On a curious variant of Lie_n

Sheila Sundaram

University of Minnesota and ICERM

2025 Nov 11

Computation in Representation Theory Workshop
ICERM

Outline

- The multilinear component of the free Lie algebra on n generators
- The representation Lie_n of the symmetric group $\mathfrak{S}_n$
- Properties of Lie_n (classical)
- A different $\mathfrak{S}_n$-module: The variant $\mathrm{Lie}_n^{(2)}$
- Properties of $\mathrm{Lie}_n^{(2)}$

Ground field is always the field of complex numbers.

The free Lie algebra

$V :=$ a vector space of dimension n, with basis $\{a_1, \ldots, a_n\}$ (the "alphabet").
$T(V) :=$ the tensor algebra over V ("words of finite length in the alphabet A").
The free Lie algebra $\mathcal{L}(V)$ over V is the smallest subalgebra of $T(V)$ containing V and closed under the bracket operation

$$[u, v] = uv - vu$$

(concatenate words and extend linearly). This is a *Lie bracket*: it satisfies *antisymmetry*:

$$[u, v] = -[v, u]$$

and the *Jacobi identity*:

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0.$$

The multilinear component of degree n

$\mathcal{L}(V)$ has a multigrading indexed by n-tuples $(m_1, \ldots, m_n)$ consisting of all possible bracketings with m_i occurrences of a_i. The multilinear component is the graded piece corresponding to $(1, 1, \ldots, 1)$.

It can be viewed as the subspace of $\mathcal{L}(V)$ spanned by all possible bracketings of n distinct letters, say $\{1, \ldots, n\}$.

It carries an action of $\mathfrak{S}_n$, the representation Lie_n.

Theorem: A basis for Lie_n is the set of left-normed bracketings (Garsia's *left combs*) consisting of brackets of permutations $\sigma \in \mathfrak{S}_n$ of the form

$$[\ldots[[\sigma(1), \sigma(2)], \sigma(3)], \ldots], \sigma(n)]$$

such that $\sigma(1) = 1$.

Hence Lie_n has dimension $(n-1)!$

$\mathfrak{S}_n$ acts by replacement of letters.

Ex 1: $n = 2$, $\mathrm{Lie}_2 = \mathrm{span}\{[1,2]\} = \mathrm{span}\{12 - 21\}$.
$(1,2) \cdot [1,2] = [2,1] = -[1,2]$, so $\mathfrak{S}_2$ acts like the sign representation.

Ex 2: $n = 3$, $\mathrm{Lie}_3 = \mathrm{span}\{[[1,2],3],\ [[1,3],2]\}$
$= \mathrm{span}\{123 - 213 - 312 + 321,\ 132 - 312 - 213 + 231\}$.
$(1,2) \cdot [[1,2],3] = [[2,1],3] = -[[1,2],3]$
$(1,2) \cdot [[1,3],2] = [[2,3],1] = -[[1,2],3] + [[1,3],2]$

so trace $(1,2) = 0$.

Similarly one checks that trace $(1,2,3) = -1$, so $\mathfrak{S}_3$ acts like the 2-dimensional irreducible indexed by the partition $(2,1)$.

Background and notation

$GL(V)$ is the group of invertible linear transformations of a finite-dimensional complex vector space V.

Symmetric functions in m variables are characters $\mathrm{ch}\,\phi$ of polynomial representations ϕ of GL_m, $\phi : GL_m \to GL(V)$, where GL_m is the general linear group of m by m invertible matrices over $\mathbb{C}$.

Symmetric functions of degree n, via the Frobenius characteristic map ch, are in correspondence with the representation ring of the symmetric group $\mathfrak{S}_n$.

Plethysm of symmetric functions $f[g]$ corresponds to composition of $GL(V)$ representations :

If $GL_n \xrightarrow{\phi} GL_m \xrightarrow{\psi} GL_k$, and ϕ, ψ have characters $\mathrm{ch}\,\phi$, $\mathrm{ch}\,\psi$ respectively, then the character of the (composite) GL_n-representation $\mathrm{ch}\,(\psi \circ \phi)$ is $\mathrm{ch}\,\psi[\mathrm{ch}\,\phi]$.

Partitions of n and symmetric functions

- $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1)$ such that $\sum_i \lambda_i = n$ is an integer partition of n; $\ell(\lambda)$ is the number of parts k of λ.
- If $n = 0$ then $\lambda = \emptyset$ and $\ell(\lambda) = 0$.
- $p_r = \sum_i {x_i}^r$ is the rth power sum symmetric function.
- $p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots$ is the power sum symmetric function indexed by the partition λ.
- s_λ denotes the Schur function indexed by λ;
  $s_\lambda(x_1, \ldots, x_m)$ is the character of the GL_m-irreducible **Weyl module** indexed by λ, for λ of length at most m;
  $s_\emptyset = 1$ is the trivial representation of GL_m.
- s_λ is the Frobenius characteristic of the $\mathfrak{S}_n$-irreducible **Specht module** $\mathbb{S}^\lambda$ indexed by the partition λ of n.

