Saturated Newton polytope of the Kronecker product

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Kronecker coefficients

Symmetric group S_n: permutations $\pi : [1...n] \to [1...n]$ under composition.

The irreducible modules (up to isomorphisms) of S_n are **Specht modules** $\mathbb{S}_\lambda$, indexed by

integer partitions $\lambda \vdash n$: $\lambda = (\lambda_1, \ldots, \lambda_\ell), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0, \sum_{i=1}^{\ell} \lambda_i = n$
Kronecker coefficients $g(\lambda, \mu, \nu)$: the multiplicity of $\mathbb{S}_\nu$ in the tensor product

decomposition of $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}.$$

The Kronecker coefficients generalize the Littlewood-Richardson coefficients.

The **Kronecker product** $*$ of symmetric functions is defined on the Schur basis as

$$s_\lambda * s_\mu := \sum_\nu g(\lambda, \mu, \nu) s_\nu.$$

Combinatorial Interpretations

Problem (Murnaghan 1938, Lascoux, Garsia-Remmel 1980s, Stanley 2000)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$.

- μ and ν are both hooks, and when μ is a two-row partition and ν is a hook partition [Remmel, 1989]
- μ and ν are both two-row partitions, i.e. $\nu = (n-k, k), \lambda = (n-r, r)$ [Remmel-Whitehead, 1994; Rosas, 2001; Blasiak-Mulmuley-Sohoni, 2013]
- A combinatorial interpretation is found when one of the three partitions is a two-row partition $\lambda = (n-k, k)$ with $\lambda_1 \geq 2k-1$. [Ballantine-Orellana, 2006]
- A combinatorial interpretation is found when one partition is a hook, and the other partitions are arbitrary. [Blasiak, 2012; Blasiak-Liu, 2014]
- Other special cases [Bessenrodt-Bowman, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Tewari, etc]

Computational Complexity

Problem

What is the computational complexity of computing Kronecker coefficients?

- Bürgisser–Ikenmeyer (2008): Computing $g(\lambda, \mu, \nu)$ is #P-hard.

Problem

What is the complexity of deciding whether $g(\lambda, \mu, \nu) > 0$?

- Ikenmeyer–Mulmuley–Walter (2015): Deciding positivity is NP-hard.

Newton Polytope

In 2017, Monical-Tokcan-Yong initiated the study of the Newton polytopes of important polynomials in Algebraic Combinatorics.

- Let $f(x_1, \ldots, x_k) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial with nonnegative coefficients.
- Let $M_k(f) := \{\alpha \in \mathbb{Z}^k_{\geq 0} \, : \, c_\alpha > 0\}$.
- The **Newton polytope** of a polynomial $f(x_1, \ldots, x_k)$ is the convex hull of $M_k(f)$ in $\mathbb{Z}^k$, denoted by $N_k(f) := Conv(M_k(f))$.

Newton Polytope

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Definition (SNP)

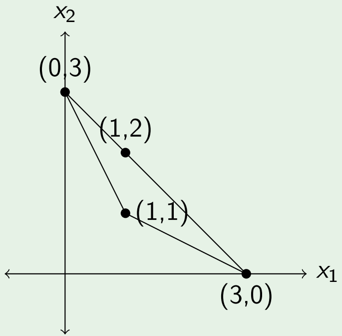
- A polynomial $f(x_1, \ldots, x_k)$ has a *saturated Newton polytope* (SNP) if $M_k(f) = N_k(f)$.
- A symmetric function f has a saturated Newton polytope if its specialization $f(x_1, \ldots, x_k)$ has a SNP for all $k \geq 1$.

Example

$f(x_1, x_2) = x_1x_2 + x_1$ has SNP as the segment $\{(1,1),(1,0)\}$ contains no interior lattice points.

Example

$f(x_1, x_2) = x_1^3 + 3x_1x_2 + x_2^3$



$f(x_1, x_2) = x_1^3 + 3x_1x_2 + x_2^3$ does not have SNP.

