① I CERM Talk

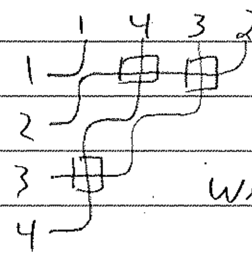
12 November 2025

$(n-1)!$ Formulas for Double Schubert Polynomials

Schubert $\mathfrak{S}_w(x)$ $\left(=\mathfrak{S}_w(x;0)\right)$ & double Schubert Polynomials $\mathfrak{S}_w(x;y)$ represent classes in ordinary & equivariant cohomology of the flag manifold.

Very early (1990's) Combinatorial formulas were given by Billey-Jockush-Stanley, Bergeron-Billey & Fomin-Kirillov.

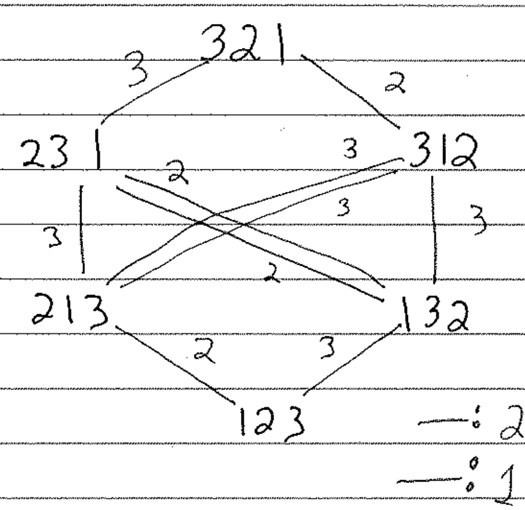
$$\mathfrak{S}_w(x;y) = \sum_{A \in PD(w)} wt(A)$$



$w = 1432$

$wt(A) = (x_1-y_2)(x_1-y_3)(x_3-y_1)$

2002 Bergeron-S
Gave an expression in terms of lexicographic chains $w \longrightarrow w_0$ in Bruhat Order ($w_0 = n, n-1, \ldots, 2, 1$)



Lex: 1) $w = u_0 \xrightarrow{1} u_1 \xrightarrow{2} \cdots \xrightarrow{n-1} u_{n-1} = w_0$

2) In $u_{k-1} \xrightarrow{k} u_k$
- In k-Bruhat Order
- Labels increase

(Pieri chains)

α_k = length of k^{th} chain

$\delta := x_1^{n-1} \cdots x_{n-1}$

Fact:

lexicographic Pieri chains of weight α = coefficient of $x^{\delta-\alpha}$ in $\mathfrak{S}_w(x)$.

(2) ICERM Talk 12 November 2024

Date

$(n-1)!$ Formulas for Double Schubert Polynomials

By Pieri Formula # Lex Pieri chains $w \to w_0$ of weight α
$\deg(\mathfrak{S}_w \cdot h_{\alpha_1}(x_1) \cdot h_{\alpha_2}(x_1,x_2) \cdot \cdots \cdot h_{\alpha_{n-1}}(x_1,\dots,x_{n-1}))$

(1) Coefficient has a geometric interpretation
(2) Formally, $(n-1)!$ different chain formula & combinatorial bijections!
(Not pursued)

2003 Lenart-S. Gave a weight-preserving bijection between PD & Lexicographic Pieri chains $w \to w_0$

2004 Knutson-Miller Gave a remarkable geometric interpretation of Pipe-dream formula

2021 Lam-Lee-Shimozono introduced what are called bumpless Pipe dreams & gave a different formula

(*) $$\mathfrak{S}_w(x;y) = \sum_{A \in BPD(w)} wt(A)$$

2023 Gao-Huang Gave a weight-preserving bijection PD $\leftrightarrow$ BPD

2023 Knutson-Udell Introduce Hybrid Pipe Dreams interpolating between PD & BPD
& $2^?$ Hybrid Pipe Dream formulas for $\mathfrak{S}_w(x;y)$