Symmetric and Exterior Powers

- The homogeneous symmetric function h_n of degree n is the Frobenius characteristic of the trivial representation of $\mathfrak{S}_n$, indexed by the partition (n).

  It is also the character of $GL(V)$ acting on the nth symmetric power $Sym^n(V)$.
- The elementary symmetric function e_n of degree n is the Frobenius characteristic of the sign representation of $\mathfrak{S}_n$, indexed by the partition (1^n).

  It is also the character of $GL(V)$ acting on the nth exterior power $\bigwedge^n(V)$.
- h_1^n is the Frobenius characteristic of the regular representation of $\mathfrak{S}_n$.
  It is also the character of $GL(V)$ acting on the nth tensor power $V^{\otimes n}$.
- The involution ω in the ring of symmetric functions, corresponding to tensoring with the sign representation for $\mathfrak{S}_n$, is defined by

$$\omega(h_n) = e_n.$$

Suppose g is a nonnegative sum of monomials.
If the multiset of monomials occurring in g is $\{\{\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2}, \ldots\}\}$, then

$$f[g] := f(\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2}, \ldots)$$

Symmetric functions f and g are plethystic inverses if

$$f[g] = g[f] = h_1.$$

Associativity of plethysm implies that $f[g] = h_1 \iff g[f] = h_1$.

Proposition

If f is a symmetric function with no constant term, and nonzero coefficient for the degree one term, then f has a plethystic inverse.

Plethysm and Schur-Weyl duality – notation

Define $H := \sum_{n\geq 0} h_n = 1 + h_1 + h_2 + \cdots$ and $E := \sum_{n\geq 0} e_n = 1 + e_1 + e_2 + \cdots$.

If $F = \sum_{i\geq 1} f_i = f_1 + f_2 + \cdots$ is the $GL(V)$-character on $W = \oplus_i W_i$, then:

$H[F]$ is the $GL(V)$-character of the symmetric algebra $Sym(W)$.

$E[F]$ is the $GL(V)$-character of the exterior algebra $\bigwedge W$.

If λ is the partition with m_i parts equal to i, then define

$H_\lambda[F] :=$ the character of the piece $\otimes_i Sym^{m_i}(W_i)$ of the symmetric algebra $Sym(W)$

.

$E_\lambda[F] :=$ the character of the piece $\otimes_i \bigwedge^{m_i}(W_i)$ of the exterior algebra $\bigwedge W$.

Also
$(1 - h_1)^{-1} = \sum_{n\geq 0} h_1^n$ is the GL(V)-character on the full tensor algebra $T(V)$.

The regular representation of a finite group G

$$Reg_G := 1\uparrow_e^G = \bigoplus_{\chi \text{ irreducible repn of } G} (\dim \chi)\chi.$$

For the symmetric group $\mathfrak{S}_n$:

Theorem (Reg0)

$$Reg_{\mathfrak{S}_n} = \bigoplus_{\lambda \vdash n} f^\lambda \chi^\lambda,$$

with Frobenius characteristic

$$h_1^n = \mathrm{ch}\, Reg_{\mathfrak{S}_n} = \sum_{\lambda \vdash n} f^\lambda s_\lambda.$$

Here
- λ is an integer partition of n,
- $f^\lambda = |\{\text{standard Young tableaux of shape } \lambda\}|$, and
- s_λ is the Schur function indexed by λ, so $s_\lambda = \mathrm{ch}\, \chi^\lambda$.

The regular representation of $\mathfrak{S}_n$ — (I)

$C_n :=$ the cyclic subgroup of $\mathfrak{S}_n$ generated by the long cycle $\sigma = (1\,2\,\ldots\,n)$.

$\omega_n :=$ a primitive nth root of unity.

For $1 \le k \le n$, $\sigma \mapsto \omega_n^k$ yields a one-dimensional representation of C_n, and these are all the distinct irreducibles, so

$$Reg_{C_n} = \bigoplus_{k=1}^{n} (\omega_n^k).$$

This in turn gives a decomposition of the regular representation of $\mathfrak{S}_n$:

Theorem (Reg1)

$$Reg_{\mathfrak{S}_n} = \bigoplus_{k=1}^{n} (\omega_n^k)\uparrow_{C_n}^{\mathfrak{S}_n}$$

$C_n :=$ cyclic subgroup of $\mathfrak{S}_n$ generated by $\sigma = (1\,2\,\ldots\,n)$, $\omega_n := \exp\frac{2i\pi}{n}$.

Theorem (Klyachko, 1974)

The induced representation $\omega_n\uparrow_{C_n}^{\mathfrak{S}_n}$ is isomorphic to the representation Lie_n of $\mathfrak{S}_n$ on the multilinear component of the free Lie algebra on n generators.

