Modified nonsymmetric Macdonald polynomials

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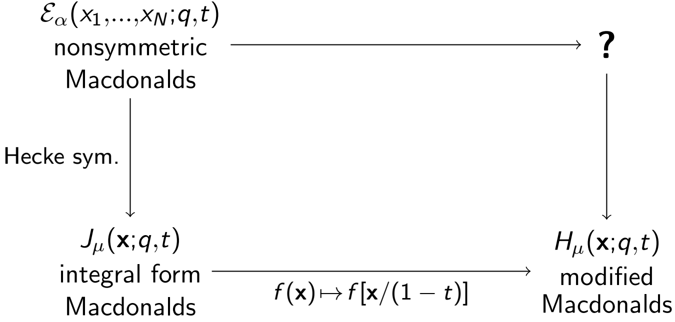
The missing corner

Can the theory of plethystically modified Macdonald polynomials $H_\mu(\mathbf{x}; q, t)$ be lifted to the nonsymmetric setting?



Macdonald polynomials

- Macdonald polynomials $P_\mu(\mathbf{x}; q, t)$ form a basis for $\Lambda_{\mathbb{Q}(q,t)}(\mathbf{x})$.
- Integral form Macdonald polynomials $J_\mu(\mathbf{x}; q, t) = c_\mu P_\mu(\mathbf{x}; q, t)$ have coefficients in $\mathbb{Z}[q, t]$.
- Plethystically modified Macdonald polynomials
$$H_\mu(\mathbf{x}; q, t) = J_\mu[\mathbf{x}/(1-t); q, t].$$
- Macdonald positivity: the H_μ are Schur positive.
- $H_\mu(\mathbf{x}; 1, 1) = (s_1)^n = h_{(1^n)}$.
- $t^{n(\mu)} H_\mu(\mathbf{x}; q, t^{-1}) =$ Frobenius series of the Garsia-Haiman module M_μ, a $\mathbb{Q}\mathcal{S}_n$-submodule of $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ of dimension $n!$.

$$H_{31} = t\, s_4 + (1 + qt + q^2 t) s_{31} + (q + tq^2) s_{22} + (q + q^2 + q^3 t) s_{211} + q^3\, s_{1111}$$

	q^0	q^1	q^2	q^3
t^1	s_4	s_{31}	$s_{31} + s_{22}$	s_{211}
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Nonsymmetric Macdonald polynomials

The Cherednik operators $Y_1, \ldots, Y_N$ act on $\mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$.

$$T_i f = s_i f + (1-t)x_i \frac{f - s_i f}{x_i - x_{i+1}}, \qquad \text{(Demazure-Lusztig operators)}$$

$$\Phi f = f(x_2, \ldots, x_N, q x_1),$$

$$Y_i = t^{-i+1} T_{i-1} \cdots T_1 x_1 \Phi T_{N-1}^{-1} \cdots T_i^{-1}.$$

Def. The *nonsymmetric Macdonald polynomials* $E_\alpha(x_1, \ldots, x_N; q, t)$ are the joint eigenfunctions of the commuting operators $Y_1, \ldots, Y_N$.

- $\{E_\alpha\}_{\alpha \in \mathbb{Z}^N}$ forms a basis for $\mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$.
- Knop introduced *integral form nonsymmetric Macdonald polynomials* $\mathcal{E}_\alpha = c_\alpha E_\alpha$ which lie in $\mathbb{Z}[q,t][x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$.

The missing corner

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$$\begin{array}{ccc}
\mathcal{E}_\alpha(x_1, \ldots, x_N; q, t) & \longrightarrow & \textbf{?} \\
\Big\downarrow {\scriptstyle \text{Hecke sym.}} & & \Big\downarrow \\
J_\mu(\mathbf{x}; q, t) & \xrightarrow[f(\mathbf{x}) \mapsto f[\mathbf{x}/(1-t)]]{} & H_\mu(\mathbf{x}; q, t)
\end{array}$$

Features of the plethystically modified Macdonald polynomials $H_\mu(\mathbf{x}; q, t)$:

- Macdonald positivity: the H_μ are Schur positive.
- Frobenius series of the Garsia-Haiman modules.
- ∇ operator and shuffle theorems.
- H_μ is a positive sum of ribbon LLT polynomials.

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Related work

$$\begin{array}{ccc} \mathcal{E}_\alpha(x_1, \ldots, x_N; q, t) & \longrightarrow & \textbf{?} \\ \Big\downarrow{\scriptstyle \text{Hecke sym.}} & & \Big\downarrow \\ J_\mu(\mathbf{x}; q, t) & \xrightarrow[f(\mathbf{x}) \mapsto f[\mathbf{x}/(1-t)]]{} & H_\mu(\mathbf{x}; q, t) \end{array}$$

- Sanderson (2000) showed the $\mathcal{E}_\alpha|_{t=0}$ are affine Demazure characters.
- Assaf-Gonzalez (2019) showed the $\mathcal{E}_\alpha|_{t=0}$ are key positive.
- Knop (2007) formulated a positivity conjecture for a stable version of $\mathcal{E}_\alpha$ involving Kazhdan-Lusztig theory.
- Lapointe (2022) formulated another positivity conjecture for a stable version of $\mathcal{E}_\alpha$.
- Related work by Goodberry and Orr, and Bechtloff Weising and Orr.

