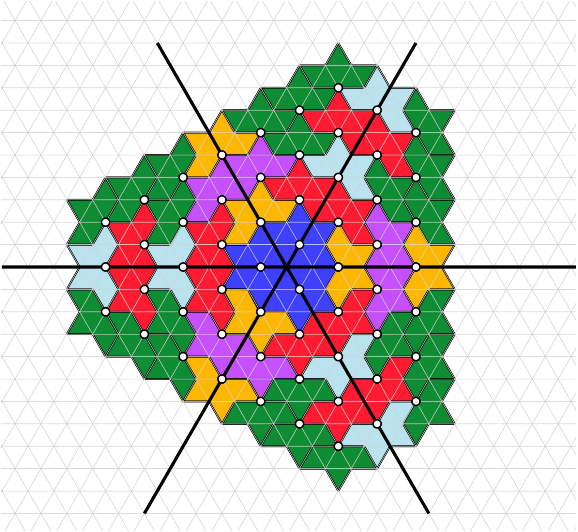

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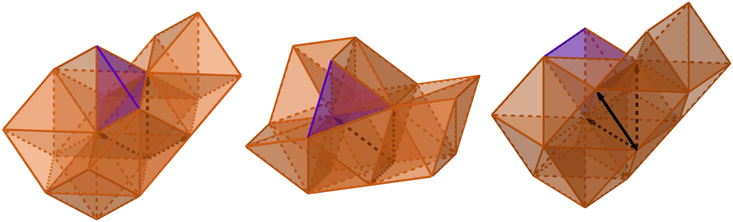

Theorem (Castillo, de la Fuente, L., Plaza in "Paper BOAT")

*Let $\lambda \in \Lambda^+$ and $(\lambda_1, \ldots, \lambda_n)$ its coordinates in the basis of fundamental weights. For a fixed $a \in \mathcal{AMIGO}$ the function $q_a : \mathbb{N}^n \to \mathbb{N}$ defined as $(\lambda_1, \ldots, \lambda_n) \mapsto | \leq \theta_a(\lambda)|$, agrees with a **quasi-polynomial** of degree n on a cofinite subset of $\mathbb{N}^n$.*

(Quasi)-polynomiality

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Conjecture

For any affine Weyl group the map q_a is a geometric polynomial.

Perspectives

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Perspectives

1. All lower intervals (growing codimension of the cells)
2. THE WORLD (all intervals) (outside from W_f).

The classification

Second big problem: Classify Bruhat intervals modulo poset isomorphisms.

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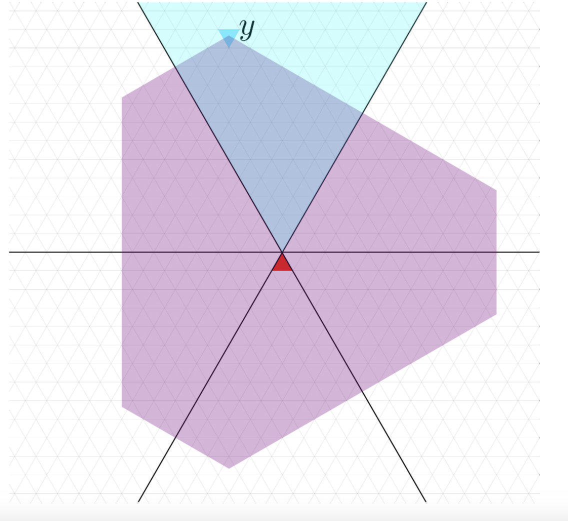
Question 8 (*I*80, *WK*70, *D*40) Prove that if W and W' are two Coxeter groups, $[x, y]$ and $[x', y']$ Bruhat intervals in W and W', respectively, then if $[x, y]$ is isomorphic as a poset to $[x', y']$, then $p_{x,y} = p_{x',y'}$.

Geometry of the intervals



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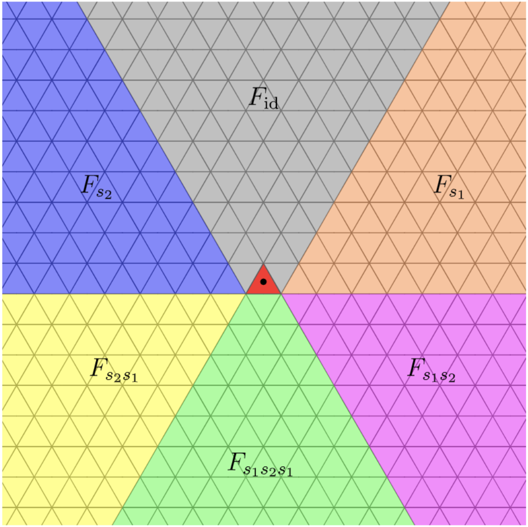