Angeline Brandt (1944): (Recall $p_d = x_1^d + x_2^d + \cdots$)

$$\mathrm{ch}\,\mathrm{Lie}_n = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{\frac{n}{d}}$$

If $\lambda \vdash n$ has m_i parts equal to i, then the *higher Lie module* is

$$H_\lambda[\mathrm{Lie}] := h_{m_1}[\mathrm{Lie}_1]h_{m_2}[\mathrm{Lie}_2]\ldots$$

For $\lambda = (4,3,3,2)$: $H_{(4^1,3^2,2^1)}[\mathrm{Lie}] = h_1[\mathrm{Lie}_4]\cdot h_2[\mathrm{Lie}_3]\cdot h_1[\mathrm{Lie}_2]$

In particular: $H_{(n)}[\mathrm{Lie}] = \mathrm{Lie}_n$, $H_{(1^n)}[\mathrm{Lie}] = h_n$.

Thrall's theorem

Let $\mathrm{Lie} := \sum_{n\geq 1} \mathrm{Lie}_n = \mathrm{Lie}_1 + \mathrm{Lie}_2 + \cdots$.
Recall $H := \sum_{n\geq 0} h_n = 1 + h_1 + h_2 + \cdots$ and $E = \sum_{n\geq 0} e_n = 1 + e_1 + e_2 + \cdots$.

Theorem (Robert Thrall 1942)

$$H[\mathrm{Lie}] = (1 - h_1)^{-1}.$$

$$Sym(L(V)) \overset{PBW}{\simeq} U(L(V)) \simeq T(V)$$

$$\textit{Equivalently,} \quad h_1^n = \sum_{\lambda \vdash n} H_\lambda[\mathrm{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\mathrm{Lie}_1] h_{m_2}[\mathrm{Lie}_2] \ldots,$$

where λ *has* m_i *parts equal to* i.

The **dimension** of the higher Lie module $H_\lambda[\mathrm{Lie}]$ is the number of permutations in $\mathfrak{S}_n$ with cycle-type λ.

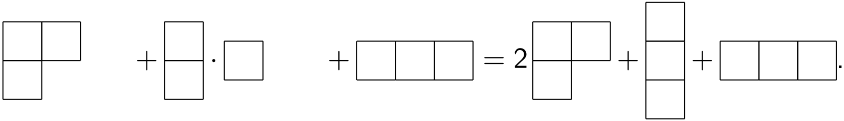
The regular representation of $\mathfrak{S}_n$ – (II)

$$h_1^n = \text{ch}\, Reg_{\mathfrak{S}_n} = \sum_{\lambda \vdash n} H_\lambda[\text{Lie}] = \sum_{\lambda \vdash n} h_{m_1}[\text{Lie}_1] h_{m_2}[\text{Lie}_2] \ldots,$$

where λ has m_i parts equal to i.

Example:

$$\begin{aligned} h_1^3 = \text{ch}\, Reg_{\mathfrak{S}_3} &= H_{(3)}[\text{Lie}] + H_{(2,1)}[\text{Lie}] + H_{(1^3)}[\text{Lie}] \\ &= \text{Lie}_3 \qquad + \text{Lie}_2\text{Lie}_1 \qquad + h_3. \end{aligned}$$

So $Reg_{\mathfrak{S}_3} =$ 

The variant $\mathrm{Lie}_n^{(2)}$

Fix $n \geq 1$. Let k_n be the highest power of 2 dividing n. Define $\mathrm{Lie}_n^{(2)}$ to be the induced module

$$Lie_n^{(2)} := \left(\omega_n^{2^{k_n}}\right) \uparrow_{C_n}^{\mathfrak{S}_n}$$

Ex: $\mathrm{Lie}_2^{(2)} = \mathbb{1}_{\mathfrak{S}_2}$, $\mathrm{Lie}_3^{(2)} = \mathrm{Lie}_3$, $\mathrm{Lie}_4^{(2)} = \mathbb{1} \uparrow_{C_4}^{\mathfrak{S}_4}$.

Properties shared with Lie_n:

- $\mathrm{Lie}_n^{(2)}$ has dimension $(n-1)!$
- it is a submodule of the regular representation of $\mathfrak{S}_n$
- its restriction to $\mathfrak{S}_{n-1}$ is the regular representation
- its character on σ is nonzero $\Rightarrow$ σ has all cycles of the same length.

Properties: Lie_n and $\mathrm{Lie}_n^{(2)}$

Proposition (S, 2020)

$\mathrm{Lie}_n^{(2)}$ is isomorphic to

- *Lie_n, if n is odd;*
- *the conjugacy action on the class of n-cycles, if n is a power of 2;*
- *$\mathrm{Lie}_n \otimes \mathrm{sgn}_n$, if n is twice an odd number.*

Write Lie_n (resp. $\mathrm{Lie}_n^{(2)}$) for both the module and its Frobenius characteristic. Recall: If $\lambda \vdash n$ has m_i parts equal to i, then one can define *higher* Lie *modules*

$$H_\lambda[\mathrm{Lie}] = h_{m_1}[\mathrm{Lie}_1]h_{m_2}[\mathrm{Lie}_2]\cdots,$$

$$E_\lambda[\mathrm{Lie}] = e_{m_1}[\mathrm{Lie}_1]e_{m_2}[\mathrm{Lie}_2]\cdots$$

Can replace Lie with $\mathrm{Lie}^{(2)} := \mathrm{Lie}_1^{(2)} + \mathrm{Lie}_2^{(2)} + \cdots$, giving *higher* $\mathrm{Lie}^{(2)}$ *modules*

