Igor Pak, UCLA

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Vanishing of Schubert coefficients

Computation in Representation Theory

ICERM, Brown University

Main Theorem (P.-Robichaux, 2025)

Positivity of Schubert coefficients can be decided in probabilistic polynomial time

Plan of the talk

1) *What is this about?*

2) *Why should one care?*

3) *How to prove this?*

Structure constants

$\mathcal{A} \leftarrow$ set of combinatorial objects

$\mathcal{R} = \mathbb{Z}\langle e_\alpha : \alpha \in \mathcal{A}\rangle \leftarrow$ ring with a linear basis

$$e_\alpha \cdot e_\beta = \sum_{\gamma \in \mathcal{A}} c^\gamma_{\alpha\beta}\, e_\gamma \leftarrow \text{structure constants}$$

Two problems:

$[c^\gamma_{\alpha\beta} =^? 0] \leftarrow$ *the vanishing problem*

$[c^\gamma_{\alpha\beta} >^? 0] \leftarrow$ *the positivity problem*

Littlewood-Richardson coefficients

$\Lambda = \mathbb{Z}\langle s_\lambda \rangle \leftarrow$ symmetric functions with *Schur functions* as a basis

$c^\lambda_{\mu\nu} = \big|\mathrm{LR}(\lambda/\mu, \nu)\big| \in \mathbb{N} \leftarrow$ LR coefficients count the number of LR tableaux

Theorem [De Loera–McAllister'06, Mulmuley–Narayanan–Sohoni'12]

LR Vanishing $\in$ P

Two step proof:

(1) $c^\lambda_{\mu\nu} = \big|\mathrm{GT}(\lambda/\mu, \nu) \cap \mathbb{Z}^d\big| \leftarrow$ number of integer points in *Gelfand–Tsetlin polytopes*

(2) $c^\lambda_{\mu\nu} > 0 \iff c^{k\lambda}_{k\mu\, k\nu} > 0$ for any $k \leftarrow$ *Knutson–Tao saturation theorem* (1999)

Littlewood-Richardson coefficients

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$c^\lambda_{\mu\nu} = |\mathrm{LR}(\lambda/\mu, \nu)| \in \mathbb{N} \leftarrow$ LR coefficients count the number of LR tableaux

Theorem [De Loera–McAllister'06, Mulmuley–Narayanan–Sohoni'12]

LR Vanishing $\in$ P

Note:

For LR polynomials, the vanishing is also in P (Adve–Robichaux–Yong'19), via

(1) structure constants are counted by integer points in Thomas–Yong polytopes (2018)

(2) saturation property proved by Anderson–Richmond–Yong (2013)

Clebsch–Gordan coefficients

$\lambda, \mu, \nu \leftarrow$ highest weights of a semisimple Lie algebra $\mathfrak{g}$

$V_\lambda \otimes V_\mu = \oplus_\nu C^\lambda_{\mu\nu} V_\nu \leftarrow$ Clebsch–Gordan coefficients

Conjecture: CG Vanishing $\in$ P

Known:

(1) $C^\lambda_{\mu\nu} = \left|\mathrm{BZ}(\lambda/\mu, \nu) \cap \mathbb{Z}^d\right| \leftarrow$ number of integer points in *Berenstein–Zelevinsky polytopes*

(2) saturation fails in other types (Èlashvili, 1992)

Kronecker coefficients

$\chi^\lambda, \chi^\mu, \chi^\nu \leftarrow$ irreducible S_n characters

$\chi^\lambda \cdot \chi^\mu = \sum_\nu g(\lambda, \mu, \nu) \chi^\nu \leftarrow$ *Kronecker coefficients*

Theorem [Ikenmeyer–Mulmuley–Walter'17]

Kronecker positivity is NP–hard

Notes:

- Open whether Kronecker coefficients are in #P (i.e., have a combinatorial interpretation)
- Saturation easily fails for Kronecker coefficients (Rosas'01)
- Saturation fails for reduced Kronecker coefficients (P.–Panova'20)
- IMW theorem holds even for reduced Kronecker coefficients (P.–Panova'20, Ikenmeyer–Panova'24)

Structure constants for Schur P-functions

$P_\lambda(\boldsymbol{x}) = P_\lambda(\boldsymbol{x}, -1) \in \mathbb{Z}[p_1, p_3, p_5, \ldots] \leftarrow$ Schur P-functions (Schur, 1911)

correspond to spin (projective) irreducible characters of S_n

$P_\mu \cdot P_\nu = \sum_\lambda f^\lambda_{\mu\nu} P_\lambda \leftarrow$ structure constants

Notes:

- $f^\lambda_{\mu\nu} =$ number of certain shifted tableaux (Serrano'09, Cho'12, Nguyen'22, etc.)
- saturation fails (Robichaux–Yadav–Yong'22)
- corresponds to Schubert structure constants for Grassmannians in types B/D

Conjecture: Schur P Positivity $\left[f^\lambda_{\mu\nu} >^? 0\right] \in \mathrm{P}$

Theorem [P.–Robichaux'25]: $\left[f^\lambda_{\mu\nu} >^? 0\right] \in \mathrm{RP}$

(positivity of Schur P structure constants can be decided in probabilistic polynomial time)

Schubert coefficients

$$\mathbb{Z}[x_1, x_2, \ldots] = \mathbb{Z}\langle \mathfrak{S}_w : w \in S_\infty \rangle \quad \leftarrow \quad \textit{Schubert polynomials}$$

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_\infty} c^w_{u,v} \, \mathfrak{S}_w \quad \leftarrow \quad \textit{Schubert coefficients}$$

Schubert polynomial $\mathfrak{S}_w \in \mathbb{N}[x_1, x_2, \ldots]$ is defined as

$$\mathfrak{S}_w(\boldsymbol{x}) := \sum_{H \in \mathrm{RC}(w)} \boldsymbol{x}^H \quad \text{where} \quad \boldsymbol{x}^H := \prod_{(i,j)\,:\,H(i,j) = \boxplus} x_i \,.$$

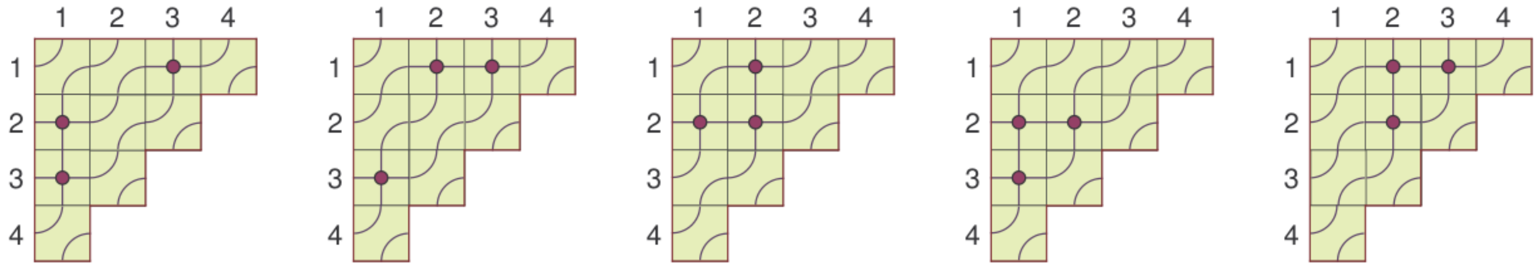

FIGURE 1.2. Graphs in RC(1432) and the corresponding Schubert polynomial $\mathfrak{S}_{1432} = x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1^2x_2$ with monomials in this order.

Schubert coefficients

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SCHUBERT POSITIVITY $[c^u_{vw} >^? 0]$ is equivalent to:

Input: $n \times n$ integral matrices (a_{ij}), (b_{ij}), (c_{ij})

Decide: $\forall U_\bullet, V_\bullet \in \mathcal{F}_n$ s.t. $U_i \cap V_{n-i} = \{\mathbf{0}\}$ for all $1 \leq i < n$,

$\exists W_\bullet \in \mathcal{F}_n$ s.t. $\dim(W_i \cap E_j) = a_{ij}$, $\dim(W_i \cap U_j) \geq b_{ij}$

and $\dim(W_i \cap V_j) \geq c_{ij}$, for all $1 \leq i, j \leq n$

Schubert coefficients

$$\mathbb{Z}[x_1, x_2, \ldots] = \mathbb{Z}\langle \mathfrak{S}_w : w \in S_\infty \rangle \quad \leftarrow \quad \textit{Schubert polynomials}$$

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_\infty} c^w_{u,v} \mathfrak{S}_w \quad \leftarrow \quad \textit{Schubert coefficients}$$

Theorem [P.–Robichaux'25+]: *Saturation fails for Schubert coefficients.*

Here $u, v, w \in S_n$ are given by their Lehmer code, and have two descents.

Notes:

- Saturation was conjectured by Kirillov (2004).
- Proof is based on St. Dizier–Yong vanishing condition (2022).

Schubert coefficients

$$c^w_{uv} = c(u, v, w_\circ w)$$

Main Theorem [P.–Robichaux'25]:

For each $Y \in \{A, B, C, D\}$, SCHUBERT POSITIVITY in type Y is in RP.

