Affine Higher Bruhat Orders

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Outline

Classical Permutations and Affine Permutations

Defn. A *permutation* in S_n is a bijection $w : [n] \to [n]$, or equivalently a total order $w = w(1)w(2)\dots w(n)$ on $\{1, 2, \dots, n\}$.

Defn. An *affine permutation* in $\widetilde{S}_n$ is a bijection $w : \mathbb{Z} \to \mathbb{Z}$ s.t.

1. n-**periodicity**: for all $i \in \mathbb{Z}$, we have $w(i+n) = w(i) + n$,
2. **binomial window sum**: $\sum_{i \in [n]} w(i) = \binom{n+1}{2}$.

Examples. $4123, 1720 \in \widetilde{S}_4$ determine 4-periodic total orders on $\mathbb{Z}$

$$4123 = (\dots, -5, 0, -3, -2, -1, 4, 1, 2, 3, 8, 5, 6, 7, 12, 9, \dots)$$

$$1720 = (\dots, -8, -3, 3, -2, -4, 1, 7, 2, 0, 5, 11, 6, 4, 9, 15, \dots)$$

Coxeter Group Structure

Fact. The *Affine Symmetric Group* $\widetilde{S}_n$ is a Coxeter group. The generators are the *adjacent transpositions* $s_1, \ldots, s_n$ with indices taken mod n so $s_0 = s_n$.

The adjacent transpositions satisfy the "Coxeter relations":

1. *involution*: $s_i^2 = \mathrm{id}$ for all $i\mathbb{Z}$,
2. *braid*: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i \in \mathbb{Z}$
3. *commutation*: $s_i s_j = s_j s_i$ for $i, j \in \mathbb{Z}$ s.t. $j \not\equiv i, i+1, i-1 \mod n$.

Example. $w = 1720 = (\ldots, -2, -4, 1, 7, 2, 0, 5, 11, 6, \ldots) \in \widetilde{S}_4$.
Swap values in positions 1,2:

$$1720 s_1 = 7120 = (\ldots, -2, -4, 7, 1, 2, 0, 11, 5, 6, \ldots)$$

Swap in positions 4,5:

$$1720 s_4 = (\ldots, -2, 1, -4, 7, 2, 5, 0, 11, 6, \ldots) = -4725$$

Weak Bruhat Orders

Def. The *weak (Bruhat) order* is the partial order on $\widetilde{S}_n$ defined by the covering relations of the form $w \lessdot ws_i$ if $w(i) < w(i+1)$.

Fact. The saturated chains in the weak Bruhat order on the interval $[id, w]$ are in bijection with the minimal length expressions $w = s_{a_1} s_{a_2} \cdots s_{a_p}$.

Fact. $v \leq w \iff \mathrm{Inv}(v) \subset \mathrm{Inv}(w)$.

Fact. The Hasse diagram of weak Bruhat order on $S_n = [id, w_0]$ is the 1-skeleton of the permutahedron.

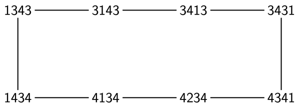
Reduced Expressions and Reduced Words

Def. A *reduced word* for $w \in \widetilde{S}_n$ is a minimal length word $a_1 a_2 \cdots a_p$ on the alphabet $[n]$ such that $s_{a_1} s_{a_2} \cdots s_{a_p} = w$, and p is the *length* of w, denoted $\ell(w) = p = \mathrm{Inv}(w)$.

Def. Let $R(w)$ be the graph on the reduced words for w with an edge connecting two reduced words if they differ by a braid or commutation relation.

Fact. (Matsumoto–Tits) The graph $R(w)$ is connected.