Filling in the missing corner

We fill in the missing corner with a *nonsymmetric plethysm map* Π_r and *modified r-nonsymmetric Macdonald polynomials* ${}_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}(\mathbf{x};q,t)$.

$$\begin{array}{ccc}
{}^{\mathrm{stable}}\mathcal{E}_{\eta|\lambda}(\mathbf{x};q,t) & \xrightarrow{\quad \Pi_r \quad} & {}_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}(\mathbf{x};q,t) \\
\text{Hecke sym.} \downarrow & & \downarrow \text{Weyl sym.} \\
J_{(\eta;\lambda)_+}(\mathbf{x};q,t) & \xrightarrow[f(\mathbf{x}) \mapsto f[\mathbf{x}/(1-t)]]{} & H_{(\eta;\lambda)_+}(\mathbf{x};q,t)
\end{array}$$

- $\mathcal{P}(r) = \mathbb{Q}(q,t)[x_1,\ldots,x_r] \otimes \Lambda_{\mathbb{Q}(q,t)}(x_{r+1},\ldots)$.
- $\mathbf{x} = x_1, x_2, \ldots$.
- The ${}_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}(\mathbf{x};q,t)$, for $(\eta|\lambda) \in \mathbb{N}^r \times \mathrm{Par}$, form a basis for $\mathcal{P}(r)$.
- $(\eta;\lambda)_+$ is the partition rearrangement of the concatenation $(\eta;\lambda)$.

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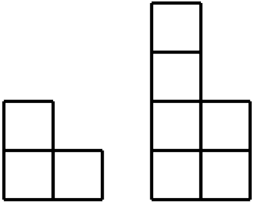
- $(\eta; \lambda)_+$ is the partition rearrangement of the concatenation $(\eta; \lambda)$.

Flagged fillings

- For $\beta \in \mathbb{N}^d$, the *column diagram* of β, $\mathrm{cdg}(\beta)$, consists of d bottom-justified columns of heights $\beta_1, \ldots, \beta_d$.
- An *r-flagged filling* of $\mathrm{cdg}(\beta)$ is a map $T\colon \mathrm{cdg}(\beta) \to \mathbb{Z}_+$ such that the box in the bottom of column i (if it exists) is $\leq i$, for $i = 1, \ldots, r$.



$\mathrm{cdg}((2,1,0,4,2))$

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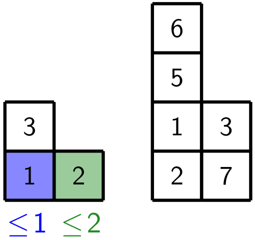
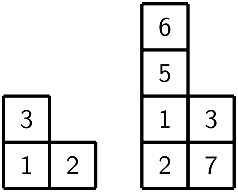


An r-flagged filling T, for $r = 2$

Inversions

- An *attacking pair* is a pair $a, b \in \mathrm{cdg}(\beta)$ such that
  - b is strictly to the left of and in the same row as a, or
  - b is one row below and strictly to the right of a.
- An *attacking inversion* is an attacking pair (a, b) with $T(a) > T(b)$.
- $\mathrm{inv}(T) = \#$ of attacking inversions of T.



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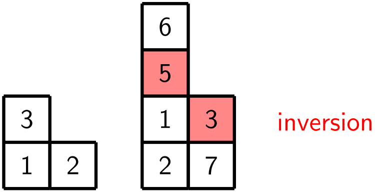

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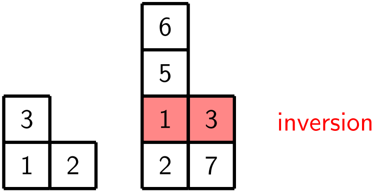

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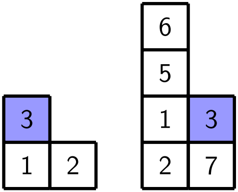

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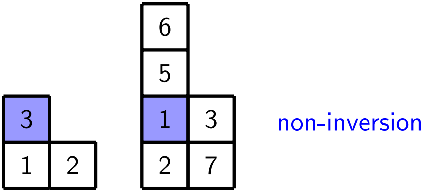

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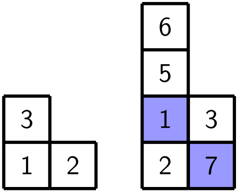

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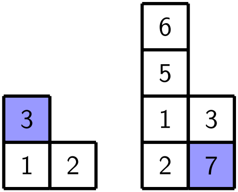

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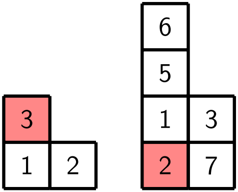

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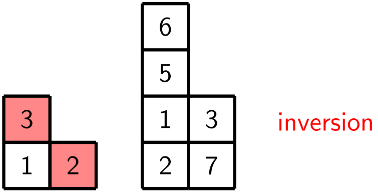

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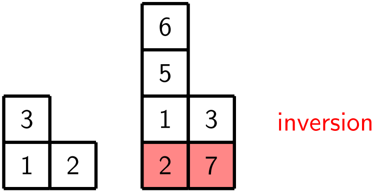

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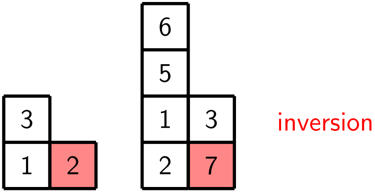

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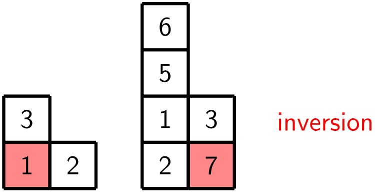

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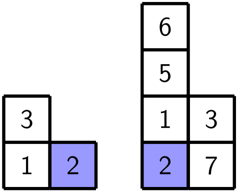

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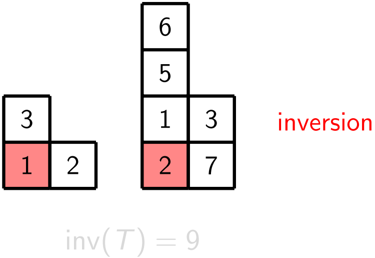

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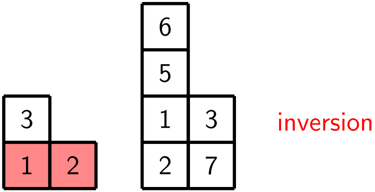

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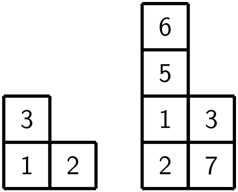

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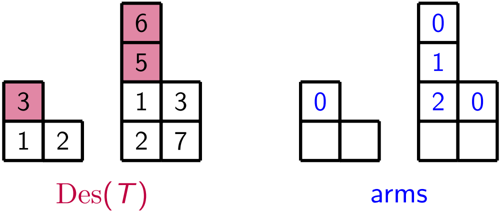
Modified r-nonsymmetric Macdonald polynomials ${}_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}$

- $\mathrm{Des}(T)$ = set of boxes $b \in \mathrm{cdg}(\beta)$ such that $T(b) > T(\mathrm{south}(b))$.
- $\mathrm{arm}(b)$ = # of boxes above and in the same column as b.