Newton polytope of polynomials in Algebraic Combinatorics

- Schur and skew Schur functions, the Stanley symmetric functions and Macdonald polynomials have SNP. [Monical–Tokcan–Yong 2017]
- Schubert polynomials and key polynomials have SNP. [Fink–Mëszäros-St Dizier 2018]
- Double Schubert polynomials has SNP. [Castillo–Cid–Ruiz–Mohammadi–Montaño 2023]
- Grothendieck case was resolved for Grassmannian permutations but remains open in general [Escobar–Yong 2017, Mëszäros-St Dizier 2020]
- in chromatic case, it is known in the case of claw-free incomparability graphs and false in other cases [Monical, Matherne–Morales–Selove 2024]
- non-symmetric Macdonald polynomials has SNP [Black–Weising 2025]

Newton polytope of Kronecker product

Conjecture (Monical–Tokcan–Yong 2017)

The Kronecker product $s_\lambda * s_\mu = \sum_\nu g(\lambda,\mu,\nu) s_\nu$ has a saturated Newton polytope.

$$s_\lambda = \sum_\mu K_{\lambda,\mu}\, m_\mu$$

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu}\, s_\nu$$

$$s_\lambda * s_\mu = \sum_\nu g(\lambda,\mu,\nu)\, s_\nu$$

- Kostka numbers (positivity governed by dominance order $\Rightarrow$ ensures SNP of s_λ)
- Littlewood–Richardson coefficients (LR rule for $s_\lambda s_\mu$ $\Rightarrow$ positivity and SNP of the product)
- Kronecker coefficients

Newton polytope of the Kronecker product

Theorem (Panova–Z)

Let $\lambda, \mu \vdash n$ *with* $\ell(\lambda), \ell(\mu) \leq 2$. *Then* $s_\lambda * s_\mu(x_1, \ldots, x_k)$ *has a saturated Newton polytope for every* k.

Theorem (Panova–Z)

Let $\lambda, \mu \vdash n$ *with* $\ell(\lambda) \leq 2$ *and* $\ell(\mu) \leq 3$. *If* $\mu_1 \geq \lambda_1$, *then the Kronecker product* $s_\mu * s_\lambda(x_1, \ldots, x_k)$ *has a saturated Newton polytope for every* k.

Lemma (Monical–Tokcan–Yong)

Suppose that $\lambda \vdash n$ and $\mu \vdash n$ are such that there is a partition λ with $g(\lambda,\mu,\nu) > 0$ and for every τ, such that $g(\lambda,\mu,\tau) > 0$ we have $\tau \prec \nu$ in the dominance order. Then $M_k(s_\lambda * s_\mu) = M_k(s_\nu) = N_k(s_\nu)$ for all k and $s_\lambda * s_\mu$ has a saturated Newton polytope.

Proof idea: showing that among all partitions $\nu \vdash n$ such that $g(\lambda,\mu,\nu) > 0$, there is a unique maximal term in dominance order.

Theorem (Rosas 2001)

Let β, γ, and α be partitions of n, where $\beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_1, \gamma_2)$ are two two-row partitions and let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a partition of length less than or equal to 4. Assume that $\gamma_2 \leq \beta_2$. Then

$$g(\beta,\gamma,\alpha) = \left(\phi(a,b,a+b+1,c) - \phi(a,b,a+b+c+d+2,c)\right)(\gamma_2, \beta_2+1),$$

where $a = \alpha_3 + \alpha_4$, $b = \alpha_2 - \alpha_3$, $c = \min(\alpha_1 - \alpha_2, \alpha_3 - \alpha_4)$ and $d = |\alpha_1 + \alpha_4 - \alpha_2 - \alpha_3|$.

Conjecture (Monical–Tokcan–Yong 2017)

The Kronecker product $s_\lambda * s_\mu = \sum_\nu g(\lambda, \mu, \nu) s_\nu$ has a saturated Newton polytope.

Question

Does Kronecker product $s_\lambda * s_\mu$ have a saturated Newton polytope when there is no unique maximal term?

Let λ and μ be partitions of n, where $\mu = (\mu_1, \mu_2, \mu_3)$ and $\lambda = (\lambda_1, \lambda_2)$. If $\mu_1 < \lambda_1$, does the Kronecker product $s_\lambda * s_\mu$ have SNP?