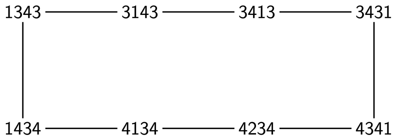
Example. $w = 21543 \in S_5$ and $\widetilde{S}_5$



Commutation Classes of Reduced Expressions

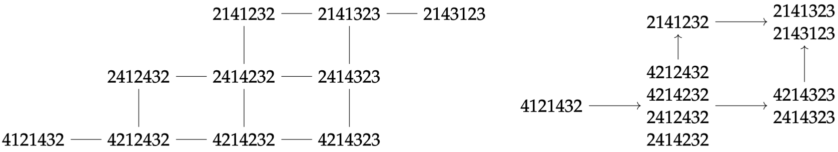
Def. Two reduced words are *commutation-equivalent* if the differ by a sequence of commutation relations.

Def. Let $G(w)$ be the directed graph obtained from $R(w)$ by orienting the braid edges so $i(i+1)i \to (i+1)i(i+1)$ and contracting edges corresponding with commutation relations.



Elias's Conjecture

Example. $R(w)$ and $G(w)$ for $w = 1720 \in \widetilde{S}_n$



Conjecture. (Ben Elias) For $w \in \widetilde{S}_n$, $G(w)$ is acyclic.

Previously known to hold for $w_0 = n(n-1)\ldots 21$ by work of Manin-Schechtman.1989 and for $w \in S_n$ by work of Assaf.2019.

Mannin-Schechtman's Admissible Orders $\mathcal{A}(n,k)$

Notation.

- $\binom{[n]}{k}$ = size k subsets of $[n] = \{1,2,3,\ldots,n\}$
- For $X = [x_1 < x_2 < \cdots < x_k] \in \binom{[n]}{k}$, let $X_i = [x_1 < \cdots < \widehat{x_i} < \cdots < x_k]$
- $P(X) = \{X_1, X_2, \ldots, X_k\}$ = $k-1$-subsets of X = *Packet of X*
- $\mathcal{A}(n,k)$ = all *admissible orders* of $\binom{[n]}{k}$

Def. A total order on $\binom{[n]}{k}$ is *admissible* if every packet is either in lexicographic (lex) order or the reverse (antilex).

Example. $\mathcal{A}(4,3) = \{(123, 124, 134, 234), \quad (234, 134, 124, 123)\}$

2-Admissible Orders and Reflection Orders

There are 16 admissible total orders on $\binom{[4]}{2}$ including:

$(12, 34, 14, 24, 13, 23), (34, 12, 14, 24, 13, 23), (34, 24, 14, 12, 13, 23)$

Fact.: The 2-admissible sequences are exactly Dyer's "reflection orders" for the permutation $w_0 = n \ldots 21$, which are in bijection with reduced words for w_0.

Higher Analogs of Coxeter Relations and $R(w)$ graphs

Commutation. $\rho, \tau \in \mathcal{A}(n,k)$ differ by a *commutation move* if

$$\rho = (\alpha,\ X, Y,\ \beta), \qquad \tau = (\alpha,\ Y, X,\ \beta)$$

and X, Y *commute*, meaning $P(X) \cap P(Y) = \varnothing$.

Braid Analog. $\rho, \tau \in \mathcal{A}(n,k)$ differ by a *packet flip* for X if

$$\rho = (\alpha,\ X_1, X_2, \ldots, X_{k+1},\ \beta)$$
$$\tau = (\alpha,\ X_{k+1}, \ldots, X_2, X_1,\ \beta).$$

Thm.(M-S) The admissible orders on $\binom{[n]}{k}$ are connected by commutation moves and packet flips.

Higher Order Inversion Sets

Def. The *reversal/inversion set* of $\rho \in \mathcal{A}(n,k)$ is determined by the set of all antilex packets in the total order ρ

$$\mathrm{Rev}(\rho) = \{X \in \binom{[n]}{k+1} \mid X_1 <_\rho X_2 <_\rho \cdots <_\rho X_{k+1})\}$$

Example. For $\rho = (34, 12, 14, 24, 13, 23) \in \mathcal{A}(4,2)$,

- $\mathrm{Rev}(\rho)$ = ???
- $[\rho]$ = commutation class of ρ = ???

