Algebraic combinatorics, nonnegative combinatorial interpretations, and computational complexity theory

Christian Ikenmeyer

WARWICK
THE UNIVERSITY OF WARWICK

ICERM, Brown University, Computation in Representation Theory

2025-Nov-10

Agenda

1. Numbers embedded in other numbers as special cases
2. Circuits and #P
3. Outside #P
4. The character of the symmetric group

Agenda

1. Numbers embedded in other numbers as special cases
2. Circuits and #P
3. Outside #P
4. The character of the symmetric group

Standard Young tableau:

- numbers $1, 2, \ldots$ into the boxes, each number once
- increase left to right, and increase top to bottom

f^λ = number of standard Young tableaux of shape λ.

$$f^{(2,1)} = 2 \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Semistandard Young tableau:

- numbers $1, 2, \ldots$ into the boxes
- increase left to right, and non-decrease top to bottom

Def. Kostka number

Given $\lambda \vdash d$ and $\mu \vDash d$.

$$K_{\lambda,\mu} \;=\; \text{the number of semistandard Young tableaux of shape } \lambda \text{ and content } \mu$$

Example: $\lambda = (4, 3)$, $\mu = (2, 2, 3)$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array}$$

$$f^\lambda \;=\; K_{\lambda,(1^{|\lambda|})}$$

Partial order of special cases: $\textsc{StdTab} \;\leq\; \textsc{Kostka}$

Def. Littlewood–Richardson coefficient

Given partitions $|\alpha| = |\beta| + |\gamma|$. The Littlewood–Richardson coefficient $c^{\alpha}_{\beta,\gamma}$ is the number of

- semistandard skew Young tableaux of shape α/β
- and content γ
- such that the rowwise top-to-bottom right-to-left reading word w satisfies:
  For all prefixes w', for all i, the number of i in w' is at least the number of $i+1$ in w'.

Example: $\alpha = (4,3,2)$, $\beta = (2,1)$, $\gamma = (3,2,1)$

·	·	1	1
·	1	2	
2	3		

·	·	1	1
·	2	2	
1	3		

Def. Kostka number

Given $\lambda \vdash d$ and $\mu \vDash d$. $K_{\lambda,\mu} =$ the number of semistandard Young tableaux of shape λ and content μ

Def. Littlewood–Richardson coefficient

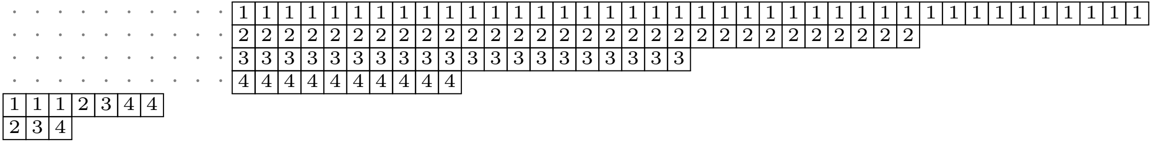
Given partitions $|\alpha| = |\beta| + |\gamma|$. The LRC $c^{\alpha}_{\beta,\gamma}$ is the number of semistandard skew Young tableaux of shape α/β and content γ such that the rowwise right-to-left reading word w satisfies: For all prefixes w', for all i, the number of i in w' is at least the number of $i+1$ in w'.

$n = \ell(\mu)$. $\Delta = (n, n-1, n-2, \ldots, 1)$.

$$K_{\lambda,\mu} \;=\; c^{(d^n)+d\Delta+\lambda}_{(d^n),\, d\Delta+\mu}$$

Example: $\lambda = (7,3)$, $\mu = (3,2,2,3)$, $d = 10$, $n = 4$, $d\Delta = (40, 30, 20, 10)$

1	1	1	2	3	4	4
2	3	4				



StdTab $\leq$ Kostka $\leq$ Littlewood–Richardson

For $\lambda \vdash n$, let $[\lambda]$ be the irreducible S_n-representation of type λ in characteristic zero.

Def. Kronecker coefficient

Given $\lambda, \mu \vdash n$.

$$[\lambda] \otimes [\mu] \;\simeq\; \bigoplus_{\nu \vdash n} [\nu]^{\oplus k(\lambda,\mu,\nu)}$$

The multiplicity $k(\lambda, \mu, \nu)$ is called the **Kronecker coefficient**.

Richard Stanley's "Positivity problems and conjectures in algebraic combinatorics" (2000):

- Problem 10: Give a nonnegative combinatorial interpretation for the Kronecker coefficient.

