I will not give a formal definition!

A brief review of Abelian categorifications, M. Khovanov, V. Mazorchuk, C. Stroppel

Theory Appl. Cat. 22 (2009), 479–508 ArXiv: math/0702746

All categories/functors will be $\Bbbk$-linear, $\Bbbk = \bar{\Bbbk}$ char. $p \geq 0$

All Abelian categories will be locally finite $\Rightarrow$ K_0 is free $\mathbb{Z}$-module with basis $[L]$ (isoclasses of irreps)

$\mathcal{R} \circlearrowleft E, F, \dots$ exact functors $\circlearrowleft$ $x, \tau, \dots$ natural transformations (the main point!)

categorify $\Updownarrow$ decategorify

$K_0(\mathcal{R}) \circlearrowleft e, f, \dots$ induced $\mathbb{Z}$-linear maps

Example 0 Let S be the symmetric groupoid $\coprod_{n\geq 0} S_n$

Its groupoid algebra is $\Bbbk S$ $\quad \bigoplus_{n\geq 0} \Bbbk S_n$

Take $R = \Bbbk S\text{-mod}_{fd}$ $\quad \bigoplus_{n\geq 0} \Bbbk S_n\text{-mod}_{fd}$

What endofunctors / natural transformations come to mind?

$\uparrow = \mathrm{id}_{\mathrm{ind}}$

$\mathrm{ind} = \Bbbk S\, i^* \otimes_{\Bbbk S} -$ $\quad \bigoplus_{n\geq 0} \Bbbk S_{n+1} \otimes_{\Bbbk S_n} -$

$i : S \to S$ is $| \otimes -$ $\quad S_n \hookrightarrow S_{n+1}$

$\downarrow = \mathrm{id}_{\mathrm{res}}$

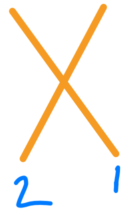
$\mathrm{res} = i^*$, naturally right adjoint to ind

(also left adjoint)

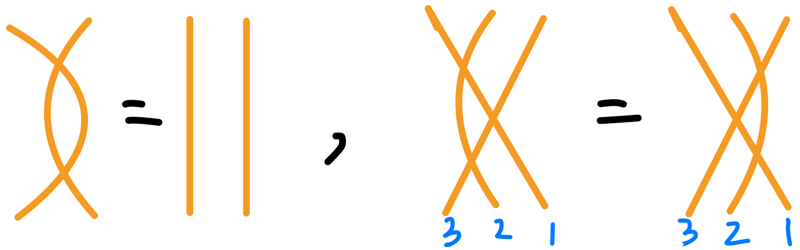
Better: S is the free symmetric strict monoidal cat. with one generating object $|$

$\mathrm{Ob}\, S \longleftrightarrow \mathbb{N}$

$\mathrm{id}_1 = |$ identity endo.

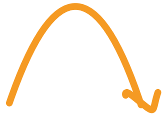
$c_{1,1} =$  symmetric braiding

Relations:



$\mathrm{End}(|^{\otimes n}) = S_n$

The adjunction (ind, res) defines natural tfs.

 : ind∘res ⇒ id counit

 : id ⇒ res∘ind unit

 : ind∘ind ⇒ ind∘ind

× s_{n+1} (last transposition)

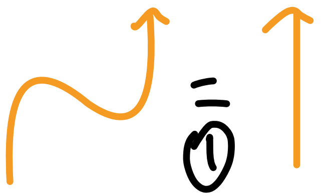
$\Bbbk S_{n+2} \otimes_{\Bbbk S_n} -$

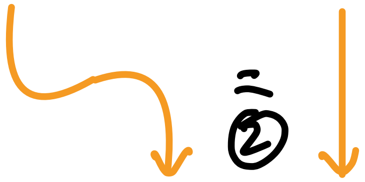
 : ind ⇒ ind

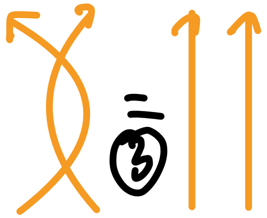
× $x_{n+1} = \sum_{i=1}^{n} (i\ \ n+1)$

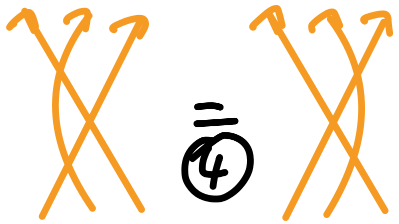
$\Bbbk S_{n+1} \otimes_{\Bbbk S_n} -$ Jucys–Murphy

