Computing Lusztig's q-Weight Multiplicities Beyond Type A

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Computation in Representation Theory

Joint work with Adam Schultze (Lewis University), contains work with other collaborators.
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Irreducible representations of semisimple Lie algebras $\mathfrak{g}$

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Character: $\mathrm{ch}\, V(\lambda) := \sum_\mu K_{\lambda,\mu}\, x^\mu \,.$

Crystal graphs

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The vertices of $B(\lambda)$ correspond to the (crystal) basis elements of $V(\lambda)$, and the edges (labeled α_i, $i \in I$) to the action of crystal operators (modified versions of the Chevalley generators of the quantum algebra).

Tableau models: type A

The vertices of $B(\lambda)_\mu$ (corresponding to the μ-weight space $V(\lambda)_\mu$) can be labeled by semistandard Young tableaux of shape λ and content μ (denoted $\mathsf{SSYT}(\lambda, \mu)$):

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$$\lambda=(4,2,2,1)\,,\;\mu=(3,2,2,1,1)\qquad b=\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3\\\cline{1-2} 3&4\\\cline{1-2} 5\\\cline{1-1}\end{array}\;.$$

Tableau models: types $B - D$

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Fact. In type C there are bijections between KN$\leftrightarrow$King [Sheats, 1999] and King$\leftrightarrow$SSOT [Lee, 2025].

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Its definition is involved: it is a sum of local energies, computed based on the combinatorial R-matrix. So we need a simpler computation.

KR crystals (cont.)

Type A_{n-1}. We have $B^{k,1} \simeq B(\omega_k) = B(1^k)$, so the vertices of $B^{\otimes \mathbf{p}}$ are represented as column-strict fillings (with integers $1, \ldots, n$) of the diagram with columns of heights $p_1, p_2, \ldots$.

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Types $B - D$. $B^{\otimes \mathbf{p}}$ is represented as a sequence of Kashiwara-Nakashima columns.

Lusztig's q-weight multiplicities

Lusztig's q-analogue of weight multiplicities $K_{\lambda,\mu}(q)$ of $K_{\lambda,\mu} = K_{\lambda,\mu}(1)$ is obtained via a q-analogue of the Weyl character formula:

$$\mathrm{ch}_q\, V(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} x^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R^+} (1 - q x^{-\alpha})} = \sum_{\mu} K_{\lambda,\mu}(q) x^{\mu}\,,$$

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For $\mu \in P^+$ (assumed throughout), we have $K_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$ (Kostka-Foulkes polynomial in type A).

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- they are related to the Hall-Littlewood polynomials;
- they are related to the energy function on Kirillov-Reshetikhin (KR) crystals (next).

Computing Lusztig's q-weight multiplicities

"Gold standard:"

In type A, $K_{\lambda,\mu}(q)$ is calculated by the Lascoux-Schützenberger charge statistic on SSYT of shape λ and content μ:

$$K_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{charge}(T)} .$$

Other classical types: important long-standing problem

There are only partial results:

- $K_{\lambda,0}(q)$ of type C_n computed via crystal graphs [Lecouvey-L., 2020], extending a type A formula of Lascoux-Leclerc-Thibon;

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- $K_{\lambda,\mu}(q)$ of type C_n for λ a row [Dolega-Gerber-Torres, 2020], via a charge statistic conjectured by Lecouvey;
- $K_{\lambda,\mu}(q)$ of type C_2 [Patimo-Torres, 2025], via the atomic decomposition of the respective crystals [Lecouvey-L., 2021].

Reviving a traditional approach. Main goal

Facts. (1) In type A, there is a well-known duality [Schur, Nakayashiki-Yamada] between q-weight multiplicities (charge on SSYT) and tensor product multiplicities (energy function on tensor products of column-shape KR crystals of affine type A).

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Need the correspondence between the quantum alcove model and the respective tableaux models [L.-Schultze].