Thm.(M-S) $\mathrm{Rev}(\rho) = \mathrm{Rev}(\tau)$ if and only if $[\rho] = [\tau]$.

Manin-Schechtman's Higher Bruhat Orders

The *Higher Bruhat Order* $\mathcal{B}(n,k) = \{[\rho] \mid \rho \in \mathcal{A}(n,k)\}$, along with the transitive closure of the order relations $[\rho] < [\tau]$ if ρ and τ differ by a packet flip that is lex ordered in ρ and antilex in τ.

Thm.(Manin-Schechtman, 1989) Let $1 \le k \le n$.

1. $\mathcal{B}(n,k)$ is poset.
2. $\mathcal{B}(n,k)$ is ranked by $|\mathrm{Rev}(\rho)|$.
3. $\mathcal{B}(n,k)$ has a unique minimal and maximal elements:

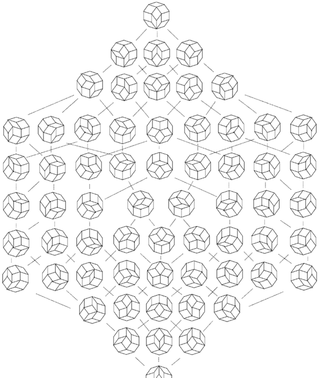
$$\mathrm{Rev}(\hat{0}) = \varnothing, \qquad \mathrm{Rev}(\hat{1}) = \binom{[n]}{k+1}.$$

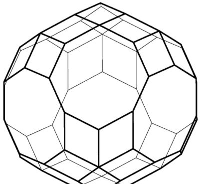
4. There is a bijection between maximal chains of $\mathcal{B}(n,k)$ and $\mathcal{A}(n,k+1)$ given by single step inclusion of reversal sets.

Connections to Weak Bruhat Order

Corollary. $\mathcal{B}(n, 1)$ is isomorphic to the weak Bruhat order on S_n.
Corollary. The Hasse diagram of $\mathcal{B}(n, 2)$ is $G(n, n-1, \ldots, 1)$.

Example. The 62 elements of $\mathcal{B}(5, 2)$ from "Zonotopes associated with higher Bruhat orders" by Felsner and Ziegler, 2001, and its corresponding polytopal realization.





Motivation

Manin-Schechtman first defined “higher order” analogs of the braid groups to be the fundamental groups of manifolds $U(n,k)$ given as the complement of discriminantal hyperplane arrangements. Packet flips and commutations relate to the order in which a path can move across an intersection of hyperplanes, so give rise to higher order Coxeter relations in the fundamental groups.

Other related work including. Athanasiadis, Bayer-Brandt, Chau, Elias, Felsner-Ziegler, Galashin-Postnikov, Hothem, Rambau-Reiner, ShelleyAbrahamson-Vijaykumar, Ziegler.

Ziegler's Consistent Sets

Defn. A subset $R \subset \binom{[n]}{k}$ is *consistent* provided $P(X) \cap R$ contains a prefix or suffix of lex order on $P(X) = \{x_1, X_2, \ldots, X_{k+1}\}$ for every $X \in \binom{[n]}{k+1}$.

$\mathcal{C}(n, k)$ = poset of all consistent subsets of $\binom{[n]}{k}$ ordered by *single step inclusion*.

Examples. For $n = 4, k = 3$, $\{\}$, $\{234, 134\}$ and $\{123, 124, 134\}$ are consistent, but $\{124, 134\}$ is not consistent.