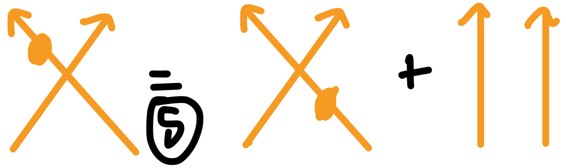
Relations

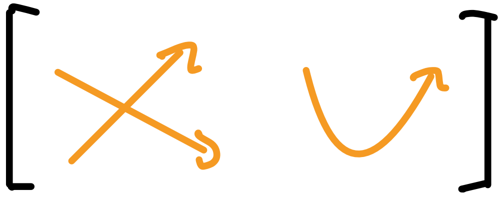
 ,

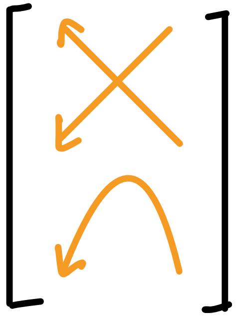
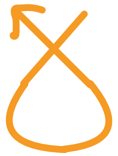
 ,

 ,

 ,

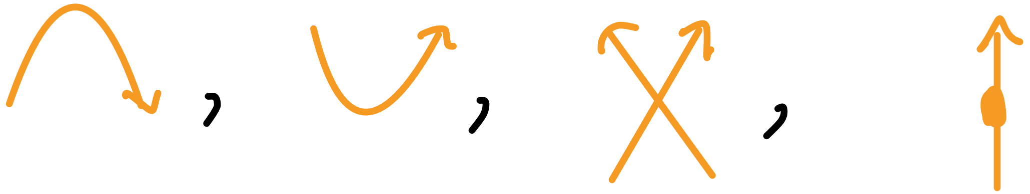
 ,

 : ind∘res ⊕ id $\overset{\sim}{\underset{⑥}{\Longrightarrow}}$ res∘ind

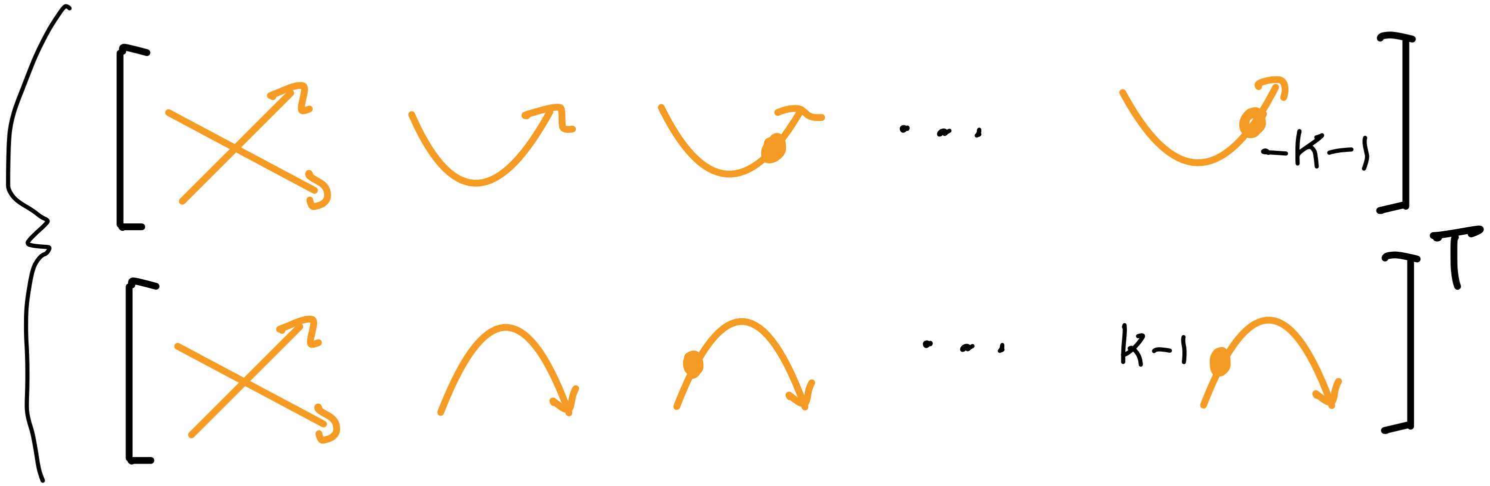
 is the inverse matrix,  := 

These are the other adjunction!

<u>Definition</u> Fix $k \in \mathbb{Z}$, central charge. The Heisenberg category Heis_k is the strict monoidal cat w/ generating objects E, F ($id_E =$ , $id_F =$ ), generating morphisms



and relations ①–⑤ as before and generalized ⑥:



is invertible $(k \leq 0)$

is invertible $(k \geq 0)$.

Note: $k=-1$ case is due to Khovanov, (2014) as on previous slide. General: MacKaay-Savage Brundan (2018)

Definition A <u>Heisenberg categorification</u> is a locally finite Abelian category $\mathcal{R}$ plus adjoint pair (E, F) of endofunctors with given adjunction
plus nat. t.f.s $: E^2 \Rightarrow E^2$ $: E \Rightarrow E$
satisfying Heis_k-relations for some $k \in \mathbb{Z}$.

i.e. $\mathcal{R}$ is a strict left Heis_k-module category

Examples

① $\Bbbk S\text{-mod}_{fd}$, $k = -1$

② $\mathrm{Rep}(GL_n)$, $k = 0$ Rational reps of group scheme $GL_n / \Bbbk$

$E = V \otimes -$ ($V = \Bbbk^n$ = natural module)

is action of Casimir tensor $\sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}$

③ Pretty much any classically studied "type A" Abelian cat. of representations

Now I'm going to list some general results which apply to all Heisenberg categorifications

A long history – $\underbrace{\text{Kleshchev, Grojnowski, Vazirani,}}_{1990s}$ $\underbrace{\text{Chuang-Rouquier}}_{2004}$, ...