③ ICERM Talk

Date 12 November 2025

$(n-1)!$ Formulas for Double Schubert Polynomials

Point de départ Yu : 2024

① Gives a combinatorial bijection to deduce $(n-1)!$ chain formulas for $\mathfrak{S}_w(x)$

② Weight-preserving map HPD $\longrightarrow 2^{n-1}$ Pieri chains

B-S type Chain Formulas for double Schubert Polynomials

There are two steps to obtain such formulas; In principal, they apply to Schubert classes in all types (A, B, C, D) of flag manifolds in all flavours: cohomology, K-theory & equivariant versions.

1) Substitution / restriction formulas : Pulling indeterminates out one-at-a-time.

$$\psi_k : R[x_1, \ldots, x_n] \longrightarrow R[x_1, \ldots, x_{n-1}][z]$$

$$\psi_k f = f(x_1, \ldots, x_{k-1}, z, x_k, \ldots, x_{n-1})$$

From first principles $\exists$ formula of the form

$$\psi_k(\mathfrak{S}_w(x;y)) = \sum_{\substack{u \in S_{n-1} \\ a \in \mathbb{N}}} C^{a,u}_w(y)\, \mathfrak{S}_a(z,y') \cdot \mathfrak{S}_u(x;y)$$

$$\mathfrak{S}_{1,a}(z, y') = \prod_{i=1}^{a} (z - y_i') \qquad y' \subseteq \{y_1, \ldots, y_n\}$$

④ ICERM Talk 12 November 2025

$(n-1)!$ Formulas for Double Schubert Polynomials

2017 Adeyemo – S.
The geometry of permutation patterns
$\rightsquigarrow$ A similar formula w/ more summands

1) Indexed by a subsum of Pieri formula $\mathbb{F}l_{n+m}$ $(m \geq a)$
2) $d^{a,u}_{w}(y)$ coeffs in the Pieri formula
3) $y' = \{y_n, y_{n+1}, \ldots, y_{n+a-1}\}$

2019 LRSY Gave a useful equivariant Pieri formula:

$$\mathfrak{S}_u(x;y) \cdot h_{k,a}(x;y) = \sum_{u \xrightarrow{k} w} c^{w}_{u,(k,a)}(y) \, \mathfrak{S}_w(x;y)$$

$$c^{w}_{u,(k,a)}(y) = h_{|A|,b}(y_A ; y_B) \qquad b = a - (\ell(w) - \ell(u))$$

Theorem S-Yu

The LRSY Pieri formula in the A-S pattern formula admits a simplification

$$\psi_k \mathfrak{S}_u(x;y) = \sum_{u \xrightarrow{k} w} \prod (z - y'_i) \, \mathfrak{S}_{w'}(x;y)$$

$w(k) = n$ $w' = w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_{n-1}$

$y' = \{ y_j \mid j = u(l) = w(l) \ \& \ l > k \}$
$= \mathrm{Fix}(k,u,w)$

ICERM Talk 12 November 2025

Date

$(n-1)!$ Formulas for Double Schubert Polynomials

Some re-indexing gives

$$\mathfrak{S}_u(x;y) = \sum_{\substack{u \xrightarrow{k} w \\ w(k)=n}} \prod_{j \in \mathrm{Fix}(k,u,w)} (x_k - y_j)\, \mathfrak{S}_{w|}(x_|\,;y)$$

& Eventually a formula $k_0 = k_1, \ldots, k_{n-1} \in S_{n-1}$

$$\mathfrak{S}_u(x;y) = \sum_{*} (x;y)^{\delta} = \sum_{k_0\text{-Chains } \Lambda} w_t(\Lambda)$$

$* \quad u_0 \xrightarrow{k_1} u_1 \to \cdots \xrightarrow{k_{n-1}} u_{n-1} = w_0$

$$\delta = \prod_{k_i} \prod_{\circledast} (x_{k_i} - y_j)$$

(Has many consequences.)