Properties: Lie_n versus $\mathrm{Lie}_n^{(2)}$

Theorem (S, 2020)

The symmetric function $\mathrm{Lie}_n^{(2)}$ *satisfies plethystic identities analogous to* Lie_n.

$$\sum_{\lambda\vdash n} H_\lambda[\mathrm{Lie}] = h_1^n \qquad \sum_{\lambda\vdash n} E_\lambda[\mathrm{Lie}^{(2)}] = h_1^n \qquad (1)$$

$$H\left[\sum_{n\geq 1}(-1)^{n-1}\omega(\mathrm{Lie}_n)\right] = 1 + h_1 \qquad E\left[\sum_{n\geq 1}(-1)^{n-1}\omega(\mathrm{Lie}_n^{(2)})\right] = 1 + h_1 \qquad (2)$$

$$\text{If } n\geq 2,\ \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)} E_\lambda[\mathrm{Lie}] = 0 \qquad \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)} H_\lambda[\mathrm{Lie}^{(2)}] = 0 \qquad (3)$$

$$\text{If } n\geq 2,\ \sum_{\lambda\vdash n} E_\lambda[\mathrm{Lie}] = 2e_2 h_1^{n-2} \qquad \sum_{\lambda\vdash n} H_\lambda[\mathrm{Lie}^{(2)}] = \sum_{\lambda\vdash n, \lambda_i = 2^{a_i}} p_\lambda \qquad (4)$$

$$H_\lambda[\mathrm{Lie}] = h_{m_1}[\mathrm{Lie}_1]h_{m_2}[\mathrm{Lie}_2]\cdots, \quad E_\lambda[\mathrm{Lie}] = e_{m_1}[\mathrm{Lie}_1]e_{m_2}[\mathrm{Lie}_2]\cdots$$

Dissecting the properties (1)

$$(1) \qquad \sum_{\lambda \vdash n} H_\lambda[\mathrm{Lie}] = h_1^n; \qquad \sum_{\lambda \vdash n} E_\lambda[\mathrm{Lie}^{(2)}] = h_1^n$$

By Thrall, the higher Lie modules decompose the regular representation of $\mathfrak{S}_n$. For GL_n this says that the full tensor algebra decomposes as a symmetric algebra over the free Lie algebra.

The variant $\mathrm{Lie}^{(2)}$ apparently gives a GL_n-decomposition of the full tensor algebra as an exterior algebra over some other (?) object.

Dimensions:
$\dim H_\lambda[\mathrm{Lie}] = \dim E_\lambda[\mathrm{Lie}^{(2)}] = |\{\sigma \in \mathfrak{S}_n : \text{ cycle type of } \sigma \text{ is } \lambda\}|$

Dissecting the properties (2)

(2) $\quad H\left[\sum_{n\geq 1}(-1)^{n-1}\omega(\mathrm{Lie}_n)\right] = 1 + h_1; \qquad E\left[\sum_{n\geq 1}(-1)^{n-1}\omega(\mathrm{Lie}_n^{(2)})\right] = 1 + h_1;$

$\omega(\mathrm{Lie}_n)$ gives the plethystic inverse of $\sum_{n\geq 1} h_n$, a result Cadogan (1971) obtained via cycle index computations on graphs.

Similarly, $\omega(\mathrm{Lie}_n^{(2)})$ gives the plethystic inverse of $\sum_{n\geq 1} e_n$.

Equivalently, these are plethystic inverses:

Theorem (S, 2020)

$$\sum_{n\geq 1} \mathrm{Lie}_n \textit{ and } \sum_{n\geq 1}(-1)^{n-1} e_n.$$

$$\sum_{n\geq 1} \mathrm{Lie}_n^{(2)} \textit{ and } \sum_{n\geq 1}(-1)^{n-1} h_n.$$

Dissecting the properties (3)

Let Π_n denote the lattice of *set partitions* of an n-element set. Let $\mathbf{Whit}_r(\Pi_n)$ denote the rth Whitney homology of Π_n, $0 \leq r \leq n-1$. Then $\mathbf{Whit}_r(\Pi_n)$ coincides with the rth graded piece of the *Orlik-Solomon algebra* of the braid matroid of type A_{n-1}.

Theorem (Lehrer and Solomon 1986)

As an $\mathfrak{S}_n$-module, $\omega(\mathbf{Whit}_r(\Pi_n))$ has Frobenius characteristic $\sum_{\lambda \vdash n,\ \ell(\lambda)=n-r} E_\lambda[Lie]$.

$\dim \mathbf{Whit}_r(\Pi_n) = c(n, n-r)$, the signless Stirling number of the first kind.

$$(3)\ \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} E_\lambda[\mathrm{Lie}] = 0 \quad (n \geq 2) \quad \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} H_\lambda[\mathrm{Lie}^{(2)}] = 0.$$

Lie: Asserts, up to sign, the $\mathfrak{S}_n$-equivariant **acyclicity** of the Whitney homology complex for the partition lattice, i.e. of the Orlik-Solomon algebra of type A_{n-1}.

