

# The minimal Rickard complexes of braids on two strands

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# Outline

- Introduction
  - the minimal Rouquier complexes for braids on two strands
- Main result
  - the minimal Rickard complexes for braids on two strands
- An application related to the free loop space of the Grassmannian
  - also, how I found these minimal complexes

# Introduction

Rouquier's categorification of the braid group assigns

$$\begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} \begin{array}{c} 1 \\ \diagup \\ 1 \end{array} := \begin{array}{c} 1 \\ \hline 1 \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} \xleftarrow{Z_{01}} t^{-1} q^1 \begin{array}{c} 1 \\ \diagup \\ 1 \end{array} \begin{array}{c} 2 \\ \diagdown \\ 1 \end{array} \begin{array}{c} 1 \\ \diagup \\ 1 \end{array}$$

$$\begin{array}{c} 1 \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} := t^1 q^{-1} \begin{array}{c} 1 \\ \diagup \\ 1 \end{array} \begin{array}{c} 2 \\ \diagdown \\ 1 \end{array} \begin{array}{c} 1 \\ \diagup \\ 1 \end{array} \xleftarrow{Z_{10}} \begin{array}{c} 1 \\ \hline 1 \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array}$$

The webs are shorthand for the Bott–Samelson bimodules

$$\begin{array}{c} 1 \\ \hline 1 \end{array} \begin{array}{c} 1 \\ \hline 1 \end{array} := \mathbf{Z} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes_{\mathbf{Z}[x_1, x_2]} \mathbf{Z} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{array}{c} 1 \\ \diagup \\ 1 \end{array} \begin{array}{c} 2 \\ \diagdown \\ 1 \end{array} \begin{array}{c} 1 \\ \diagup \\ 1 \end{array} := q^{-1} \mathbf{Z} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes_{\mathbf{Z}[e_1, e_2]} \mathbf{Z} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$Z_{01}: 1 \mapsto 1$  and  $Z_{10}: 1 \mapsto 1 \otimes x_2 - x_1 \otimes 1$  are generators of their hom spaces. Tensor products satisfy the braid relations, so  $\beta \in \text{Br}_n$  determines a complex  $F_\beta$ . The triply-graded homology and the  $\mathfrak{sl}_N$  homology of  $\bar{\beta}$  are determined by  $F_\beta$ .

- $\text{HHH}(\bar{\beta})$ : take Hochschild homology of the bimodules, then homology.
- $\text{KR}_N(\bar{\beta})$ : pass to  $\mathfrak{sl}_N$  foams category, form closure, then homology.

# Introduction

Exercise 19.25 of Elias–Makisumi–Thiel–Williamson “Introduction to Soergel bimodules”: There is a homotopy equivalence

$$\begin{array}{c}
 \begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} \begin{array}{c} 1 \\ \diagup \\ 1 \end{array} = \begin{array}{c} t^{-1}q^1 \text{ (cup) } \\ \downarrow Z_{01} \otimes \text{Id} \\ \text{parallel strands} \end{array} \xleftarrow{\text{Id} \otimes Z_{01}} \begin{array}{c} t^{-2}q^2 \text{ (cap) } \\ \downarrow Z_{01} \otimes \text{Id} \\ \text{parallel strands} \end{array} \\
 \simeq \text{parallel strands} \xleftarrow{Z_{01}} t^{-1}q^1 \text{ (cup) } \xleftarrow{1 \otimes x_1 - x_1 \otimes 1} t^{-2}q^3 \text{ (cup) }
 \end{array}$$

This three-term complex is *minimal*: it has no contractible direct summand. Any equivalent complex admits a deformation retract onto it.

The minimal complex of  $\begin{array}{c} 1 \\ \diagdown \\ 1 \end{array} \begin{array}{c} 1 \\ \diagup \\ 1 \end{array}$  is

$$\text{parallel strands} \xleftarrow{Z_{01}} t^{-1}q^1 \text{ (cup) } \xleftarrow{1 \otimes x_1 - x_1 \otimes 1} t^{-2}q^3 \text{ (cup) } \xleftarrow{Z_{10}Z_{01}} t^{-3}q^5 \text{ (cup) }$$