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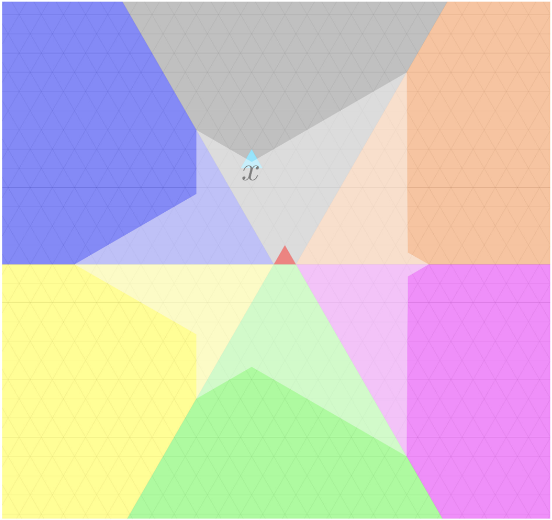
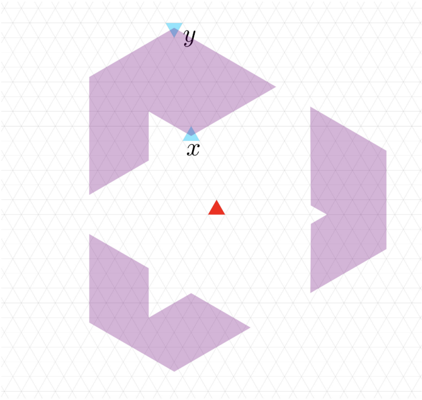


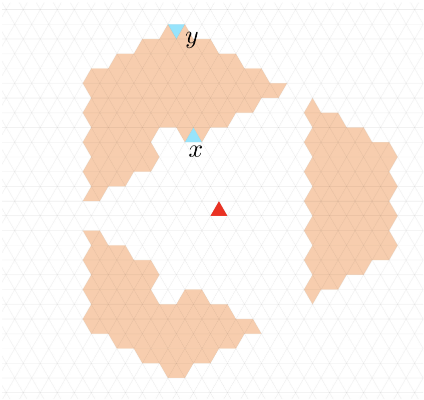
Geometry of the intervals

Theorem (Burrull, L.,Villegas) in "Shape and class of Bruhat intervals"

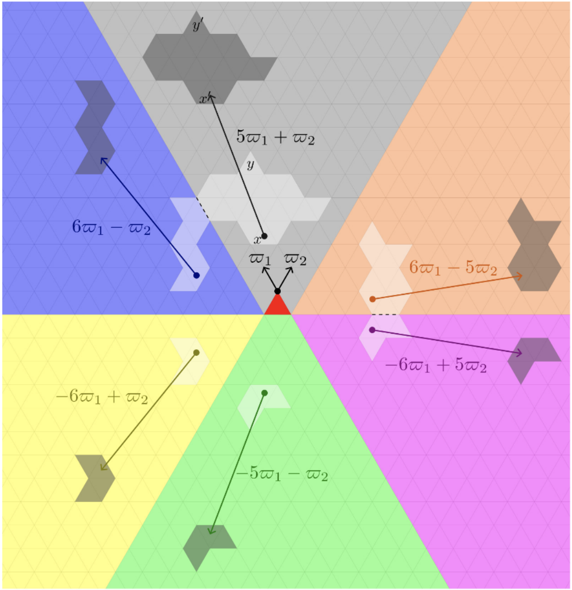
The shape of the interval $[x, y]$ is illustrated in the following figure.

Geometry of the intervals





Translations



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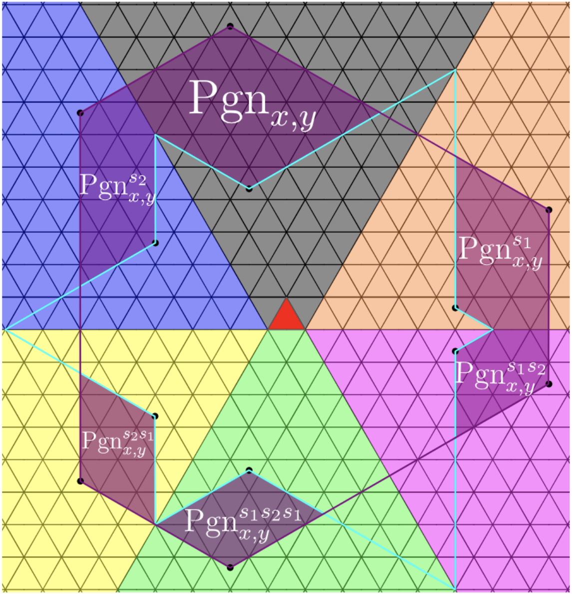

A Theorem and a Conjecture

Theorem (Burrull, L.,Villegas) in "Shape and class of Bruhat intervals"

Let x, x', y, y' be in the dominant chamber of $\widetilde{A}_2$. If the intervals $[x, y] \cong [x', y']$ are isomorphic as posets, they are related by a translation modulo “silly isomorphisms”.

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Suboptimal conjecture (Burrull, L.,Villegas) in "Shape and class of Bruhat intervals"

Let x, x', y, y' be in the dominant chamber of any affine Weyl group W. If the intervals $[x, y] \cong [x', y']$ are isomorphic as posets, they are related by a translation modulo “silly isomorphisms”

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A Theorem and a Conjecture

Theorem (Burrull, L.,Villegas) in "Shape and class of Bruhat intervals"

Let x, x', y, y' be in the dominant chamber of $\widetilde{A}_2$. If the intervals $[x, y] \cong [x', y']$ are isomorphic as posets, they are related by a translation modulo "silly isomorphisms".

This implies the combinatorial invariance conjecture for these elements.

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Let x, x', y, y' be in the dominant chamber of any affine Weyl group W. If the intervals $[x, y] \cong [x', y']$ are isomorphic as posets, they are related by a translation modulo "silly isomorphisms" **IF** x is a translation of x' by a weight (outside from W_f).

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$$[x, y] \cong [x', y'] \iff D_{[x,y]} \cong D_{[x',y']}.$$

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GAGE

How deep does the rabbit hole really go?