Consistent Sets Characterize Reversal Sets

Thm. (Ziegler.1993) For $1 \leq k \leq n$, there is a poset isomorphism

$$\text{Rev} :\to \mathcal{C}(n, k+1)$$

Thm.(Ziegler.1993) $\mathcal{B}(n,k) \approx \mathcal{C}(n, k+1)$ is also isomorphic to

1. the set of extensions of the cyclic arrangement $X_c^{n,n-k-1}$ by a new pseudo-hyperplane in general position, ordered by single step inclusion of their vertex sets, and
2. the poset of all uniform single element extensions of the corresponding alternating oriented matroid.

Consistent Sets and Extensions of the Cyclic Arrangement

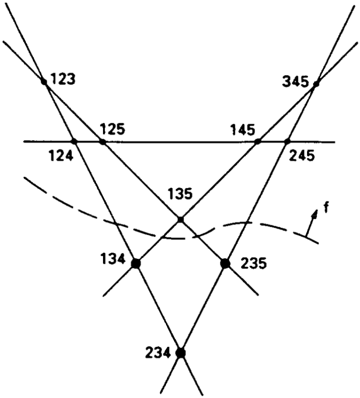


Fig. 1. The cyclic arrangement $X_c^{5,2}$ with a pseudoline extension f and the corresponding vertex set.

From Ziegler, "Higher Bruhat Orders and Cyclic Hyperplane Arrangements"

Affine Higher Bruhat Orders

Elias and Hothem. Can the higher Bruhat orders be extended to other intervals in the weak order on $\widetilde{S}_n$?

Challenges.

1. There is no analog of the longest permutation w_0.
2. Affine permutations are periodic total orders on $\mathbb{Z}$.
3. Subsets of $\mathbb{Z}$ may contain congruent elements.

Affine Analog of k-Subsets of $[n]$

- Let $\sim_n$ be an equivalence relation on $\binom{\mathbb{Z}}{k}$ where

$$\{x_1, x_2, \cdots, x_k\} \sim_n \{y_1 < y_2 < \cdots < y_k\}$$

  if $k > 1$ and there exists an integer m such that

$$\{x_1, x_2, \ldots, x_k\} = \{y_1 + mn, y_2 + mn, \ldots, y_k + mn\}.$$

  If $k = 1$, then $\{x\} \sim_n \{y\}$ if and only if $x = y$.

- Consider the equivalence classes of $\binom{\mathbb{Z}}{k}$

$$\binom{\mathbb{Z}}{k}_n = \{[x_1 < \ldots < x_k] : x_i \not\equiv x_j \pmod{n} \text{ for } 1 \le i < j \le k\}.$$

- For $X \in \binom{\mathbb{Z}}{k}_n$, the *packet* of X is $P(X) = \{X_1, X_2, \ldots, X_k\}$.

k-inversions

Def. The *k-inversion set* of $w \in \widetilde{S}_n$ is

$$\mathrm{Inv}_k(w) = \left\{[x_1, \ldots, x_k] \in \binom{\mathbb{Z}}{k}_n : w^{-1}(x_1) > \ldots > w^{-1}(x_k)\right\}.$$

Example. For $7120 = (\ldots, 3, -3, -2, -4, 7, 1, 2, 0, 11, 5, 6, 4, \ldots)$,

$$\mathrm{Inv}_2(7120) = \{[2,7],[2,3],[1,7],[1,3],[0,7],[0,3],[0,2],[0,1]\}$$

$$\mathrm{Inv}_3(7120) = \{[0,2,7],[0,2,3],[0,1,7],[0,1,3]\}.$$

$$\mathrm{Inv}_4(7120) = \{\}.$$

Nice properties:.

1. If $|P(X) \cap \mathrm{Inv}_k(w)| > 2$, then $P(X) \cap \mathrm{Inv}_k(w) = P(X)$.
2. If $[x_1, \ldots, x_k], [y_1, \ldots, y_k] \in \mathrm{Inv}_k(w)$ have the same set of congruence classes, then $x_i \equiv y_i \mod n \forall i$.