$\mathrm{Lie}^{(2)}$: Similar interpretation? Why are the symmetric powers of $\mathrm{Lie}_n^{(2)}$ acyclic?

Dissecting the properties (4)

$$(4)\ \sum_{\lambda\vdash n} E_\lambda[\mathrm{Lie}] = 2e_2 h_1^{n-2} \quad (n\geq 2) \quad \sum_{\lambda\vdash n} H_\lambda[\mathrm{Lie}^{(2)}] = \sum_{\lambda\vdash n, \lambda_i=2^{a_i}} p_\lambda.$$

Lie: Up to sign, the $\mathfrak{S}_n$-action on the full Orlik-Solomon algebra/total Whitney homology of Π_n is twice a permutation module (Lehrer 1987).

$\mathrm{Lie}^{(2)}$: Interesting consequence of (4) is that the multiplicity-free

sum of power sums p_λ, where every part of λ is a power of 2,

is Schur positive (in fact, twice a Schur positive function).

(Original Motivation)

An equivariant Whitney homology result

Let $\mathbf{Whit}_r(P)$ denote the rth *Whitney homology* of the poset P.

Theorem (S 1994)

If P is a Cohen-Macaulay poset of rank r, with a group of automorphisms G, then for each $i \leq r$ (over a field of characteristic zero), this alternating sum is a true G-module:

$$\mathbf{Whit}_i(P) - \mathbf{Whit}_{i-1}(P) + \cdots + (-1)^i \mathbf{Whit}_0(P)$$

Recall $\omega(\mathbf{Whit}_r(\Pi_n)) = \sum_{\lambda \vdash n,\ \ell(\lambda)=n-r} E_\lambda[Lie]$.
Hence we know that for each $i \leq n-1$, the *truncated* alternating sum

$$\sum_{r=0}^{i} (-1)^{i-r} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=n-r}} E_\lambda[\mathrm{Lie}]$$

is a true $\mathfrak{S}_n$-module, of dimension $c(n, n-i) - c(n, n-i+1) + \cdots + (-1)^i c(n, n)$.
(Can describe the module precisely.)

A counterpart for $\mathrm{Lie}_n^{(2)}$

In analogy with $\omega(\mathbf{Whit}_r(\Pi_n)) = \sum_{\lambda\vdash n,\ \ell(\lambda)=n-r} E_\lambda[Lie]$:
Define a new module $\mathrm{Vh}_r(n) := \sum_{\lambda\vdash n,\ \ell(\lambda)=n-r} H_\lambda[\mathrm{Lie}^{(2)}]$. Then

$$(\mathrm{Lie}): \ \mathbf{Whit}_{n-1}(\Pi_n) - \mathbf{Whit}_{n-2}(\Pi_n) + \cdots + (-1)^{n-1}\mathbf{Whit}_0(\Pi_n) = 0$$

$$(\mathrm{Lie}^{(2)}): \ \mathrm{Vh}_{n-1}(n) - \mathrm{Vh}_{n-2}(n) + \mathrm{Vh}_{n-3}(n) - \cdots + (-1)^{n-1}\mathrm{Vh}_0(n) = 0, \quad n \geq 2$$

In analogy with Lie_n:

Conjecture (S, 2020 – checked with Maple for $n \leq 9$)

For each $i \leq n-1$, the truncated *alternating sum of modules* $U_i(n) := \mathrm{Vh}_i(n) - \mathrm{Vh}_{i-1}(n) + \cdots + (-1)^{n-1}\mathrm{Vh}_0(n)$ *is a true $\mathfrak{S}_n$-module.*

Again, $\dim U_i(n) = \sum_{r=0}^{i}(-1)^{i-r}c(n, n-r)$.

True for $i \leq 3$ and all n

$$
\begin{aligned}
U_0(n) &= \mathrm{Vh}_0(n) = s_{(n)};\\
U_1(n) &= \mathrm{Vh}_1(n) - \mathrm{Vh}_0(n) = s_{(n-1,1)} + s_{(n-2,2)};\\
U_2(n) &= \mathrm{Vh}_2(n) - U_1(n)\\
&= h_{n-3}s_{(2,1)} + h_{n-4}\big(h_4 + s_{(2,2)}\big) - s_{(n-1,1)} - s_{(n-2,2)};\\
U_3(n) &= \mathrm{Vh}_3(n) - U_2(n)\\
&= h_{n-4}s_{(2,1^2)} + s_{(2,1)}\big(h_{n-5}h_2 - h_{n-3}\big)\\
&\qquad + h_{n-6}\big(h_6 + s_{(4,2)} + s_{(2^3)}\big) + s_{(n-1,1)} + s_{(n-2,2)}.
\end{aligned}
$$