Example

$s_{(4,4,4)} * s_{(6,6)} =$
$s_{(2,2,2,2,2,2)} + s_{(3,3,2,2,1,1)} + s_{(3,3,3,3)} + s_{(4,2,2,2,2)} + s_{(4,3,2,2,1)} + s_{(4,3,3,1,1)} +$
$s_{(4,4,1,1,1,1)} + s_{(4,4,2,2)} + s_{(4,4,4)} + s_{(5,2,2,1,1,1)} + s_{(5,3,2,1,1)} + s_{(5,3,3,1)} + s_{(5,4,2,1)} +$
$s_{(5,5,1,1)} + s_{(6,2,2,2)} + s_{(6,3,1,1,1)} + s_{(6,3,2,1)} + s_{(6,4,2)} + s_{(6,6)} + s_{(7,3,1,1)} + s_{(7,4,1)} + s_{(8,2,2)}$

Two and three-row families

Theorem (Panova–Z)

Let $\lambda, \mu \vdash n$ with $\ell(\lambda) \leq 2$ and $\ell(\mu) \leq 3$. Then $s_\lambda * s_\mu(x_1, x_2, x_3)$ has a saturated Newton polytope.

Symmetric function techniques

The Kronecker coefficients can be equivalently defined using

$$s_\nu[x \cdot y] = \sum_{\lambda,\mu} g(\lambda,\mu,\nu) s_\lambda(x) s_\mu(y)$$

where $[x \cdot y] := (x_1y_1, x_1y_2, \ldots, x_2y_1, \ldots)$.

triple Cauchy identity:

$$\sum_{\lambda,\mu,\nu} g(\lambda,\mu,\nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k}$$

$$\prod_{i,j,k} \frac{1}{1 - x_i y_j z_k} = \sum_\alpha h_\alpha(xy) m_\alpha(z)$$

$$\begin{aligned}\langle s_\lambda(x) * s_\mu(y), h_{\mathbf{a}}[x \cdot y] \rangle &= \langle s_\lambda(x) * s_\mu(y), \prod_i \sum_{\alpha^i \vdash a_i} s_{\alpha^i}(x) s_{\alpha^i}(y) \rangle \\ &= \sum_{\alpha^i \vdash a_i, i=1,\ldots} c^\lambda_{\alpha^1\alpha^2\ldots} c^\mu_{\alpha^1\alpha^2\ldots}\end{aligned}$$

Monomial expansion via multi-LR coefficients

The coefficient of $x_1^{a_1} x_2^{a_2} \dots$ in $s_\lambda * s_\mu(x)$ is

$$\sum_{\alpha^i \vdash a_i, i=1,\dots} c^{\lambda}_{\alpha^1 \alpha^2 \dots} c^{\mu}_{\alpha^1 \alpha^2 \dots}$$

Let
$P(\mu; \mathbf{a}) := \{(\alpha^1, \alpha^2, \cdots, \alpha^k) \in \mathbb{Z}_{\geq 0}^{\ell(\mu)k} : c^{\mu}_{\alpha^1 \alpha^2 \dots} > 0, |\alpha^i| = a_i \text{ for all } i = 1, \dots, k\}.$

The set of monomial degrees $\mathbf{a} = (a_1, \dots, a_k)$ appearing in $s_\lambda * s_\mu$ is given as

$$M_k(s_\lambda * s_\mu) = \{\mathbf{a} \in \mathbb{Z}_{\geq 0}^k : P(\lambda; \mathbf{a}) \cap P(\mu; \mathbf{a}) \neq \emptyset\}.$$

We want to understand the set of points: $M_k(s_\lambda * s_\mu)$.

Horn Inequalities

Theorem (Klyachko 1998, Knutson-Tao 1999)

Let $\lambda, \mu, \nu \in \mathbb{N}^r$ with weakly decreasing component. Then $c^{\lambda}_{\mu,\nu} > 0$ if and only if $|\lambda| = |\mu| + |\nu|$ and

$$\sum_{i \in I} \lambda_i \leq \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k$$

for all LR-consistent triples $I, J, K \subset [r]$.