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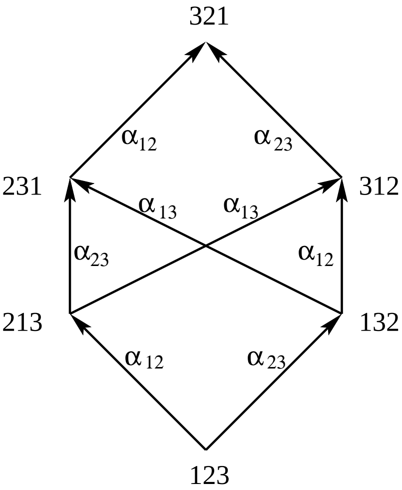
$$w \xrightarrow{\alpha} ws_\alpha \,, \qquad \text{where}$$

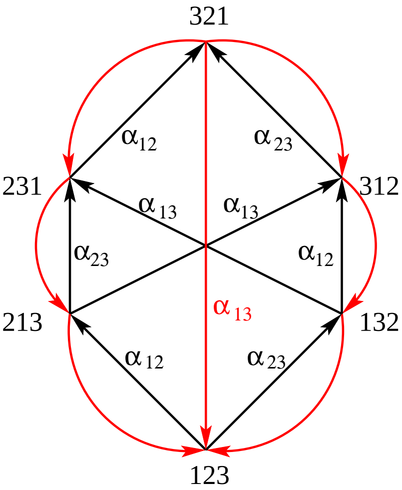
$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(covers of Bruhat order)}\,, \qquad \text{or}$$
$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1\,.$$

(If $\alpha^\vee = \sum_i c_i \alpha_i^\vee$, then $\mathrm{ht}(\alpha^\vee) := \sum_i c_i$.)

Hasse diagram of the Bruhat order for S_3:



Quantum Bruhat graph for S_3:



The realization of KR crystals of arbitrary type (cont.)

Definition. Given $\mu \in P^+$, we start by fixing *any* corresponding sequence of positive roots called a μ-chain

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Let $r_i := s_{\beta_i}$, and consider the set $\mathcal{A}(\Gamma)$ of admissible subsets:

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, m\}\,,$$

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such that the following is a path in QBG(W):

$$Id \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} r_{j_1} \ldots r_{j_s}\,.$$

Main results

Theorem. [L.-Naito-Sagaki-Schilling-Shimozono] *Given a composition* $\mathbf{p} = (p_1, p_2, \ldots)$ *and an arbitrary Lie type, consider a* μ*-chain* Γ*, where*

$$\mu := \omega_{p_1} + \omega_{p_2} + \ldots .$$

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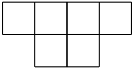
Theorem. [L.-Schultze] *In all classical types there is a forgetful bijection from* $\mathcal{A}(\Gamma)$ *to the tableau model (sequence of Kashiwara-Nakashima columns). The inverse map is explicitly constructed via a greedy algorithm.*

Example in type $A_2^{(1)}$

$\mathbf{p} = (1,2,2,1) =$  $; \quad \mu = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4,2,0).$

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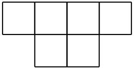
$\mathbf{p} = (1,2,2,1) =$  $; \quad \mu = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4,2,0).$

A μ-chain as a concatenation of ω_1-, ω_2-, ω_2-, and ω_1-chains:

$$\Gamma = (\ \ (1,2), \ \ (1,3) \ \mid \ (2,3), \ \ (1,3) \ \mid \ (2,3), \ \ (1,3) \ \mid \ (1,2), \ \ (1,3) \ \ ) .$$

Example in type $A_2^{(1)}$ (cont.)

Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$\left(\ \underline{(1,2)},\ \underline{(1,3)} \ \middle|\ \underline{(2,3)},\ (1,3) \ \middle|\ (2,3),\ \underline{(1,3)} \ \middle|\ \underline{(1,2)},\ \underline{(1,3)} \ \right).$$

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Claim: J is admissible.