(Everything in this talk also applies to problem 9, the plethysm coefficient.)

Def. reduced Kronecker coefficient

$$\overline{k}(\alpha,\beta,\gamma) \;=\; \lim_{n\to\infty} k(\,(n-|\alpha|,\alpha),\;(n-|\beta|,\beta),\;(n-|\gamma|,\gamma)\,)$$

Murnaghan 1938:

for $|\alpha| = |\beta| + |\gamma|$ we have $c^{\alpha}_{\beta,\gamma} \;=\; \overline{k}(\alpha,\beta,\gamma)$

Kirillov' problem (2004): Find a nonnegative combinatorial interpretation for $\overline{k}$

Partial order of special cases:

StdTab $\leq$ Kostka $\leq$ Littlewood–Richardson $\leq$ reduced Kronecker $\leq$ Kronecker

With Greta Panova 2023:

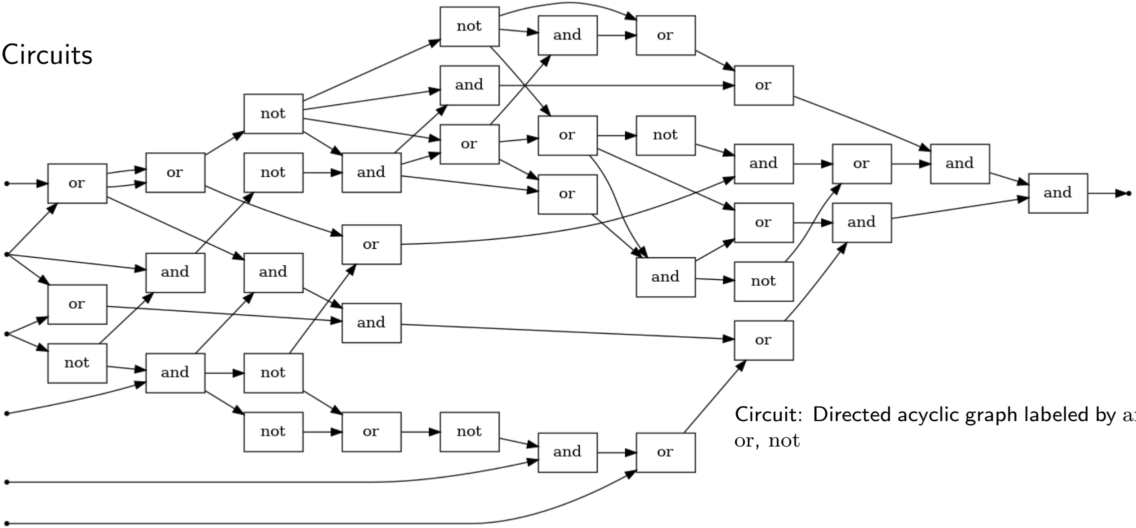
$$k(\lambda,\mu,\nu) \;=\; \overline{k}\Big(\nu_1^{\ell(\lambda)}+\lambda,\; \nu_1^{\ell(\mu)}+\mu,\; (\nu_1^{\ell(\lambda)+\ell(\mu)},\nu)\Big)$$

StdTab $\leq$ Kostka $\leq$ Littlewood–Richardson $\leq$ reduced Kronecker $\overset{!}{=}$ Kronecker

Stanley's problem 10 is equivalent to Kirillov's problem!

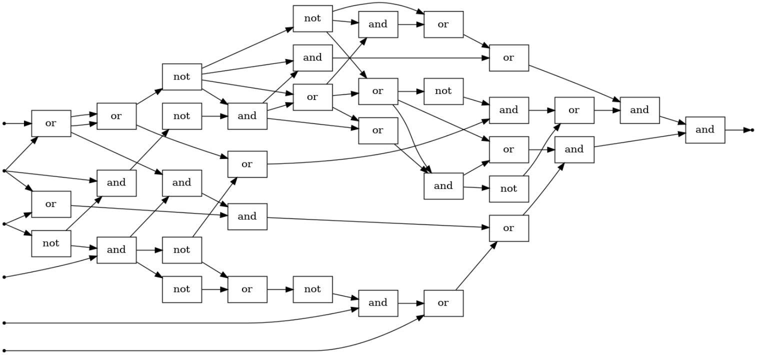
Agenda

Circuits



Circuit: Directed acyclic graph labeled by and, or, not

Define $\#\textsc{CircuitSat}(C) =$ number of inputs $w \in \{0,1\}^n$ such that $C(w) = 1$



$\#\text{CIRCUITSAT}(C)$ = number of inputs $w \in \{0,1\}^n$ such that $C(w) = 1$

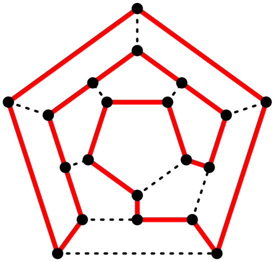
- If $\#\text{CIRCUITSAT}$ is a special case of X, then X is NP-hard to compute.