$c^{\nu}_{\alpha^1,\alpha^2,\alpha^3}\, c^{\mu}_{\alpha^1,\alpha^2,\alpha^3} > 0 \iff$

$\max\{\alpha^1_1, \alpha^2_1, \alpha^3_1, \alpha^1_2 + \alpha^2_2, \alpha^1_2 + \alpha^3_2, \alpha^2_2 + \alpha^3_2\} \leq \nu_1,$

$\max\{\alpha^1_2, \alpha^2_2, \alpha^3_2\} \leq \nu_2,$

$\alpha^1_2 + \alpha^2_2 + \alpha^3_2 \leq \mu_2,$

$\max\{\alpha^1_1 + \alpha^2_2 + \alpha^3_2, \alpha^1_2 + \alpha^2_1 + \alpha^3_2, \alpha^1_2 + \alpha^2_2 + \alpha^3_1\} \leq \min\{\nu_1 + \nu_3, \mu_1\},$

$\max\{\alpha^1_1 + \alpha^2_1 + \alpha^3_2, \alpha^1_2 + \alpha^2_1 + \alpha^3_1, \alpha^1_1 + \alpha^2_2 + \alpha^3_1\} \leq \nu_1 + \nu_2,$

$\max\{\alpha^1_1 + \alpha^1_2 + \alpha^2_2 + \alpha^3_2, \alpha^1_2 + \alpha^2_1 + \alpha^2_2 + \alpha^3_2, \alpha^1_2 + \alpha^2_2 + \alpha^3_1 + \alpha^3_2\} \leq \nu_1 + \nu_2.$

Inequalities

$$
\begin{aligned}
\mathcal{P}(\mu,\nu,(a_1,a_2,a_3)) := \{(x,y,z)\in\mathbb{R}^3 \mid\ & \\
a_1 - \min(\nu_2,\mu_2,\tfrac{a_1}{2}) \le x \le \min(a_1,\nu_1),\ & \\
a_2 - \min(\nu_2,\mu_2,\tfrac{a_2}{2}) \le y \le \min(a_2,\nu_1),\ & \\
a_3 - \min(\nu_2,\mu_2,\tfrac{a_3}{2}) \le z \le \min(a_3,\nu_1),\ & \\
\max(\nu_3, a_1+a_2-\nu_1) \le x+y,\ & \\
\max(\nu_3, a_1+a_3-\nu_1) \le x+z,\ & \\
\max(\nu_3, a_2+a_3-\nu_1) \le y+z,\ & \\
\mu_1 \le x+y+z,\ & \\
\max(\nu_2,\mu_2) - a_1 \le -x+y+z \le \nu_1+\nu_2-a_1,\ & \\
\max(\nu_2,\mu_2) - a_2 \le x-y+z \le \nu_1+\nu_2-a_2,\ & \\
\max(\nu_2,\mu_2) - a_3 \le x+y-z \le \nu_1+\nu_2-a_3\}.\ &
\end{aligned}
$$

Inequalities

$$\begin{aligned}
\mathcal{P}(\mu,\nu,(a_1,a_2,a_3)) := \{(x,y,z)\in\mathbb{R}^3 \mid \\
a_1-\min(\nu_2,\mu_2,\tfrac{a_1}{2}) \le x \le \min(a_1,\nu_1),\\
a_2-\min(\nu_2,\mu_2,\tfrac{a_2}{2}) \le y \le \min(a_2,\nu_1),\\
a_3-\min(\nu_2,\mu_2,\tfrac{a_3}{2}) \le z \le \min(a_3,\nu_1),\\
\max(\nu_3,a_1+a_2-\nu_1) \le x+y,\\
\max(\nu_3,a_1+a_3-\nu_1) \le x+z,\\
\max(\nu_3,a_2+a_3-\nu_1) \le y+z,\\
\mu_1 \le x+y+z,\\
\max(\nu_2,\mu_2)-a_1 \le -x+y+z \le \nu_1+\nu_2-a_1,\\
\max(\nu_2,\mu_2)-a_2 \le x-y+z \le \nu_1+\nu_2-a_2,\\
\max(\nu_2,\mu_2)-a_3 \le x+y-z \le \nu_1+\nu_2-a_3\}.
\end{aligned}$$

The set of monomial degrees $\mathbf{a}=(a_1,\dots,a_k)$ appearing in $s_\lambda * s_\mu$ is given as

$$M_k(s_\lambda * s_\mu) = \{\mathbf{a}\in\mathbb{Z}^k_{\ge 0} : \mathcal{P}(\mu,\nu,(a_1,a_2,a_3))\cap\mathbb{Z}^3 \neq \emptyset\}.$$

Proposition

Suppose that $\mathcal{P}(\mu, \nu, \mathbf{a}^i) \neq \emptyset$ for some vectors $\mathbf{a}^i$, $i = 1, \ldots, 4$ and $\mathbf{c} = \sum_i t_i \mathbf{a}^i$ for some $t_i \in [0, 1]$ with $t_1 + t_2 + t_3 + t_4 = 1$. Then $\mathcal{P}(\mu, \nu, \mathbf{c}) \neq \emptyset$.