Data for $n = 6$

Table: Alternating sums $U_k(n)$ of $h_k[\mathrm{Lie}^{(2)}]$ for $n = 6$

k	$U_k(6)$
0	(6)
1	$(5,1) + (4,2)$
2	$(6) + (5,1) + 2(4,2) + (4,1^2) + 2(3,2,1) + (2^3)$
3	$(6) + (5,1) + 3(4,2) + 2(4,1^2) + (3^2) + 3(3,2,1) + 2(3,1^3) + 2(2^2,1^2)$
4	$\mathrm{Lie}_6^{(2)} = (5,1)+2(4,2)+(4,1^2)+3(3,2,1)+2(3,1^3)+(2^3)+(2^2,1^2)+(2,1^4)$

Data for $n = 7$

Table: Alternating sums $U_k(n)$ of $h_k[\mathrm{Lie}^{(2)}]$ for $n = 7$

k	$U_k(7)$
0	(7)
1	$(6,1)+(5,2)$
2	$(7)+(6,1)+2(5,2)+(5,1^2)+(4,3)+2(4,2,1)+(3,2^2)$
3	$(7)+2(6,1)+3(5,2)+2(5,1^2)+3(4,3)+5(4,2,1)+2(4,1^3)+2(3^2,1)+3(3,2^2)+3(3,2,1^2)+2(2^3,1)$
4	$2(6,1)+4(5,2)+3(5,1^2)+3(4,3)+8(4,2,1)+3(4,1^3)+4(3^2,1)+5(3,2^2)+7(3,2,1^2)+3(3,1^4)+3(2^3,1)+2(2^2,1^3)$
5	$\mathrm{Lie}_7^{(2)} = \mathrm{Lie}_7 = (6,1)+2(5,2)+2(5,1^2)+2(4,3)+5(4,2,1)+3(4,1^3)+3(3^2,1)+3(3,2^2)+5(3,2,1^2)+2(3,1^4)+2(2^2,1)+2(2^2,1^3)+(2,1^5)$

The derived series

The ($\mathfrak{S}_n$-equivariant) derived series of the free Lie algebra:
Let $\kappa = \sum_{n\geq 2} s_{(n-1,1)} = s_{(1,1)} + s_{(2,1)} + \cdots$
Let $\mathrm{Lie}_{\geq 2} = \mathrm{Lie}_2 + \mathrm{Lie}_3 + \cdots$.
FACT:

$$\mathrm{Lie}_{\geq 2} = \kappa + \kappa[\kappa] + \kappa[\kappa[\kappa]] + \cdots$$

Derived series of any Lie algebra L is $L \supset [L, L] \supset [[L, L], [L, L]] \supset \cdots$

Comparing derived series

Theorem (S, 2020: Analogue of the derived series filtration of the free Lie algebra)

$$\mathrm{Lie}_{\geq 2} = \kappa + \kappa[\kappa] + \kappa[\kappa[\kappa]] + \cdots$$

$$\mathrm{Lie}_{\geq 2} = \mathrm{Lie}[\kappa]$$

The degree n term in

$$\sum_{r \geq 0} (-1)^{n-r} e_{n-r}[\mathrm{Lie}_{\geq 2}]$$

is $(-1)^{n-1} s_{(n-1,1)}$

$$\mathrm{Lie}^{(2)}_{\geq 2} = \omega(\kappa) + \omega(\kappa)[\omega(\kappa)] + \omega(\kappa)[\omega(\kappa)[\omega(\kappa)]] + \cdots$$

$$\mathrm{Lie}^{(2)}_{\geq 2} = \mathrm{Lie}^{(2)}[\omega(\kappa)]$$

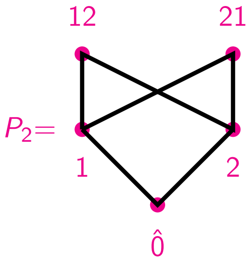
The degree n term in

$$\sum_{r \geq 0} (-1)^{n-r} h_{n-r}[\mathrm{Lie}^{(2)}_{\geq 2}]$$

is $(-1)^{n-1} s_{(2,1^{n-2})}$

The complex of injective words

Injective words in the alphabet $\{1, 2, \ldots, n\}$ form a poset P_n under subword order: u is a subword of v if u can be obtained by erasing letters in v.



Frank Farmer (1979) showed: the order complex of $P_n \cup \{\hat{1}\}$ has the homotopy type of a wedge of spheres in the top dimension.

Vic Reiner and Peter Webb (2004) computed the $\mathfrak{S}_n$-action on the top homology of the order complex of $P_n \cup \{\hat{1}\}$.

Theorem (Reiner-Webb, 2004)

The $\mathfrak{S}_n$-homology module $\tilde{H}(P_n)$ satisfies the recurrence

$$\tilde{H}(P_n) = \tilde{H}(P_{n-1})\uparrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} + (-1)^n \mathbb{1}_n.$$

dim $\tilde{H}(P_n)$ is the number of *derangements* (permutations without fixed points) in $\mathfrak{S}_n$.