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Claim: J is admissible. Indeed, the corresponding QBG path is:

$$\begin{array}{c}\boxed{1}\\ \\ \boxed{2}\\ \boxed{3}\end{array} \begin{array}{c}<\\ \\ \rightarrow\end{array} \begin{array}{c}\boxed{2}\\ \\ \boxed{1}\\ \boxed{3}\end{array} \begin{array}{c}<\\ \\ \rightarrow\end{array} \begin{array}{c}\boxed{3}\\ \\ \boxed{1}\\ \boxed{2}\end{array} \ \Big| \ \begin{array}{c}\boxed{3}\\ \boxed{1}\\ \\ \boxed{2}\end{array} \begin{array}{c}<\\ \\ \rightarrow\end{array} \begin{array}{c}\boxed{3}\\ \boxed{2}\\ \\ \boxed{1}\end{array} \ \Big| \ \begin{array}{c}\boxed{3}\\ \boxed{2}\\ \\ \boxed{1}\end{array} \begin{array}{c}>\\ \\ \rightarrow\end{array} \begin{array}{c}\boxed{1}\\ \boxed{2}\\ \\ \boxed{3}\end{array} \ \Big| \ \begin{array}{c}\boxed{1}\\ \\ \boxed{2}\\ \boxed{3}\end{array} \begin{array}{c}<\\ \\ \rightarrow\end{array} \begin{array}{c}\boxed{2}\\ \\ \boxed{1}\\ \boxed{3}\end{array} \begin{array}{c}<\\ \\ \rightarrow\end{array} \begin{array}{c}\boxed{3}\\ \\ \boxed{1}\\ \boxed{2}\end{array}.$$

Example in type $A_2^{(1)}$ (cont.)

Let $J = \{1,2,3,6,7,8\}$.

$$\left(\ \underline{(1,2)},\ \underline{(1,3)}\ \middle|\ \underline{(2,3)},\ (1,3)\ \middle|\ (2,3),\ \underline{(1,3)}\ \middle|\ \underline{(1,2)},\ \underline{(1,3)}\ \right).$$

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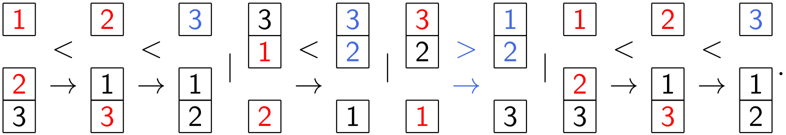

The corresponding element $b \in B^{\otimes \mathbf{p}} = B^{1,1} \otimes B^{2,1} \otimes B^{2,1} \otimes B^{1,1}$:

$$J \xrightarrow{\text{fill}} \boxed{3} \otimes \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \otimes \boxed{3} \xrightarrow{\text{sort}} \boxed{3} \otimes \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \otimes \boxed{3} = b\,.$$

Example in type $A_2^{(1)}$ (cont.)

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Inverse map: invert "sort", needed for rederiving the type A charge;

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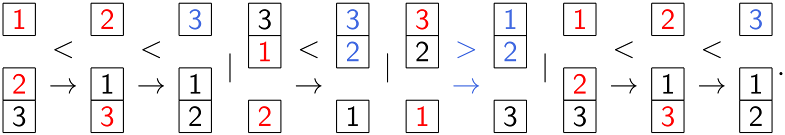

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Inverse map: invert "sort", needed for rederiving the type A charge; then relate all pairs of consecutive columns in $\mathrm{sort}^{-1}(b)$ (for $b \in B^{\otimes \mathbf{p}}$) by QBG paths, constructed via a greedy algorithm.

The energy function in arbitrary type

Definition. For $J \in \mathcal{A}(\Gamma)$, define the statistic

$$\mathrm{height}(J) := \sum_{j \in J^-} h_j \,,$$

where

$$J^- := \{ j_i \ : \ r_{j_1} \ldots r_{j_{i-1}} > r_{j_1} \ldots r_{j_{i-1}} r_{j_i} \} \,.$$

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Theorem. [L.-Naito-Sagaki-Schilling-Shimozono] *Given J in $\mathcal{A}(\Gamma)$, which is identified with $B^{\otimes \mathbf{p}}$, the energy function at the vertex J is given by* $\mathrm{height}(J)$.