Define $\text{GAPCIRCUITSAT}(C, D) = \#\text{CIRCUITSAT}(C) - \#\text{CIRCUITSAT}(D) \in \mathbb{Z}$

- If $\text{GAPCIRCUITSAT}(C, D)^2$ has a nonnegative combinatorial interpretation, then each of those objects proves that $\#\text{CIRCUITSAT}(C) \neq \#\text{CIRCUITSAT}(D)$. The existence of such proofs would imply PH $= \Sigma_2^{\mathsf{p}}$.
- If $(\text{GAPCIRCUITSAT})^2$ is a special case of X, then X does not have a not have a nonnegative combinatorial interpretation, provided PH $\neq \Sigma_2^{\mathsf{p}}$.

- #CircuitSat can be replaced by many other problems!

#HamiltonianCycle: Given a graph, determine the number of cycles that use each vertex exactly once.



- #HamiltonianCycle is a special case of #CircuitSat (via a polytime algorithm converted into a circuit).
- #CircuitSat is a special case of #HamiltonianCycle (via a combinatorial construction).

Such a relationship holds for many counting versions of NP-complete problems.

Recall: $\textsc{GapCircuitSat}(C, D) = \#\textsc{CircuitSat}(C) - \#\textsc{CircuitSat}(D) \in \mathbb{Z}$

With Mulmuley and Walter 2017:

$\#\textsc{CircuitSat}$ is a special case of $\textsc{Kronecker}$.

- Kronecker coefficients for extremal weights: Combinatorial interpretation as 3-dim contingency tables
- Gardner–Gritzmann–Prangenberg "On the computational complexity of reconstructing lattice sets from their X-rays", Discrete Mathematics, 1999

$\textsc{Kronecker}$ is a special case of $\textsc{GapCircuitSat}$.

Big open question

Is $\textsc{Kronecker}$ a special case of $\#\textsc{CircuitSat}$?

This would give a nonnegative combinatorial interpretation of $\textsc{Kronecker}$.
This is equivalent to $\textsc{Kronecker} \in \#\mathrm{P}$.

The verifier V inspecting the (x, w)

$$V\left(\left[\ydiagram{4,3}, (3,2,2)\right], \ytableaushort{1212,233}\right) = 0 \qquad V\left(\left[\ydiagram{4,3}, (3,2,2)\right], \diamond\right) = 0$$

$$V\left(\left[\ydiagram{4,3}, (3,2,2)\right], \ytableaushort{1112,233}\right) = 1$$

$$V\left(\left[\ydiagram{4,3}, (3,2,2)\right], \ytableaushort{23,11123}\right) = 0$$

$$V\left(\left[\ydiagram{4,3}, (3,2,2)\right], \ytableaushort{1113,223}\right) = 1$$

Def. #P

$f : \{0,1\}^* \to \mathbb{N}$ is in #P if

- there exists a polynomial time algorithm $V : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$
- and a polynomial p

such that for all $x \in \{0,1\}^*$ we have:
$$f(x) = \sum_{w \in \{0,1\}^{p(\mathsf{len}(w))}} V(x, w)$$

Def. #P

$f : \{0,1\}^* \to \mathbb{N}$ is in #P if

- there exists a polynomial time algorithm $V : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$
- and a polynomial p

such that for all $w \in \{0,1\}^*$ we have: $$f(x) = \sum_{w \in \{0,1\}^{p(\mathsf{len}(w))}} V(x,w)$$

Remarks:

- The witnesses w are of polynomially bounded length, otherwise they could contain the whole computation transcript of the Kronecker coefficient
- V has limited time, otherwise V could compute the Kronecker coefficient directly
- In some situations, V should get more/less power, for example to distinguish non-isomorphic knots. The definition can be adjusted, but this gives a different class.

The history of #P w.r.t. algebraic combinatorics

#P was defined by Leslie Valiant in 1979, for complexity theoretic reasons.

"Nonnegative combinatorial interpretation" = #P for Kronecker/plethysm coefficients was initiated by Ketan Mulmuley.