Theorem (S, 2020)

Let $n \geq 2$. For $k \geq 1$ let Δ_n^k denote the degree n term in $e_k[\mathrm{Lie}_{\geq 2}^{(2)}]$. Define $\Delta_n = \sum_{k\geq 1} \Delta_n^k$ for $n \geq 2$, and $\Delta_1 = 0, \Delta_0 = 1$. Then

1. $\Delta_n = \sum_{k=0}^n (-1)^k h_1^{n-k} h_k = h_1 \Delta_{n-1} + (-1)^n h_n$;
2. *For $n \geq 2$, Δ_n coincides with the homology representation $\tilde{H}(P_n)$ on the complex of injective words in the alphabet $\{1, 2, \ldots, n\}$.*

Hodge decomposition of Hanlon and Hersh

Let D_n^k be the degree n term in $h_k[Lie_{\geq 2}]$. Then

Theorem (Phil Hanlon-Patricia Hersh 2004: Hodge decomposition)

The $\mathfrak{S}_n$-equivariant Hodge decomposition of Δ_n is $\Delta_n = \sum_{k\geq 1} \omega(D_n^k)$.

- This is not the same as the decomposition $\Delta_n = \sum_{k\geq 1} \Delta_n^k$ arising from $Lie_n^{(2)}$.

In general, $\Delta_n^k \neq \omega(D_n^k)$, even though

$$\sum_{k\geq 1} \Delta_n^k = \Delta_n = \sum_{k\geq 1} \omega(D_n^k).$$

Ex: $n = 4$.

$\Delta_4 = h_1^2 h_2 - h_1 h_3 + h_4 = (4) + (3,1) + (2^2) + (2,1^2)$.

Also $\Delta_4^2 = e_2[h_2] = (3,1)$, $\Delta_4^1 = \mathrm{Lie}_4^{(2)} = (4) + (2^2) + (2,1^2)$.

The two Hodge pieces of Hanlon and Hersh, however, are:
$\omega(h_2[\mathrm{Lie}_2]) = (2^2) + (4)$ and $\omega(h_1[\mathrm{Lie}_4]) = (3,1) + (2,1^2)$.

Analogue of Whitehouse lifts

FACT: the $\mathfrak{S}_{n-1}$-module Lie_{n-1} lifts to a true $\mathfrak{S}_n$-module W_n, the *Whitehouse* module (up to sign), defined by $\mathrm{ch}\, W_n = h_1 \mathrm{Lie}_{n-1} - \mathrm{Lie}_n$.

Conjecture (S, 2020, checked up to $n = 32$ with Stembridge's SF package)

The symmetric function $h_1 \mathrm{Lie}_{n-1}^{(2)} - \mathrm{Lie}_n^{(2)}$ is Schur positive if and only if n is NOT a power of 2. Equivalently, $\mathrm{Lie}_{n-1}^{(2)} \uparrow^{\mathfrak{S}_n} - \mathrm{Lie}_n^{(2)}$ is a true $\mathfrak{S}_n$-module which lifts $\mathrm{Lie}_{n-1}^{(2)}$, if and only if n is not a power of 2.

(If n is a power of 2, $\mathrm{Lie}_{n-1}^{(2)} = \mathrm{Lie}_{n-1}$ and $\mathrm{Lie}_n^{(2)} = \mathbb{1}\uparrow_{C_n}^{S_n}$.)

If n is a power of 2, then $\mathrm{Lie}_{n-1}^{(2)} = \mathrm{Lie}_{n-1}$, so it has the Whitehouse lift W_n.

Relating Lie and $\mathrm{Lie}^{(2)}$

$T = $

1	3	**4**	9	10
2	6	**7**		
5	8			

major index of T $= 1 + 4 + 7 = 12$

Work of H.O. Foulkes (1972), Stanley (1999), Kraskiewicz and Weyman (2001) imply:

- $\mathrm{Lie}_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}}$ $\qquad$ $\mathrm{Lie}_n^{(2)} = \frac{1}{n} \sum_{j=0}^{k} \sum_{\substack{d=2^j d_1 \\ d_1|\ell}} \phi(2^j)\mu(d_1) p_d^{\frac{n}{d}}$

  where $n = 2^k \ell, \ell$ odd.

  The multiplicity of s_λ is the number of standard Young tableaux of shape λ with major index congruent to

- 1 modulo n for Lie_n
- 2^k modulo n, where $n = 2^k \ell, \ell$ odd, for $\mathrm{Lie}_n^{(2)}$.

From Lie_n to $\mathrm{Lie}_n^{(2)}$

Theorem (S, 2020)

- $\mathrm{Lie}_n^{(2)}$ *is the degree n term in the plethysm* $\sum_{k\geq 0} \mathrm{Lie}[p_{2^k}]$
- Lie_n *is the degree n term in* $\mathrm{Lie}^{(2)} - \mathrm{Lie}^{(2)}[p_2]$.

$$\mathrm{Lie}_8^{(2)} = \mathrm{Lie}_8 + \mathrm{Lie}_4[p_2] + \mathrm{Lie}_2[p_4] + \mathrm{Lie}_1[p_8]$$

$$\mathrm{Lie}_{12}^{(2)} = \mathrm{Lie}_{12} + \mathrm{Lie}_6[p_2] + \mathrm{Lie}_3[p_4]$$