Recall:

Main goal. Rederive the type A charge (for q-weight multiplicities) via the connection to the energy function on KR crystals (tensor product multiplicities computed via the quantum alcove model).

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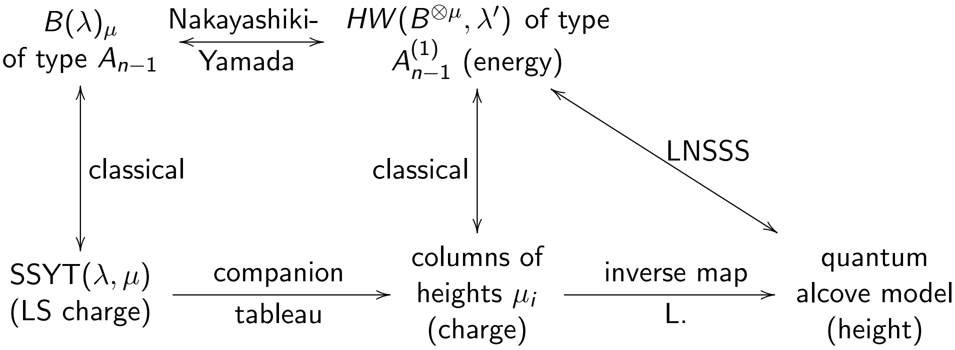
Then generalize to types C and B.

Rederiving the type A charge



Consider $B_{\mathrm{cl}}^{\otimes\mu}$ (remove the edges labeled by α_0), and its connected components which are isomorphic to $B(\lambda')$. Let $HW(B^{\otimes\mu}, \lambda')$ be the set of highest weight vertices of these components.

LS=Lascoux-Schützenberger
LNSSS=L.-Naito-Sagaki-Schilling-Shimozono

Rederiving the type A charge (cont.)

Recall: the bijection from the quantum alcove model to the tableau model is

$$\mathcal{A}(\Gamma) \xrightarrow{\text{fill}} \text{fill}(\mathcal{A}(\Gamma)) \xrightarrow{\text{sort}} B^{\otimes\mu} \, .$$

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Fact. The following rule (based on the quantum Bruhat graph) is used to construct the map sort^{-1}:

a_1	b_1
$\dots$	$\dots$
a_i	$\boldsymbol{b_i}$
a_{i+1}	
$\dots$	

,

$$b_i = \min\{b_i, b_{i+1}, \dots\}$$

in the circular order starting at a_i :

$$a_i,\ a_i+1,\ \dots,\ n,\ 1,\ \dots,\ a_i - 1\,.$$

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Example.

$$\text{sort}^{-1}\left(\begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 & \\ \cline{1-3} 6 & 4 & 4 & \\ \cline{1-3}\end{array}\right) = \begin{array}{|c|c|c|c|}\hline 3 & & & \\ \hline 5 & & & \\ \cline{1-3} 6 & & & \\ \cline{1-3}\end{array}\,.$$

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Example.

sort^{-1} of

3	2	1	2
5	3	2	
6	4	4	

=

3	3		
5	2		
6			

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Rederiving the type A charge (cont.)

Given an admissible subset J in $\mathcal{A}(\Gamma)$, with associated column-strict filling $b \in B^{\otimes\mu}$, we can read off the statistic $\mathrm{height}(J)$ (expressing the energy) from $\mathrm{sort}^{-1}(b)$:

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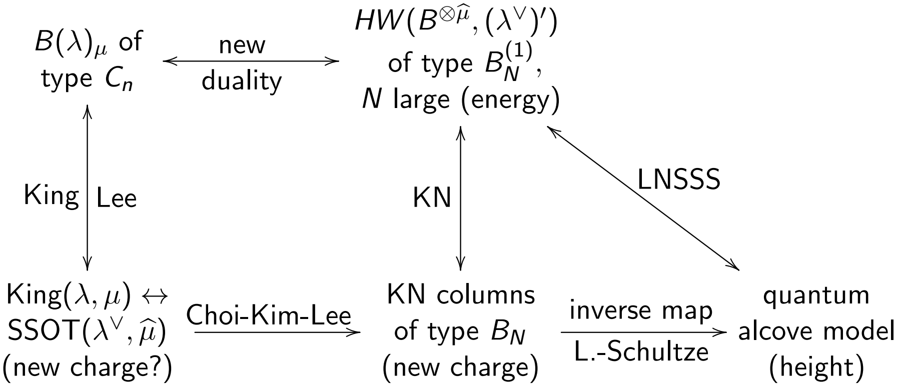
Main idea (for the new approach to charge, to be generalized). The classical charge $\Leftrightarrow$ sort^{-1} $\Leftrightarrow$ quantum Bruhat graph.

A new type C_n charge



$\lambda^\vee := (g - \lambda_n, \ldots, g - \lambda_1)$, $\widehat{\mu} := (g - \mu_1, \ldots, g - \mu_n)$, for $g \geq \lambda_1$

SSOT=semistandard oscillating tableaux

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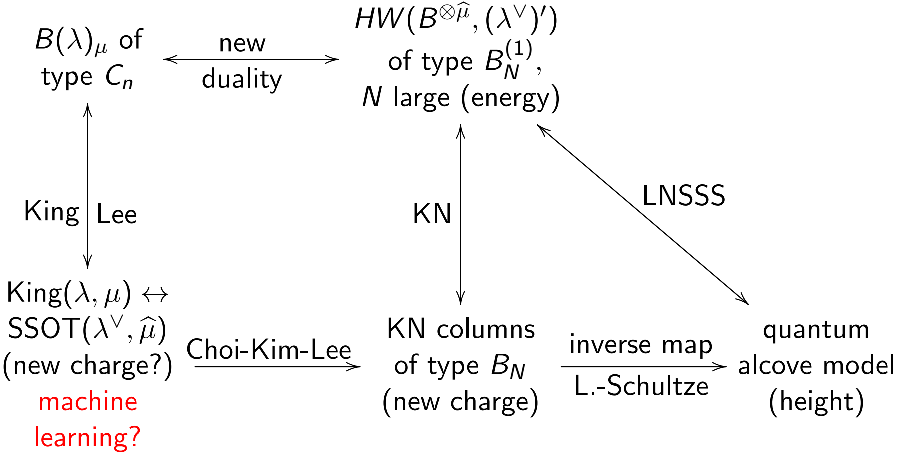

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KR crystals of type $B_N^{(1)}$: complications for the inverse map

Complication 1. Unlike in types A, C, where $B^{k,1} \simeq B(\omega_k)$, now

$$(B^{k,1})_{\mathrm{cl}} \simeq B(\omega_k) \oplus B(\omega_{k-2}) \oplus \dots .$$

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Complication 2. Unlike in type A, KN columns of type $B - D$ (realizing $B(\omega_k)$) need to be "split" into two columns of height k (this is needed to check an "admissibility" condition):

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Complication 3. Unlike in types A, C, at every intermediate step in the construction of the QBG path, we need to avoid certain configurations of two consecutive columns ("blocked-off").

A charge-type formula for the energy function in type $B_N^{(1)}$

Theorem. [L.-Schultze] *Given an admissible subset J in $\mathcal{A}(\Gamma)$, with associated column-strict filling $b \in B^{\otimes\mu}$, we can read off the statistic* $\mathrm{height}(J)$ *(expressing the energy) from*

$$\mathrm{sort}^{-1} \circ \mathrm{extend} \circ \mathrm{split}(b)\,.$$

Other results. Further work

- q-weight multiplicities of type B_n for spin (half-integer) weights are related to the energy function on KR crystals of type $D_N^{(2)}$ for large N [Choi-Kim-Lee, 2025];

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- there is a version of the quantum alcove model for the twisted affine types, but the correspondence with the tableau model is still to be worked out;
- not yet understood: q-weight multiplicities of type B_n for non-spin (integer) weights, and of type D_n.