Mulmuley arXiv:0709.0749, Sept 2007:

> *Find an explicit positive (#P-) formula in the spirit of the Littlewood–Richardson rule for the plethysm constant*

Mulmuley cites Stanley's open problem and formalizes it.

This was the usual viewpoint in geometric complexity theory at the time, see my MSc thesis, supervised by Peter Bürgisser (Diplomarbeit, 2008, similar to FPSAC 2008):

> *Note that we do not know whether KronCoeff is contained in the class #P. In fact, the latter would just express that $k(\lambda, \mu, \nu)$ counts a number of appropriate combinatorial objects (and it can be decided in polynomial time whether a given object is appropriate), which in fact is a combinatorial description of the Kronecker coefficient.*

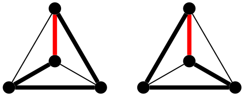
With Igor Pak 2022: First systematic study of which problems could be **outside** of #P
Very fruitful! Numerous papers by different authors by now.

Agenda

Definition $\#\mathrm{HCE}$: with Igor Pak

Input: An undirected graph G in which every vertex has degree 3, and an edge e of G.

Output: The number of Hamiltonian cycles in G that use e.



Clearly, $\#\mathrm{HCE} \in \#\mathsf{P}$

Smith's theorem

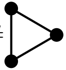
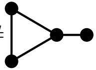
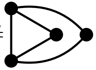
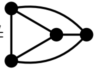
$\#\mathrm{HCE}(G, e)$ is always even.

Open problem:

Is $\frac{\#\mathrm{HCE}}{2} \overset{?}{\in} \#\mathsf{P}$

Thomason's lollipop algorithm gives a pairing, but it requires exponential time.
A polytime algorithm would imply $\#\mathrm{HCE}/2 \in \#\mathsf{P}$, by only counting the lex-smaller Hamiltonian cycle.

- Def.: A graph F is an induced subgraph of G if F is obtained from G by deleting vertices.
- An **induced copy** of H in G is an induced subgraph F of G that is isomorphic to H.
- $\#H$: Graphs $\to \mathbb{N}$, $\#H(G) =$ the number of induced copies of H in G.

$f \; = \; \#$  $- \#$  $+ \#$  $+ 2\#$  $+ 4\#$  ≥ 0

Theorem (with Bläser, Curticapean, Dörfler 2025)

A linear combination of $\#H$ is in $\#\mathsf{P}^{\circ}$ if and only if all its coefficents are in $\mathbb{N}$.

This uses a Ramsey argument.

BipartiteUnbalance

Given a bipartite graph G with $V = U \cup W$ such that

- For every edge $\{u, w\}$ we have $\deg(u) \geq \deg(w)$.

It is known that $|U| \leq |W|$.

Define $\#\textsc{BipartiteUnbalance}(G) \ = \ |W| - |U|$.

Theorem (with Pak 2023)

$\#\textsc{BipartiteUnbalance} \notin \#\mathsf{P}$ ⁛

This uses a more sophisticated circuit problem as a special case, and a Ramsey argument.

Agenda

f^λ = number of standard tableaux of shape λ

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$ via an **explicit bijection**: Robinson-Schensted correspondence

$$\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3&\\\cline{1-1}\end{array}, \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3&\\\cline{1-1}\end{array}\right), \quad \left(\begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3&\\\cline{1-1}\end{array}, \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2&\\\cline{1-1}\end{array}\right), \quad \left(\begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2&\\\cline{1-1}\end{array}, \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3&\\\cline{1-1}\end{array}\right), \quad \left(\begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2&\\\cline{1-1}\end{array}, \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2&\\\cline{1-1}\end{array}\right), \quad \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right)$$

$\chi^\lambda(\alpha)$	(1,1,1)$^{(\times 1)}$	(3)$^{(\times 2)}$	(2,1)$^{(\times 3)}$
(3)	1	1	1
(1,1,1)	1	1	-1
(2,1)	2	-1	0

Restatement of decomposition (1st column):

- $n! = \sum_{\lambda \vdash n} (\chi^\lambda(1,1,\dots,1))^2$

Another Decomposition (rows in char. table):

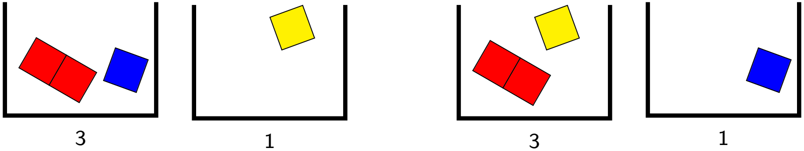
- $\forall \lambda \vdash n : \qquad n! = \sum_{\pi \in \mathfrak{S}_n} (\chi^\lambda(\pi))^2$

Does $(\chi^\lambda(\pi))^2$ have a nonnegative combinatorial interpretation?