Def. [B.-Haiman-Morse-Pun-Seelinger] The *modified r-nonsymmetric Macdonald polynomial* indexed by $\eta \in \mathbb{N}^r$ and partition λ is

$$ {}_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}(\mathbf{x}; q, t) = t^{n(\beta_+)} \sum_{\substack{r\text{-flagged fillings } T \\ \text{of } \mathrm{cdg}(\beta)}} \left(\prod_{b \in \mathrm{Des}(T)} q^{\mathrm{arm}(b)+1} t^{\mathrm{leg}(b)} \right) t^{-\mathrm{inv}(T)} \mathbf{x}^T, $$

where $\beta = (\eta; \lambda)$.

Modified r-nonsymmetric Macdonald polynomials ${}_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}$

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Example. $r=2$, $(\eta|\lambda) = (21|\varnothing)$

T	$\begin{smallmatrix}1 & \\ 1 & 1\end{smallmatrix}$	$\begin{smallmatrix}1 & \\ 1 & 2\end{smallmatrix}$	$\begin{smallmatrix}2 & \\ 1 & 1\end{smallmatrix}$	$\begin{smallmatrix}2 & \\ 1 & 2\end{smallmatrix}$	$\begin{smallmatrix}3 & \\ 1 & 1\end{smallmatrix}$	$\begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix}$
$t^{-\mathrm{inv}(T)}$	1	t^{-1}	t^{-1}	t^{-1}	t^{-1}	t^{-2}
$\prod q^{\mathrm{arm}+1} t^{\mathrm{leg}}$	1	1	qt	qt	qt	qt
$t^{n(\beta_+)}$	t	t	t	t	t	t
total q,t statistic	t	1	qt	qt	qt	q

$$ {}_{\mathrm{ns}}\mathrm{H}_{21|\varnothing}(x_1,x_2,x_3;q,t) = t\,x_1^3 + x_1^2x_2 + qt\,x_1^2x_2 + qt\,x_1x_2^2 + qt\,x_1^2x_3 + q\,x_1x_2x_3 $$

Key polynomials

Def. The *Demazure operator* π_i acts on $f \in \mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ by

$$\pi_i(f) = \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}.$$

Def. The *key polynomials* or *Demazure characters* are constructed from

- $\mathcal{D}_\lambda = \mathbf{x}^\lambda := x_1^{\lambda_1} \cdots x_N^{\lambda_N}$ for partition λ.
- $\mathcal{D}_{s_i(\alpha)} = \pi_i \mathcal{D}_\alpha$ for $\alpha_i > \alpha_{i+1}$, for any $\alpha \in \mathbb{N}^N$.

Example.

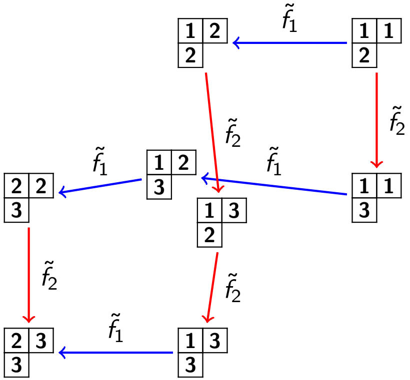
$$\mathcal{D}_{520} = x_1^5 x_2^2$$

$$\mathcal{D}_{250} = \pi_1 \mathcal{D}_{520} = \pi_1(x_1^5 x_2^2) = x_1^5 x_2^2 + x_1^4 x_2^3 + x_1^3 x_2^4 + x_1^2 x_2^5$$

$$\mathcal{D}_{205} = \pi_2 \mathcal{D}_{250} = \pi_2\left(x_1^5 x_2^2 + x_1^4 x_2^3 + x_1^3 x_2^4 + x_1^2 x_2^5\right)$$

Key polynomials and crystals

- $B(\lambda)$ = highest weight $\mathfrak{gl}_N$ crystal of highest weight λ.
- For $S \subset B(\lambda)$ and $i \in [N-1]$, $F_i S := \left\{ \tilde{f}_i^m b : b \in S,\ m \geq 0 \right\} \subset B(\lambda)$.



$B(2,1,0)$

character $s_{21}(x_1, x_2, x_3)$

Key polynomials and crystals

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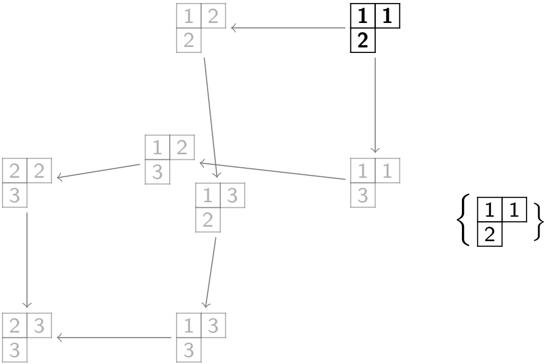

$$\mathcal{D}_{210} = x_1^2 x_2$$

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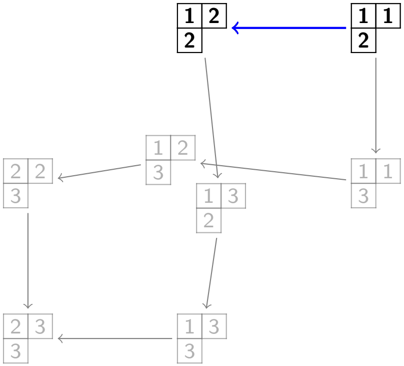