Idempotents in the free Lie algebra

Recall Klyachko's theorem: $\mathrm{Lie}_n \simeq \omega_n \uparrow_{C_n}^{\mathfrak{S}_n}$.
In the group algebra $\mathbb{C}\mathfrak{S}_n$ of $\mathfrak{S}_n$, define

- $\zeta_n = \frac{1}{n}\sum_{i=0}^{n-1} \omega_n^{-i} c^i$, where c is the long cycle $(1, 2, \ldots, n)$.
- $\theta_n := \frac{1}{n}(1-\gamma_2)(1-\gamma_3)\cdots(1-\gamma_n)$, where γ_i is the i-cycle $(i, i-1, \ldots, 1)$.
- $\kappa_n := \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \omega_n^{\mathrm{maj}(\sigma)}\sigma$, where $\mathrm{maj}(\sigma) := \sum_{\sigma(i)>\sigma(i+1)} i$

Theorem (Dynkin 1947, Specht 1948, Wever 1949; Klyachko 1974)

The group algebra elements $\zeta_n, \theta_n, \kappa_n$ are all idempotents, and θ_n, κ_n generate Lie_n as a left ideal:

$$\mathbb{C}\mathfrak{S}_n\,\zeta_n \simeq \mathrm{Lie}_n \text{ and } \mathrm{Lie}_n = \mathbb{C}\mathfrak{S}_n\,\theta_n = \mathbb{C}\mathfrak{S}_n\,\kappa_n$$

Are there idempotents like θ_n, κ_n for $\mathrm{Lie}_n^{(2)}$?

Idempotents for higher Lie modules

Recall Thrall's decomposition of the regular representation of $\mathfrak{S}_n$:

$$h_1^n = \sum_{\lambda \vdash n} H_\lambda[\mathrm{Lie}]$$

Reutenauer (1986) constructed primitive orthogonal idempotents $\{\mathcal{E}_\lambda\}_{\lambda \vdash n}$ in $\mathbb{C}\mathfrak{S}_n$:

$$\mathbb{C}\mathfrak{S}_n = \bigoplus_{\lambda \vdash n} \mathbb{C}\mathfrak{S}_n \mathcal{E}_\lambda$$

Garsia and Reutenauer (1990) showed that the idempotent $\mathcal{E}_\lambda$ generates a left ideal isomorphic to the higher Lie module $H_\lambda[\mathrm{Lie}]$.

Since we know

$$h_1^n = \sum_{\lambda \vdash n} E_\lambda[\mathrm{Lie}^{(2)}],$$

are there such idempotents for $\mathrm{Lie}^{(2)}$?

Recap

$$h_1^n = \sum_{k\geq 1} \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} H_\lambda[\mathrm{Lie}] \qquad \text{(Reutenauer idempotents)}$$

$$= \sum_{k\geq 1} \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} \omega(H_\lambda[\mathrm{Lie}]) \qquad \text{(Eulerian idempotents: Gerstenhaber-Schack, Loday, Hanlon)}$$

$$= \sum_{k\geq 1} \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} E_\lambda[\mathrm{Lie}^{(2)}] \qquad = \sum_{k\geq 1} \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} \omega(E_\lambda[\mathrm{Lie}^{(2)}])$$

$\mathrm{Lie}_n = \omega_n\uparrow_{C_n}^{S_n}$ versus $\mathrm{Lie}_n^{(2)} = \omega_n^{2^k}\uparrow_{C_n}^{\mathfrak{S}_n}$ where $n = 2^k\ell$, ℓ odd.

Can replace 2 with an odd prime q: $n = q^k\ell$, q not a divisor of ℓ.
But not many results generalise...

What is special about 2?

Summary table Lie versus $\mathrm{Lie}^{(2)}$ ($^* =$ up to sign twist)

Properties	Lie	$\mathrm{Lie}^{(2)}$
Symmetric powers	Regular repn: Thrall	powers of 2?
Exterior powers	Orlik-Solomon algebra*	Regular repn
Poset homology/matroid	partition lattice, braid matroid	?
Hyperplane arr./Configuration space?	Type A Coxeter arrangement	?
Repn induced from C_n	Yes	Yes
Irreducible decomposition	Yes	Yes
Interesting idempotents	Eulerian idempotents	?
Plethystic inverse of	$\sum_{n\geq 1}(-1)^{n-1}e_n$	$\sum_{n\geq 1}(-1)^{n-1}h_n$
Lifts from $\mathfrak{S}_n$ to $\mathfrak{S}_{n+1}$	Whitehouse module	(conjectured)
Filtration	Derived series	Yes
Complex of injective words	Hodge decomposition*	Hodge-like decomposition
Truncated signed sum of exterior/symmetric powers	A true module (Π_n: rank-selected homology)	Conjectured to be a true module

References

Papers:

S. Sundaram:

On a curious variant of the S_n-module Lie_n, Algebr. Comb. 3 (2020) no. 4

Prime power variations of higher Lie_n *modules*, J. Comb. Theory Ser. A 184 (2021)

Books:

Ian Macdonald: *Symmetric functions and Hall polynomials*

Christophe Reutenauer: *Free Lie algebras*

Richard Stanley: *EC2, Chapter 7*

THANK YOU!