Theorem (Panova–Z)

If $\mathcal{P}(\mu, \nu, \mathbf{a}) \neq \emptyset$ then it has an integer point, i.e. $\mathcal{P} \cap \mathbb{Z}^3 \neq \emptyset$.

Theorem (Panova–Z)

Let $\lambda, \mu \vdash n$ with $\ell(\lambda) \leq 3$ and $\ell(\mu) \leq 2$. Then $s_\lambda * s_\mu(x_1, x_2, x_3)$ has a saturated Newton polytope.

Limiting case of the SNP property

Theorem (Panova–Z)

Let λ, μ be partitions of the same size and $k \in \mathbb{N}$. Then the set of points

$$\bigcup_{p=1}^{\infty} \frac{1}{p} M_k(s_{p\lambda} * s_{p\mu})$$

is a convex subset of $\mathbb{Q}^k$.

Main tool: Semigroup property

Semigroup property

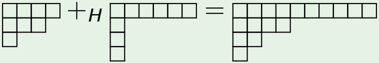
Semigroup Property (Christandl–Harrow–Mitchison 2007)

If $\alpha^1, \beta^1, \gamma^1 \vdash n$ and $\alpha^2, \beta^2, \gamma^2 \vdash m$ satisfy $g(\alpha^i, \beta^i, \gamma^i) > 0$ for $i = 1, 2$, then

$$g(\alpha^1 + \alpha^2, \beta^1 + \beta^2, \gamma^1 + \gamma^2) \geq \max\{g(\alpha^1, \beta^1, \gamma^1), g(\alpha^2, \beta^2, \gamma^2)\}.$$

Example

$\lambda = (4, 3, 1)$, $\mu = (6, 1, 1, 1)$, $\lambda + \mu = (10, 4, 2, 1)$



Given $g((2,1), (1,1,1), (2,1)) = 1$ and $g((1), (1), (1)) = 1$, by Semigroup property,

$$g((3,1), (2,1,1), (3,1)) \geq 1.$$

Positivity implications

Theorem (Panova–Z)

Suppose that $g(\lambda,\mu,\nu) > 0$ *and let* $\ell = \min\{\ell(\mu),\ell(\nu)\}$*. Then there exist nonnegative integers* $\{\alpha^i_j\}_{i\in[k],j\in[\ell]}$ *satisfying*

$$\sum_j \alpha^i_j = \lambda_i, \qquad \text{for } i \in [k];$$

$$\alpha^i_j \geq \alpha^i_{j+1}, \qquad \text{for } j \in [\ell-1],\ i \in [k];$$

$$\sum_{(i,j)\in D(I)} \alpha^i_j \leq \min\{\sum_{j\in J}\mu_j, \sum_{j\in J}\nu_j\}, \qquad \text{for every mLR-consistent } (I,J,K).$$

Positivity implications

Corollary

Suppose that $g(\lambda,\mu,\nu)>0$ and $\ell(\mu)=2$, $k=\ell(\lambda)$. Then there exist nonnegative integers $y_i \in [0, \lfloor \lambda_i/2 \rfloor]$ for $i \in [k]$, such that

$$\sum_{i\in A\cup C}\lambda_i+\sum_{i\in B}y_i-\sum_{i\in C}y_i\le \min\{\sum_{j\in J}\mu_j,\sum_{j\in J}\nu_j\}$$

for all triples of mutually disjoint sets $A\sqcup B\sqcup C\subset[k]$ and $J=\{1,\ldots,r,r+2,\ldots,r+b+1\}$, where $r=2|A|+|C|$ and $b=|B|$.

Remarks and further work

We cast doubts on whether the Kronecker product of Schur polynomials has SNP property in general.

Does the plethysm of two Schur functions $s_\lambda[s_\mu]$ have SNP?

Theorem (Paget–Wildon 2019)

Let $m \geq 2$. There is a unique partition λ, maximal in the dominance order on partitions, such that S^λ is a summand of H_μ^ν if and only if either $\nu = (n)$ or μ is rectangular and ν has exactly two parts.

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