$$F_1 \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array} \right\}$$

$$\mathcal{D}_{120} = \pi_1(x_1^2 x_2) = x_1^2 x_2 + x_1 x_2^2$$

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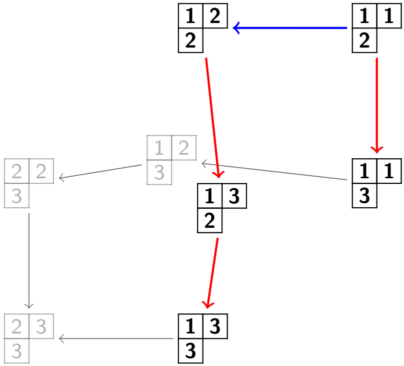

$$F_2 F_1 \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}$$

$$\mathcal{D}_{102} = \pi_2 \pi_1 (x_1^2 x_2) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2$$

Key polynomials and crystals

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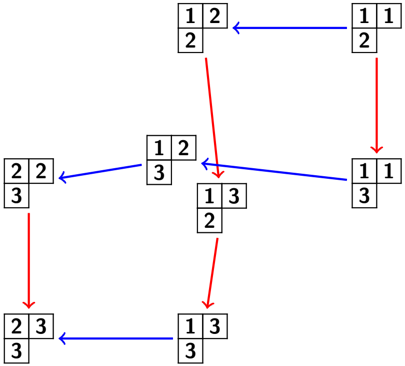

$$F_1 F_2 F_1 \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array} \right\}$$

$$\mathcal{D}_{012} = \pi_1 \pi_2 \pi_1 (x_1^2 x_2) = s_{21}(x_1, x_2, x_3)$$

Demazure atoms

- *Demazure atoms* are defined the same as keys but with $\hat{\pi}_i := \pi_i - 1$ in place of π_i.
- Demazure atoms are related to key polynomials by Bruhat order inclusion-exclusion.

$$
\begin{aligned}
\mathcal{D}_{210} &= \mathcal{A}_{210} \\
\mathcal{D}_{120} &= \mathcal{A}_{210} + \mathcal{A}_{120} \\
\mathcal{D}_{201} &= \mathcal{A}_{210} + \mathcal{A}_{201} \\
\mathcal{D}_{102} &= \mathcal{A}_{210} + \mathcal{A}_{120} + \mathcal{A}_{201} + \mathcal{A}_{102} \\
\mathcal{D}_{021} &= \mathcal{A}_{210} + \mathcal{A}_{120} + \mathcal{A}_{201} + \mathcal{A}_{021} \\
\mathcal{D}_{012} &= \mathcal{A}_{210} + \mathcal{A}_{120} + \mathcal{A}_{201} + \mathcal{A}_{102} + \mathcal{A}_{021} + \mathcal{A}_{012}
\end{aligned}
$$

Demazure atoms

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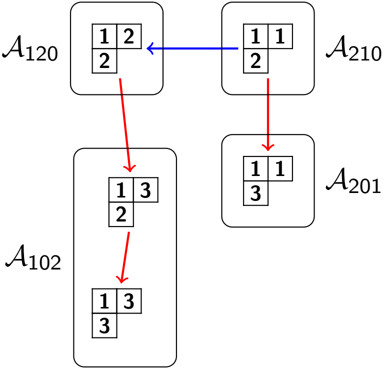


$$\mathcal{D}_{102} = \mathcal{A}_{210} + \mathcal{A}_{120} + \mathcal{A}_{201} + \mathcal{A}_{102}$$

Weyl symmetrization

- For $w = s_{i_1} s_{i_2} \cdots s_{i_m} \in S_N$ reduced, $\pi_w := \pi_{i_1} \pi_{i_2} \cdots \pi_{i_m}$.
- π_{w_0} is the *Weyl symmetrization operator*.
- "Non-partition Schur function" $s_\alpha := \pi_{w_0} \mathbf{x}^\alpha$ is $\pm$ an ordinary Schur function or 0, for any $\alpha \in \mathbb{N}^N$.
- $\pi_{w_0} \mathcal{A}_\alpha = \begin{cases} s_\alpha(x_1, \ldots, x_N) & \text{if } \alpha \text{ is a partition} \\ 0 & \text{otherwise.} \end{cases}$

Atom positivity

Macdonald positivity: the modified Macdonald polynomials $H_\mu(\mathbf{x}; q, t)$ are Schur positive.

Theorem (B.-Haiman-Morse-Pun-Seelinger)

The modified r-nonsymmetric Macdonald polynomials Weyl symmetrize to modified Macdonald polynomials:

$$\pi_{w_0}\, {}_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}(x_1, \ldots, x_N; q, t) = H_{(\eta;\lambda)_+}(x_1, \ldots, x_N; q, t),$$

where $(\eta; \lambda)_+$ is the partition rearrangement of the concatenation $(\eta; \lambda)$.

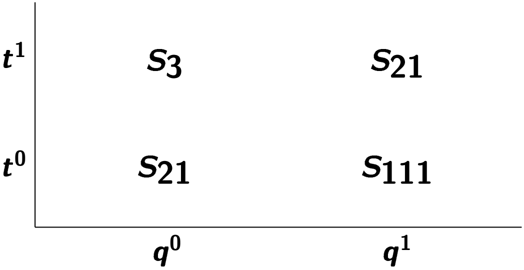
Conjecture (B.-Haiman-Morse-Pun-Seelinger)

The ${}_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}$ are Demazure atom positive.

This gives a conjectural strengthening of Macdonald positivity.