Theorem (with Pak and Panova 2023)

Let $\chi^2 : (\lambda, \pi) \mapsto (\chi^\lambda(\pi))^2$.
$(\text{GapCircuitSat})^2$ is a special case of χ^2.
More precisely, if $\chi^2 \in \#\mathsf{P}$, then $\mathsf{PH} = \Sigma_2^{\mathsf{p}}$.

$P(\alpha,\mu) :=$ # ordered set partitions,
item sizes α, bin sizes μ.
Example: $P\big((2,1,1),(3,1)\big) = 2$



$$\chi^\lambda(\alpha) := \sum_{\sigma\in\mathfrak{S}_{\ell(\lambda)}} \operatorname{sgn}(\sigma) P(\alpha,\ \lambda + (\sigma(1),\sigma(2),\sigma(3),\ldots) - (1,2,3,\ldots)).$$
i.e., bin sizes vary

Here $P(\alpha,\mu) := 0$ if any entry is negative.
Example: $\chi^{(2,1)}(1,1,1) = P((1,1,1),(2,1)+(1,2)-(1,2)) - P((1,1,1),(2,1)+(2,1)-(1,2))$
$= P((1,1,1),(2,1)) - P((1,1,1),(3,0)) \ = \ 3-1 \ = \ 2.$
(similar to Murnaghan–Nakayama)

Hepler 1994:

For large n: $\chi^{n\lambda}(n\alpha) = P(n\alpha, n\lambda) = P(\alpha,\lambda)$

- This shows #SetPartition is a special case of χ
- We know #CircuitSat is a special case of #SetPartition
  (Thm 4.4 in Garey–Johnson, 1979, "Computers and Intractability: A Guide to the Theory of NP-completeness")
- Hence: #CircuitSat is a special case of χ.

We need much more: We need $\chi^\lambda(\alpha) = P(\beta,\mu) - P(\gamma,\nu)$

Controlling the signed sum

Recall: $P(\alpha, \mu) := \#$ ordered set partitions, item sizes α, bin sizes μ. For large n: $\chi^{n\lambda}(n\alpha) = P(n\alpha, n\lambda)$

To allow a second summand:

- First item very large, $\equiv 1$ mod n. First bin very large, 0 mod n.
- second item $\equiv -1$ mod n.

$$\chi^{\lambda}(\alpha) = P(\alpha, \lambda) - P(\alpha, \lambda + (1, -1)).$$

That looks good, but both terms should be **independent**!
The bin sizes can be dependent. In fact, we could have all bin sizes to be the same.
Solution idea:

- Two independent sets A and B of items, both of total size s.
- Four additional small items: 1, 2, 5, 6
- Very large first bin and new first item: $\lambda_1 = \alpha_1 + 5$
- Large second bin: $\lambda_2 = s + 2$
- A can only go in the other bins together with 1 and 6
- B can only go in the other bins together with 2 and 5

For bin sizes λ all solutions of the form:

- First item $+5$ in bin 1
- $B + 2$ in bin 2
- $A + 1 + 6$ in the other bins

For bin sizes $\lambda + (1, -1)$ all solutions of the form:

- First item $+6$ in bin 1
- $A + 1$ in bin 2
- $B + 2 + 5$ in the other bins

Theorem (with Pak and Panova 2023)

Let $\chi^2 : (\lambda, \pi) \mapsto (\chi^\lambda(\pi))^2$.
$(\textsc{GapCircuitSat})^2$ is a special case of χ^2.
More precisely, if $\chi^2 \in \#\mathsf{P}$, then $\mathsf{PH} = \Sigma_2^{\mathsf{p}}$.

Bravyi, Chowdhury, Gosset, Havlíček, Zhu 2023:

Deciding Kronecker positivity is in QMA.

(using their construction, I with Subramanian: Kronecker is in $\#$BQP)

If the character proof method also worked for Kronecker coefficients, then NQP $\subseteq$ QMA.
This applies to all methods based on the vanishing of Kronecker coefficients.
But there is no such implication for the $\textsc{BipartiteUnbalance}$ approach, and similar approaches.

Summary

- Quantities form a "partial order of special cases"
- Sometimes one can find a special case of high computational complexity: Hardness of computing
- Even better: Sometimes one can find a special case that has no nonnegative combinatorial interpretation!
- These rely on a priori unrelated conjectures in computer science

Thank you for your attention!