Affine Inversion Sets

Thm.(Bjorner-Brenti.1995) Affine permutations in $\widetilde{S}_n$ are unique determined by their inversion sets

$$\mathrm{Inv}(w) = \{(x < y) \in [n] \times \mathbb{Z} \mid w^{-1}(x) > w^{-1}(y)\} \leftrightarrow \mathrm{Inv}_2(w)$$

Thm. If $R \subseteq \binom{\mathbb{Z}}{2}_n$, then R is the inversion set for some affine permutation in $\widetilde{S}_n$ if and only if for all $[x, y, z] \in \binom{\mathbb{Z}}{3}_n$

- $[x, z] \in R$ implies $[x, y] \in R$ or $[y, z] \in R$,
- $[x, y] \in R$ and $[y, z] \in R$ implies $[x, z] \in R$, and
- $[x, y] \in R$ implies $[x, y - en] \in R \ \forall e \in \mathbb{Z}_{\geq 0}$ s.t. $x \leq y - en \leq y$.

Proof follows from work of Dyer.2019 and Barkley-Speyer.2024.

Computer Experimentation for Affine Higher Bruhat Orders

1. Start with $k = 1$, define $\mathcal{B}_w(n, 1) := [id, w]$ in weak order using $2 - \textit{inversion}$ sets ordered by single step inclusion.
2. Define the admissible orders $\mathcal{A}_w(n, k + 1)$ to be the set of saturated chains in $B_w(n, k)$. Verify these can be encoded as total orders on $\mathrm{Inv}_{k+1}(w)$.
3. Use $\mathcal{A}_w(n, k + 1)$ to determine the refined rules for commutation on k-inversions for w.
4. Define a directed graph on the commutation classes $\{[\rho] \mid \rho \in \mathcal{A}_w(n, k + 1)\}$ with directed edges $[\rho] \to [\tau]$ if they have representatives that differ by a packet flip and $\mathrm{Rev}(\rho) \subset \mathrm{Rev}(\tau)$.
5. Verify the directed graph is acyclic with a unique sink and source.
6. Define $\mathcal{B}_w(n, k + 1)$ to be the transitive closure of the directed graph, and return to Step 2 to define $\mathcal{A}_w(n, k + 2)$.

Permanent Posets

Def: $\mathcal{P}_w(n,k)$**.** The *permanent poset* is the poset on $\mathrm{Inv}_k(w)$ with order relation $\leq_P$ given by the transitive closure of

1. *quasi-inversion relations*: if $X \in \binom{[n]}{k+1}$ is a *$k+1$-quasi-inversion* for w such that $P(X) \cap \mathrm{Inv}_k(w) = \{X_i, X_{i+1}\}$, then $X_i <_P X_{i+1}$ whenever $(k-i)$ is odd, and $X_{i+1} <_P X_i$ whenever $(k-i)$ is even.
2. *congruence relations*: for all $X \in \binom{\mathbb{Z}}{k}_n$ and $0 \leq i < k$: $X <_P X + (0^i, n^{n-i})$ if $k-i$ is odd, and $X + (0^i, n^{n-i}) <_P X$ if $k-i$ is even.

Affine Higher Bruhat Orders for $1 \le k \le n$ and $w \in \widetilde{S}_n$

Def: $\mathcal{A}_w(n,k)$**.** A total order ρ on $\mathrm{Inv}_k(w)$ is a *k-admissible order* for w if ρ is a linear extension of $\mathcal{P}_w(n,k)$ such that $\rho|_{P(X)}$ is lex or antilex order on $P(X)$ $\forall X \in \mathrm{Inv}_{k+1}(w)$.

$$\mathrm{Rev}_{\mathbf{w},\mathbf{n},\mathbf{k}}(\rho) = \{X \in \mathrm{Inv}_{k+1}(w) : \rho|_{P(X)} = (X_1, X_2, \ldots, X_{k+1})\}.$$