Atom positivity

- $\pi_{w_0} \mathrm{nsH}_{\eta|\lambda} = H_{(\eta;\lambda)_+}$.
- Conj: $_{\mathrm{ns}}\mathrm{H}_{\eta|\lambda}$ are Demazure atom positive.



symmetric Macdonald $H_{21}(x_1, x_2, x_3; q, t)$ in Schurs

Atom positivity

- $\pi_{w_0} \mathsf{nsH}_{\eta|\lambda} = H_{(\eta;\lambda)_+}$.
- Conj: $\mathsf{nsH}_{\eta|\lambda}$ are Demazure atom positive.

	q^0	q^1
t^1	$\mathcal{A}_{300}$ $\mathcal{A}_{030}$ $\mathcal{A}_{003}$	$\mathcal{A}_{210}$ $\mathcal{A}_{120}$ $\mathcal{A}_{201}$ $\mathcal{A}_{102}$ $\mathcal{A}_{021}$ $\mathcal{A}_{012}$
t^0	$\mathcal{A}_{210}$ $\mathcal{A}_{120}$ $\mathcal{A}_{201}$ $\mathcal{A}_{102}$ $\mathcal{A}_{021}$ $\mathcal{A}_{012}$	$\mathcal{A}_{111}$

symmetric Macdonald $H_{21}(x_1, x_2, x_3; q, t)$ in Demazure atoms

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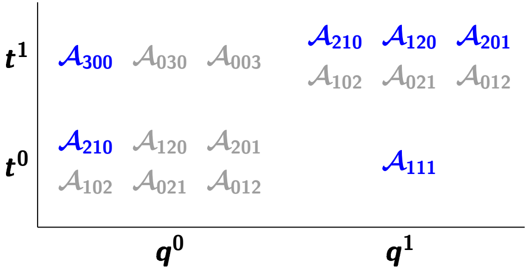
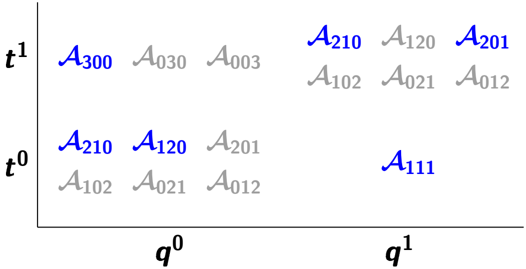


$\mathsf{nsH}_{21|\varnothing}(x_1, x_2, x_3; q, t)$ in Demazure atoms

$$\begin{aligned}\mathsf{nsH}_{21|\varnothing} &= t\,x_1^3 + x_1^2x_2 + qt\,x_1^2x_2 + qt\,x_1x_2^2 + qt\,x_1^2x_3 + q\,x_1x_2x_3 \\ &= t\,\mathcal{A}_{300} + \mathcal{A}_{210} + qt\,\mathcal{A}_{210} + qt\,\mathcal{A}_{120} + qt\,\mathcal{A}_{201} + q\,\mathcal{A}_{111}\end{aligned}$$

Atom positivity

- $\pi_{w_0}\mathrm{nsH}_{\eta|\lambda} = H_{(\eta;\lambda)_+}$.
- Conj: $\mathrm{nsH}_{\eta|\lambda}$ are Demazure atom positive.



$\mathrm{nsH}_{12|\varnothing}(x_1, x_2, x_3; q, t)$ in Demazure atoms

t-adic limit

- $\mathcal{P}(r) = \mathbb{Q}(q,t)[x_1,\ldots,x_r] \otimes \Lambda_{\mathbb{Q}(q,t)}(x_{r+1},\ldots)$.
- $\mathbf{x} = x_1, x_2, \ldots$.

Def. The sequence $g_1, g_2, \ldots$, with $g_N \in \mathbb{Q}(q,t)[x_1,\ldots,x_N]$, *converges t-adically* to $f(\mathbf{x}) \in \mathcal{P}(r)$ if for all $e \geq 0$,

$$g_N(x_1,\ldots,x_N) - f(x_1,\ldots,x_N,0,0,\ldots)$$

has coefficients whose order of vanishing in t is at least e, for sufficiently large N.

Example.

- $1, 1+t, 1+t+t^2, \ldots \to \frac{1}{1-t}$.
- $x_1, tx_1 + x_2, t^2x_1 + x_2 + x_3, t^3x_1 + x_2 + x_3 + x_4 \to x_2 + x_3 + \cdots \in \mathcal{P}(1)$.

Stable nonsymmetric Macdonald polynomials

Recall $\mathcal{E}_\alpha(\mathbf{x}; q, t) =$ integral form nonsymmetric Macdonald polynomials.

Def. For $(\eta|\lambda) \in \mathbb{N}^r \times \text{Par}$, the *integral form stable r-nonsymmetric Macdonald polynomial* ${}_{\text{stable}}\mathcal{E}_{\eta|\lambda}(\mathbf{x}; q, t) \in \mathcal{P}(r)$ is given by

$$ {}_{\text{stable}}\mathcal{E}_{\eta|\lambda}(\mathbf{x}; q, t) = \lim_{n\to\infty} \mathcal{E}_{(\eta;0^n;\lambda)}(x_1, \dots, x_{r+n}, 0^{\ell(\lambda)}; q, t). $$

Remark. The ${}_{\text{stable}}\mathcal{E}_{\eta|\lambda}(\mathbf{x}; q, t)$ are integral forms of stable versions introduced by Bechtloff Weising.

Combinatorial and algebraic descriptions agree

- Define pol: $\mathbb{Q}(q,t)[x_1^{\pm 1}, \cdots, x_r^{\pm 1}] \to \mathbb{Q}(q,t)[x_1, \ldots, x_r]$ by

$$\mathrm{pol}(\mathcal{D}_\alpha) = \begin{cases} \mathcal{D}_\alpha & \text{for } \alpha \in \mathbb{N}^r \\ 0 & \text{for } \alpha \in \mathbb{Z}^r \setminus \mathbb{N}^r. \end{cases}$$

Def. The *r-nonsymmetric plethysm map* $\Pi_r : \mathcal{P}(r) \to \mathcal{P}(r)$ is given on $f(x_1, \ldots, x_r)g(\mathbf{x})$, where g is symmetric in $\mathbf{x} = x_1, x_2, \ldots,$ by

$$\Pi_r(f(x_1, \ldots, x_r)g(\mathbf{x})) = g\left[\frac{\mathbf{x}}{(1-t)}\right] \mathrm{pol}\left(\frac{f(x_1, \ldots, x_r)}{\prod_{1 \le i < j \le r}(1 - t x_i/x_j)}\right).$$