More precisely, for all $k \geq 3$ and $\varepsilon > 0$, there is a probabilistic algorithm which inputs elements $u_1, \ldots, u_k \in Y_n$ and after $O\big(kn^{8.75} \log \frac{1}{\varepsilon}\big)$ arithmetic operations outputs either:

- $c(u_1, \ldots, u_k) > 0$, which holds with probability $p = 1$, or
- $c(u_1, \ldots, u_k) = 0$, which holds with probability $p > 1 - \varepsilon$.

Note: The timing is comparable with LP algorithm for LR POSITIVITY.

Bürgisser–Ikenmeyer network flow algorithm for LR POSITIVITY is *much* faster (2013).

Schubert coefficients

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Conjecture [P.–Robichaux'25]:

Schubert Positivity is not in P unless 

Prior work

Many necessary/sufficient conditions for Schubert Positivity:

- Lascoux–Schützenberger the *number of descents condition* (1982)
- strong Bruhat order condition, ibid.
- Knutson's descent cycling condition (2001)
- Billey–Vakil permutation array condition (2008)
- St. Dizier–Yong's fillings of Rothe diagrams condition (2022)
- Hardt–Wallach's empty rows in Rothe diagrams condition (2024)
- Purbhoo's root game conditions (2006)
- Knutson and Zinn-Justin's several (!) tiling conditions (2017, 2023)

Motivation

"For applications (including real-world engineering applications) it is more important to know that [Schubert] structure constant is positive, than it is to know its actual value."

[Knutson, ICM paper, 2022]

Conjecture [P.'22] KRONECKER POSITIVITY $\notin$ PH

$\implies$ Kronecker coefficients $\notin$ #P (no combinatorial interpretation)

Conjecture [P.'22] SCHUBERT POSITIVITY $\notin$ PH

$\implies$ Schubert coefficients $\notin$ #P (no combinatorial interpretation)

$\mathrm{RP} \subseteq \mathrm{NP} \subseteq \mathrm{PH}$

Motivation

> *"For applications (including real-world engineering applications) it is more important to know that [Schubert] structure constant is positive, than it is to know its actual value."*

[Knutson, ICM paper, 2022]

Conjecture [Adve–Robichaux–Yong'22] SCHUBERT VANISHING is NP-hard

FALSE

Conjecture [P.–Robichaux'24] SCHUBERT POSITIVITY is NP-hard

FALSE

(unless PH collapses)

Evolution of our work

1. (Dec 3, 2024): SCHUBERT POSITIVITY is in AM assuming GRH (*Proc. STOC'25*)

GRH = Generalized Riemann Hypothesis

Additionally, SCHUBERT POSITIVITY is in $\mathrm{NP}_{\mathbb{C}} \cap \mathrm{P}_{\mathbb{R}}$ (unconditionally)

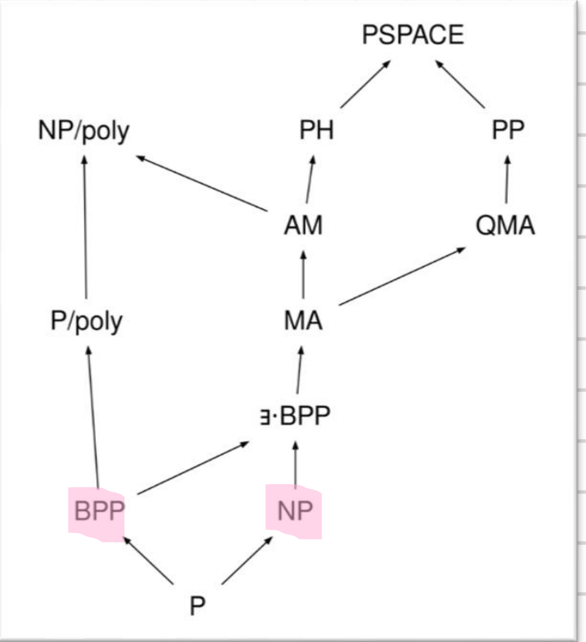
1.5 (Feb 20, 2025): uniform proof in all types (type D joint with Speyer)

2. (Apr 3, 2025): SCHUBERT POSITIVITY is in $\mathrm{AM} \cap \mathrm{coAM}$ assuming GRH

(*Forum Sigma*, 2025)

3. (Sep 19, 2025): SCHUBERT POSITIVITY is in RP (unconditionally)

Note: $\mathrm{RP} \subseteq \mathrm{NP} \cap \mathrm{BPP}$



First approach

HILBERT'S NULLSTELLENSATZ (HN):