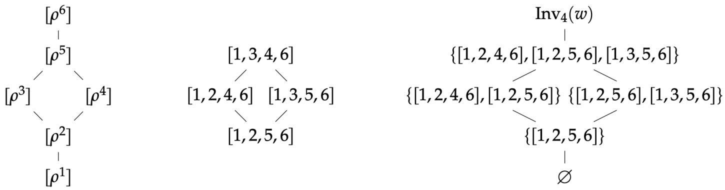
Thm.(Billey-Chau-Liu) The directed graph on $\mathcal{A}_w(n,k)/\sim_w$ where $[\rho] \to [\sigma]$ if σ can be obtained from ρ by a lex-to-antilex packet flip is acyclic.

Def: $\mathcal{B}_w(n,k)$**.** The *k^{th} higher Bruhat order for w* is the transitive closure of the acyclic graph on $\mathcal{A}_w(n,k)/\sim_w$.

Affine Consistent Sets

Def: $\mathcal{C}_w(n,k)$**.** A subset $R \subseteq \mathrm{Inv}_k(w)$ is *consistent with respect to* w if R is a lower-order ideal of $\mathcal{P}_w(n,k)$ that satisfies the *(MSZ) Condition: for any* $X \in \mathrm{Inv}_{k+1}(w)$*, the intersection* $P(X) \cap R$ *is a prefix or suffix of* $(X_1, \dots, X_{k+1})$.
Partially order $\mathcal{C}_w(n,k)$ by single step inclusion.

Example. $w = 645231 \in S_6$, $B_w(6,3)$, $P_w(6,4)$, $C_w(6,4)$:



Main Results for Affine Permutations

Thm.(Billey-Chau-Liu) For $w \in \widetilde{S}_n$, the following hold.

1. There is a natural bijection between maximal chains of $\mathcal{C}_w(n,2)$, $R(w)$, reflection orders for w, and $\mathcal{A}_w(n,2)$.
2. $\mathcal{B}_w(n,2) \cong \mathcal{C}_w(n,3)$ is a ranked poset with a unique minimal and unique maximal element.
3. The Hasse diagram of $\mathcal{B}_w(n,2) \cong \mathcal{C}_w(n,3)$ is isomorphic to $G(w)$ as a directed graph. Furthermore, the diameter of $G(w)$ as an undirected graph is $|\mathrm{Inv}_3(w)|$.

Cor. Elias's conjecture holds: $G(w)$ as a directed graph is acyclic.

Main Results for Classical Permutations

Thm .(Billey-Chau-Liu) For $1 \le k \le n$ and $w \in S_n$,

1. $\mathcal{C}_w(n,k)$ is a ranked poset with a unique minimal element and a unique maximal element.
2. There is a natural bijection between maximal chains of $\mathcal{C}_w(n,k)$ and $\mathcal{A}_w(n,k)$.
3. $\mathcal{B}_w(n,k)$ is isomorphic as a poset to $\mathcal{C}_w(n,k+1)$, and the isomorphism sends an equivalence class of admissible orders to the reversal set of the class.

Related results in Daniel Hothem's Ph.D. thesis.

Open Problem

Conjecture.(Billey-Chau-Liu) For $1 \leq k \leq n$ and $w \in \widetilde{S}_n$,

1. $\mathcal{C}_w(n,k)$ is a ranked poset with a unique minimal element and a unique maximal element.
2. There is a natural bijection between maximal chains of $\mathcal{C}_w(n,k)$ and $\mathcal{A}_w(n,k)$.
3. $\mathcal{B}_w(n,k)$ is isomorphic as a poset to $\mathcal{C}_w(n,k+1)$, and the isomorphism sends an equivalence class of admissible orders to the reversal set of the class.

Questions.

1. Does the geometry of extended cyclic hyperplane arrangments have an analog for affine permutations?
2. If so, is it related to these affine higher Bruhat orders.

Thank You for Listening!