Theorem (B.-Haiman-Morse-Pun-Seelinger)

Let $(\eta|\lambda) \in \mathbb{N}^r \times \mathrm{Par}$ *and set* $\beta = (\eta; \lambda)$. *Then*

$$\Pi_r(\mathrm{stable}\mathcal{E}_{\eta|\lambda}(\mathbf{x}; q, t)) = \mathrm{nsH}_{\eta|\lambda}(\mathbf{x}; q, t)$$

$$= t^{n(\beta_+)} \sum_{\substack{r\text{-flagged fillings } T \\ \text{of } \mathrm{cdg}(\beta)}} \left(\prod_{b \in \mathrm{Des}(T)} q^{\mathrm{arm}(b)+1} t^{\mathrm{leg}(b)} \right) t^{-\mathrm{inv}(T)} \mathbf{x}^T.$$

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Theorem (B.-Haiman-Morse-Pun-Seelinger)

Let $(\eta|\lambda) \in \mathbb{N}^r \times \mathrm{Par}$ *and set* $\beta = (\eta;\lambda)$. *Then*

$$\begin{aligned}\Pi_r(\mathrm{stable}\mathcal{E}_{\eta|\lambda}(\mathbf{x};q,t)) &= \mathrm{nsH}_{\eta|\lambda}(\mathbf{x};q,t) \\ &= t^{n(\beta_+)} \sum_{\substack{r\text{-flagged fillings } T \\ \text{of } \mathrm{cdg}(\beta)}} \Bigg(\prod_{b \in \mathrm{Des}(T)} q^{\mathrm{arm}(b)+1} t^{\mathrm{leg}(b)}\Bigg) t^{-\mathrm{inv}(T)} \mathbf{x}^T.\end{aligned}$$

Filling in the missing corner

$$\begin{array}{ccc}
{}_{\text{stable}}\mathcal{E}_{\eta|\lambda}(\mathbf{x};q,t) & \xrightarrow{\quad \Pi_r \quad} & {}_{\text{ns}}\mathrm{H}_{\eta|\lambda}(\mathbf{x};q,t) \\
\Big\downarrow \text{Hecke sym.} & & \Big\downarrow \text{Weyl sym.} \\
J_{(\eta;\lambda)_+}(\mathbf{x};q,t) & \xrightarrow[f(\mathbf{x}) \mapsto f[\mathbf{x}/(1-t)]]{} & H_{(\eta;\lambda)_+}(\mathbf{x};q,t)
\end{array}$$

- $\eta \in \mathbb{N}^r$, λ is a partition.
- $(\eta;\lambda)_+$ is the partition rearrangement of the concatenation $(\eta;\lambda)$.
- ${}_{\text{stable}}\mathcal{E}_{\eta|\lambda}(\mathbf{x};q,t)$ is the integral form stable r-nonsymmetric Macdonald polynomial.
- ${}_{\text{ns}}\mathrm{H}_{\eta|\lambda}(\mathbf{x};q,t)$ is the modified r-nonsymmetric Macdonald polynomial.
- $J_{(\eta;\lambda)_+}(\mathbf{x};q,t)$ is the integral form Macdonald polynomial.
- $H_{(\eta;\lambda)_+}(\mathbf{x};q,t)$ is the modified Macdonald polynomial.

LLT polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_\ell$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(a) = \mathrm{content}(b)$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i < j$, or
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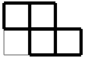
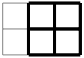
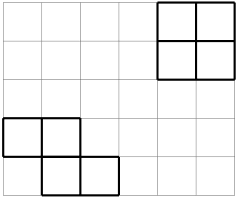
Example.

$\boldsymbol{\nu} = \Big($  $,$  $\Big)$



Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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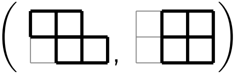
$\boldsymbol{\nu} =$ 

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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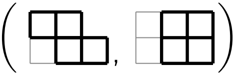
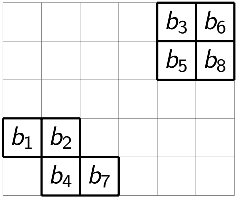
$\boldsymbol{\nu} =$ 



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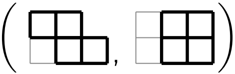
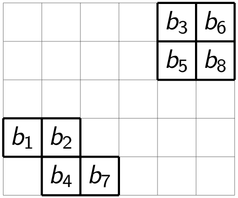
$\boldsymbol{\nu} =$ 



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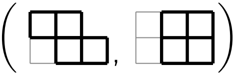
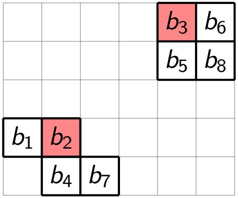
$\boldsymbol{\nu} =$ 



Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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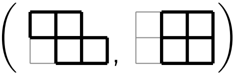
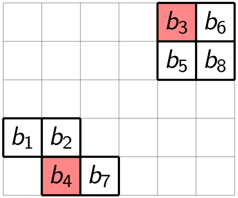
$\boldsymbol{\nu} =$ 



Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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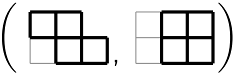
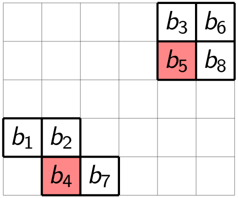
$\boldsymbol{\nu} =$ 



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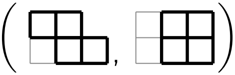
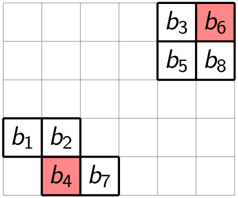
$\boldsymbol{\nu} =$ 