Given $f_i \in \mathbb{Z}[x_1, \dots, x_\ell]$, decide: $\exists \boldsymbol{x} = (x_1, \dots, x_\ell) \in \mathbb{C}^\ell$ s.t. $f_1(\boldsymbol{x}) = \dots = f_m(\boldsymbol{x}) = 0$

Theorem [Hilbert's weak Nullstellensatz]

$\neg\mathrm{HN} \iff \exists g_1, \dots, g_m \in \mathbb{Z}[x_1, \dots, x_\ell]$ s.t. $f_1 g_1 + \dots + f_m g_m = 1$

[Mayr–Meyer'82]: HN is in EXPSPACE

[Kollár'88]: HN is in PSPACE

[Koiran'96]: HN is in AM assuming GRH

First approach

PARAMETRIC HILBERT'S NULLSTELLENSATZ (HNP):

Given $f_i \in \mathbb{Z}(y_1, \ldots, y_k)[x_1, \ldots, x_\ell]$, decide: $\exists \boldsymbol{x} \in \left(\overline{\mathbb{C}[y_1, \ldots, y_k]}\right)^\ell$ s.t. $f_1(\boldsymbol{x}) = \ldots = f_m(\boldsymbol{x}) = 0$

[A+'24]: HNP is in AM assuming GRH

{A+} := Ait El Manssour, Balaji, Nosan, Shirmohammadi and Worrell

Main Lemma [P–Robichaux]: SCHUBERT POSITIVITY reduces to HNP

Proof idea: Rewrite $c^w_{u,v} = \#\{X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{\widetilde{w}}(G_\bullet)\}$ as an instance of HNP. $\square$

Second approach

$\mathsf{G} = \mathsf{G}_Y \leftarrow$ simply connected semisimple complex Lie group of type Y

$\mathcal{W} \leftarrow$ Weyl group

$\mathsf{B} \leftarrow$ Borel subgroup, $\mathsf{B}_- := w_\circ B w_\circ$

$\mathsf{G} \subset V \leftarrow$ ambient vector space

$\mathsf{N} \subset \mathsf{G} \leftarrow$ subgroup of unipotent matrices

$\mathfrak{n} \leftarrow$ Lie algebra of N

$R_w := \mathfrak{n} \cap w\mathsf{B}_- w^{-1} \leftarrow$ vector spaces corresponding to $w \in \mathcal{W}$

Theorem [Purbhoo'06]

For generic $\rho_1, \ldots, \rho_k \in \mathsf{N}$ and $u_1, \ldots, u_k \in \mathcal{W}$, we have: $c(u_1, \ldots, u_k) > 0$

$$\iff \quad \rho_1 R_{u_1} \rho_1^{-1} + \ldots + \rho_k R_{u_k} \rho_k^{-1} = \rho_1 R_{u_1} \rho_1^{-1} \oplus \ldots \oplus \rho_k R_{u_k} \rho_k^{-1}$$

Polynomial identity testing

Theorem [Schwarz–Zippel Lemma'79]

Let $\mathbb{F}$ *be a field,* $S \subset \mathbb{F}$ *be a finite set. Let* $Q \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ *be a non-zero polynomial with degree* $d \geq 0$ *over* $\mathbb{F}$. *Then:*

$$\mathbf{P}\left[Q(c_1, c_2, \ldots, c_n) = 0\right] \;\leq\; \frac{d}{|S|}$$

where the probability is over random, independent and uniform choices of $c_1, c_2, \ldots, c_n \in S$.

Schwartz–Zippel lemma

From Wikipedia, the free encyclopedia

PROOF

Polynomial identity testing

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Let $\mathbb{F}$ *be a field,* $S \subset \mathbb{F}$ *be a finite set. Let* $Q \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ *be a non-zero polynomial with degree* $d \geq 0$ *over* $\mathbb{F}$. *Then:*

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where the probability is over random, independent and uniform choices of $c_1, c_2, \ldots, c_n \in S$.

Important Note: Polynomial Q can be given indirectly, e.g. as a determinant of a $k \times k$ matrix of polynomials $q_{ij}(x_1, \ldots, x_n)$. See [Lovász'79], [Valiant'80].

Second approach

Proof idea of the Main Theorem:

Rewrite "generic" condition in the Purbhoo's criterion as an instance of PIT.

Carefully assemble the pieces to obtain the desired probabilistic poly-time algorithm. □

Conjecture [P.–Robichaux'25]:

SCHUBERT POSITIVITY is not in P unless PIT can be decided quasi-polynomial time: $n^{O((\log n)^c)}$

Thank you!