① Let E_i be gen. i-eigenspace of $x : E \Rightarrow E$. Then (E_i, F_i) are biadjoint functors $\forall i$.

F_i " " " " $x : F \Rightarrow F$

Let $I = \{ i \in \Bbbk \mid E_i \neq 0 \}$, so $E = \bigoplus_{i \in I} E_i$, $F = \bigoplus_{i \in I} F_i$ ("spectrum")

Also I is closed under $i \mapsto i \pm 1$. (A_∞ or $A^{(1)}_{p-1}$)

② Let $\mathfrak{g}$ be affine KM algebra with Cartan matrix $(a_{ij})_{i,j \in I}$ $\begin{cases} 2 & i = j \\ -1 & i = j+1 \text{ xor } j-1 \\ -2 & i = j+1 = j-1 \end{cases}$

Then e_i, f_i make $\mathbb{C} \otimes_{\mathbb{Z}} K_0(R)$ into an integrable $\mathfrak{g}$-module

of level k. The weight space decomposition gives "blocks" $R = \bigoplus_{\lambda \in X} R_\lambda$. X = weight lattice

(3) Let B_λ be a set indexing full set of irreducibles in R_λ.

Let $B = \coprod_{\lambda \in X} B_\lambda$. $L(b)$

Define $\varepsilon_i(b)$ so min. poly. of $\uparrow : E_i L(b) \to E_i L(b)$ is $\prod_{i \in I} (x-i)^{\varepsilon_i(b)}$

" $\varphi_i(b)$ " " " " $\downarrow : F_i L(b) \to F_i L(b)$ is $\prod_{i \in I} (x-i)^{\varphi_i(b)}$

If $\varepsilon_i(b) \neq 0$ then $E_i L(b)$ has irred. head/socle $L(\tilde{e}_i b)$ for $\tilde{e}_i b \in B$

" $\varphi_i(b) \neq 0$ " $F_i L(b)$ " " " $L(\tilde{f}_i b)$ for $\tilde{f}_i b \in B$

The above gives B the structure of a <u>normal $\mathfrak{g}$-crystal</u> (Kashiwara).
It is the underlying crystal of basis $\{[L(b)] \mid b \in B\}$ of $K_0(R)$, which is a <u>perfect basis</u> (Berenstein-Kazhdan).

(4) For $i \in I$, let $s_i \in W$ be the simple reflection in Weyl group of $\mathfrak{g}$.
There's a singular Rouquier complex of functors $R_\lambda \to R_{s_i(\lambda)}$ yielding an equivalence $D^b(R_\lambda) \to D^b(R_{s_i(\lambda)})$ which decategorifies to usual $\exp(e_i)\exp(-f_i)\exp(e_i)$ on K_0.
So "blocks" labelled by weights in the same W-orbit are derived equivalent.

Example $R = \Bbbk S\text{-mod}_{fd}$.

What's I? $\mathbb{Z}/p\mathbb{Z} \subset \Bbbk$ so $\mathfrak{g} = \mathfrak{sl}_\infty$ or $\widehat{\mathfrak{sl}}_p$

What's $\mathbb{C} \otimes_{\mathbb{Z}} K_0(R)$? Its $V(-\omega_0)$ basic lowest weight module

What's underlying crystal? Young's partition lattice ($p=0$), or Kashiwara's $B(-\omega_0)$ $p>0$
(also Kleshchev's branching graph)

I haven't said anything new here, but see

An update on Heisenberg and Kac-Moody categorification J.B, A. Savage, B. Webster
ArXiv: 2508.14378

Partly survey — we give more details about this and other examples.

I've focussed on Heisenberg categorification, but it is a role model for several parallel theories.

Four classical families of Abelian categorifications

GL Heisenberg categorification $\longrightarrow$ Cartan types A_∞, $A^{(1)}_{p-1}$

(B.-Savage-Webster - complete)

quasi-split iquantum groups

OSp Brauer categorification (Rui-Song) $\dashrightarrow$ Cartan types A_∞, $A^{(1)}_{p-1}$ plus Satake type AIII

(B.-Savage-Wang-Webster — in progress - far to go!)

P Periplectic Brauer categorification (Many authors) $\dashrightarrow$ Electric algebras

(Stroppel-Nehme — major progress, not quite complete?)

Q Isomeric Heisenberg categorification $\xrightarrow{\checkmark}$ Cartan types A_∞, B_∞, C_∞, $A^{(1)}_{p-1}$, $A^{(2)}_{(p-1)/2}$

(B.-Savage - writing now)

END