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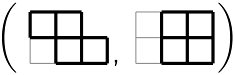
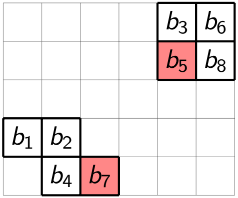
$\boldsymbol{\nu} =$ 



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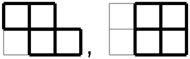
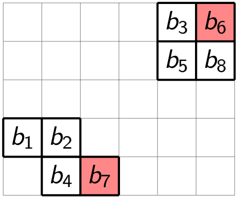
$\boldsymbol{\nu} = \Big($  $\Big)$



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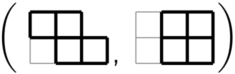
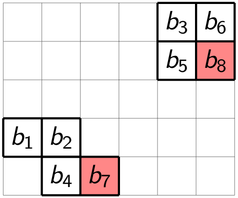
$\boldsymbol{\nu} =$ 



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- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
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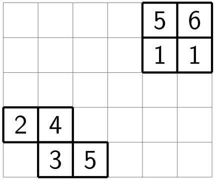
Def. The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\mathbf{x}; t) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} t^{\mathrm{inv}(T)} \mathbf{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\mathbf{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T =$ 

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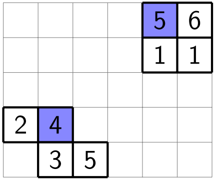
$T =$ 

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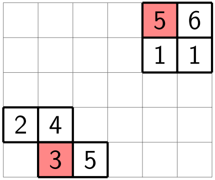

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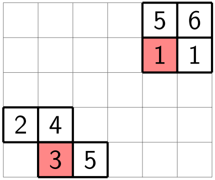
$T =$ 

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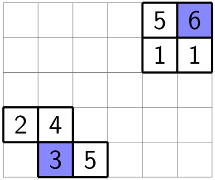

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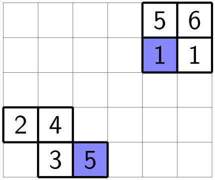

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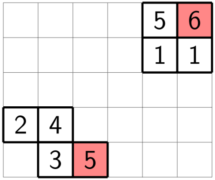

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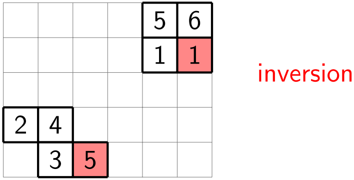

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Flagged LLT polynomials

- Let $e_1, \ldots, e_d$ be the row ends of $\boldsymbol{\nu}$, ordered in reverse reading order.
- Fix a nonnegative integer $r \le d$.
- $T \in \mathrm{SSYT}(\boldsymbol{\nu})$ is *flagged* if $T(e_i) \le i$ for $i = 1, 2, \ldots, r$.
- $\mathrm{FT}_r(\boldsymbol{\nu})$ = set of flagged semistandard tableaux on $\boldsymbol{\nu}$.

Def. The *flagged LLT polynomial* indexed by r and $\boldsymbol{\nu}$ is

$$\mathcal{G}_{r,\boldsymbol{\nu}}(\mathbf{x};t) = \sum_{T \in \mathrm{FT}_r(\boldsymbol{\nu})} t^{\mathrm{inv}(T)} \mathbf{x}^T,$$

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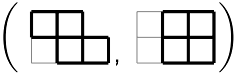
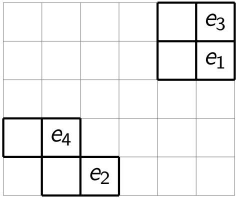
$\boldsymbol{\nu} =$ 



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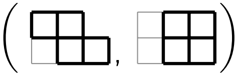
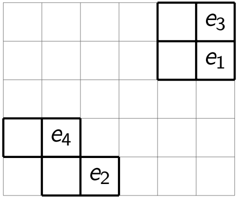
$\boldsymbol{\nu} =$ 



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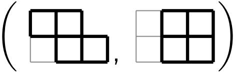
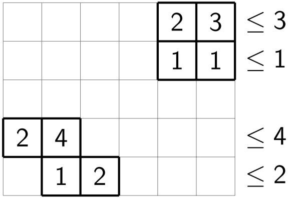
$\boldsymbol{\nu} =$ 

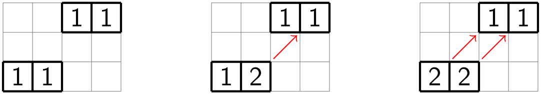


$T \in \mathrm{FT}_r(\boldsymbol{\nu})$ for $r = 4$

Flagged LLT polynomials

Example. $r = 2 \qquad \boldsymbol{\nu} = (\square\square\, , \square\square)$

T



$t^{\mathrm{inv}(T)}$ $\qquad 1 \qquad t \qquad t^2$

$$\mathcal{G}_{r,\boldsymbol{\nu}}(\mathbf{x};t) = x_1^4 + t\,x_1^3x_2 + t^2x_1^2x_2^2$$

Example. $r = 1 \qquad \boldsymbol{\nu} = (\square\, , \square\square)$

T

$\cdots$

$t^{\mathrm{inv}(T)}$ $\qquad 1 \qquad t \qquad t$

$$\mathcal{G}_{r,\boldsymbol{\nu}}(\mathbf{x};t) = x_1^3 + t\,x_1^2(x_2 + x_3 + \cdots)$$

Flagged LLT polynomials

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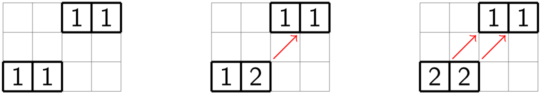

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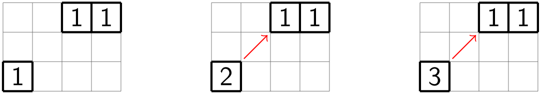

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Signed flagged LLT polynomials

- Signed alphabet $\mathcal{A} = 1 < \overline{1} < 2 < \overline{2} \cdots$
- $\mathsf{FT}_r^{\pm}(\boldsymbol{\nu})$ = fillings of $\boldsymbol{\nu}$ from $\mathcal{A}$ satisfying
  - unbarred letters weakly increase in rows, strictly increase in columns.
  - barred letters strictly increase in rows, weakly increase in columns.
  - $T(e_i) \leq i$ for $i = 1, \ldots, r$.

Def. The *signed flagged LLT polynomial* indexed by r and $\boldsymbol{\nu}$ is

$$\mathcal{G}^{\pm}_{r,\boldsymbol{\nu}}(\mathbf{x};t) = \sum_{T \in \mathsf{FT}_r^{\pm}(\boldsymbol{\nu})} t^{\mathrm{inv}(T)}(-t)^{-\#\mathrm{bar}(T)}\mathbf{x}^{|T|},$$

where $|T|$ is the result of removing all bars from T.

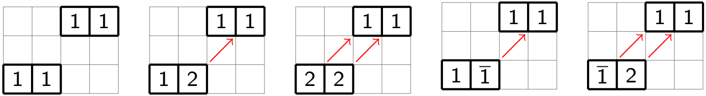
Signed flagged LLT polynomials

- Signed alphabet $\mathcal{A} = 1 < \overline{1} < 2 < \overline{2} \cdots$

$$\mathcal{G}^{\pm}_{r,\boldsymbol{\nu}}(\mathbf{x};t) = \sum_{T \in \mathrm{FT}^{\pm}_{r}(\boldsymbol{\nu})} t^{\mathrm{inv}(T)}(-t)^{-\#\mathrm{bar}(T)}\mathbf{x}^{|T|}.$$

Example. $r = 2 \qquad \boldsymbol{\nu} = (\square\square\,,\,\square\square)$

T



$t^{\mathrm{inv}(T)}$	1	t	t^2	t	t^2
$(-t)^{-\#\mathrm{bar}(T)}$	1	1	1	$-t^{-1}$	$-t^{-1}$

$$\begin{aligned} \mathcal{G}^{\pm}_{r,\boldsymbol{\nu}}(\mathbf{x};t) &= x_1^4 + t x_1^3 x_2 + t^2 x_1^2 x_2^2 - x_1^4 - t x_1^3 x_2 \\ &= t^2 x_1^2 x_2^2 \end{aligned}$$

Π_r takes signed LLTs to unsigned LLTs

Well-known fact: in the $r = 0$ (fully symmetric) case,

$$\mathcal{G}^{\pm}_{0,\boldsymbol{\nu}}(\mathbf{x}; t^{-1}) = \mathcal{G}_{0,\boldsymbol{\nu}}[\mathbf{x}(1-t); t^{-1}].$$

Theorem (B.-Haiman-Morse-Pun-Seelinger)

The r-nonsymmetric plethysm map Π_r takes signed LLTs to unsigned LLTs:

$$\Pi_r\big(\mathcal{G}^{\pm}_{r,\boldsymbol{\nu}}(\mathbf{x}; t^{-1})\big) = \mathcal{G}_{r,\boldsymbol{\nu}}(\mathbf{x}; t^{-1}).$$

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Theorem (B.-Haiman-Morse-Pun-Seelinger)

The r-nonsymmetric plethysm map Π_r takes signed LLTs to unsigned LLTs:

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Example. $r = 2 \quad \boldsymbol{\nu} = \big(\square\square\,,\,\square\square\big)$

$$\mathcal{G}^{\pm}_{r,\boldsymbol{\nu}}(\mathbf{x};t^{-1}) = t^{-2}x_1^2x_2^2$$

$$\mathcal{G}_{r,\boldsymbol{\nu}}(\mathbf{x};t^{-1}) = t^{-2}x_1^2x_2^2 + t^{-1}x_1^3x_2 + x_1^4$$

$$\begin{aligned}
\Pi_r(\mathcal{G}^{\pm}_{r,\boldsymbol{\nu}}(\mathbf{x};t^{-1})) &= \Pi_r\big(t^{-2}x_1^2x_2^2\big)\\
&= \operatorname{pol}\left(\frac{t^{-2}x_1^2x_2^2}{1-tx_1/x_2}\right)\\
&= t^{-2}\operatorname{pol}\big(x_1^2x_2^2 + tx_1^3x_2^1 + t^2x_1^4x_2^0 + t^3x_1^5x_2^{-1} + \cdots\big)\\
&= t^{-2}x_1^2x_2^2 + t^{-1}x_1^3x_2 + x_1^4 = \mathcal{G}_{r,\boldsymbol{\nu}}(\mathbf{x};t^{-1})
\end{aligned}$$

LLT and Macdonald polynomials

- We show how to convert the nonsymmetric Haglund-Haiman-Loehr formula for $\mathcal{E}_\alpha$ to a signed flagged LLTs formula for $\mathcal{E}_\alpha$.
- Signed flagged LLT formula for their stable limits ${}_{\text{stable}}\mathcal{E}_{\eta|\lambda}(\mathbf{x}; q, t)$.
- Π_r turns this into a flagged LLT formula for $H_{\eta|\lambda}(\mathbf{x}; q, t)$.
- Π_r takes Hecke symmetrization to Weyl symmetrization.

LLT and Macdonald polynomials

$\mathcal{E}_\alpha(x_1,\ldots,x_N;q,t) =$
$\sum$ signed flagged LLTs

stabilize

$\text{stable}\mathcal{E}_{\eta|\lambda}(\mathbf{x};q,t) =$
$\sum$ signed flagged LLTs

Π_r

$\text{nsH}_{\eta|\lambda}(\mathbf{x};q,t) =$
$\sum$ flagged LLTs

Hecke sym.

Weyl sym.

$J_{(\eta;\lambda)_+}(\mathbf{x};q,t) =$
$\sum$ signed LLTs

$f(\mathbf{x}) \mapsto f[\mathbf{x}/(1-t)]$

$H_{(\eta;\lambda)_+}(\mathbf{x};q,t) =$
$\sum$ LLTs