# Introduction

The minimal complexes form an infinite complex  $\mathcal{P}_1$

$$\text{---} \xleftarrow{Z_{01}} t^{-1} q^1 \text{---} \text{---} \xleftarrow{Q_1} t^{-2} q^3 \text{---} \text{---} \xleftarrow{Z_{10} Z_{01}} t^{-3} q^5 \text{---} \text{---} \xleftarrow{Q_1} \dots$$

where  $Q_1 = 1 \otimes x_1 - x_1 \otimes 1$ . The leftmost  $1 + k$  terms of  $\mathcal{P}_1$  form a subcomplex that is the minimal complex of  $\sigma^k \in \text{Br}_2$ .  $\mathcal{P}_1$  is a categorified projector:

$$\mathcal{P}_1 \otimes \mathcal{P}_1 \simeq \mathcal{P}_1 \quad \mathcal{P}_1 \otimes \text{---} \text{---} \simeq 0$$

There is a related “compact” projector  $\mathcal{K}_1$  (Hogancamp ’18)

$$\text{---} \xleftarrow{Z_{01}} t^{-1} q^1 \text{---} \text{---} \xleftarrow{Q_1} t^{-2} q^3 \text{---} \text{---} \xleftarrow{Z_{10}} t^{-3} q^5 \text{---}$$

whose  $n$ -strand generalization plays an important role in the computation of HHH of torus links (Elias–Hogancamp ’19, Hogancamp ’17, Mellit ’22, Hogancamp–Mellit ’19)

# Introduction

(Chuang–Rouquier)–Rickard complexes generalize Rouquier complexes:

$$\begin{array}{c} b \\ \diagdown \\ a \end{array} \begin{array}{c} a \\ \diagup \\ b \end{array} := \Gamma_0 \xleftarrow{\zeta_{01}} t^{-1} q^1 \Gamma_1 \xleftarrow{\zeta_{12}} \dots \xleftarrow{\zeta_{(m-1)m}} t^{-m} q^m \Gamma_m$$

$$\begin{array}{c} b \\ \diagdown \\ a \end{array} \begin{array}{c} a \\ \diagup \\ b \end{array} := t^m q^{-m} \Gamma_m \xleftarrow{\zeta_{m(m-1)}} \dots \xleftarrow{\zeta_{21}} t^1 q^{-1} \Gamma_1 \xleftarrow{\zeta_{10}} \Gamma_0$$

where  $m := \min(a, b)$  and

$$\Gamma_r := \begin{array}{ccc} & m-r & \\ b & \text{---} & a \\ & \diagdown \quad \diagup & \\ b-m+r & & a-m+r \\ & \diagup \quad \diagdown & \\ a & \text{---} & b \\ & a+b-m+r & \end{array}$$

The webs  $\Gamma_0, \dots, \Gamma_m$  represent singular Bott–Samelson bimodules where

$$R_L := \frac{\text{Sym}^b}{\otimes} \text{Sym}^a \text{ and } R_R := \frac{\text{Sym}^a}{\otimes} \text{Sym}^b \quad \text{Sym}^c = \mathbf{Z}[x_1, \dots, x_c]^{\mathfrak{S}_c}$$

i.e.  $R_L = \mathbf{Z}[x_1, \dots, x_{a+b}]^{\mathfrak{S}_b \times \mathfrak{S}_a}$  and  $R_R = \mathbf{Z}[x_1, \dots, x_{a+b}]^{\mathfrak{S}_a \times \mathfrak{S}_b}$ . The labels represent 1-column partitions (exterior powers in the  $\mathfrak{sl}_N$  category)

# Introduction

Example:  $a = b = 2$ . Webs

$$\begin{array}{c} 2 \\ \diagdown \\ 2 \end{array} \begin{array}{c} 2 \\ \diagup \\ 2 \end{array} := \begin{array}{c} \text{---} \\ \text{---} \end{array} \xleftarrow{\zeta_{01}} t^{-1} q^1 \begin{array}{c} \text{---} \\ \text{---} \end{array} \xleftarrow{\zeta_{12}} t^{-2} q^2 \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$\begin{array}{c} 2 \\ \diagdown \\ 2 \end{array} \begin{array}{c} 2 \\ \diagup \\ 2 \end{array} := t^2 q^{-2} \begin{array}{c} \diagup \\ \diagdown \end{array} \xleftarrow{\zeta_{21}} t^1 q^{-1} \begin{array}{c} \text{---} \\ \text{---} \end{array} \xleftarrow{\zeta_{10}} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

The minimal complex for  $\begin{array}{c} b \\ \diagdown \\ a \end{array} \begin{array}{c} b \\ \diagup \\ a \end{array}$  was constructed by  
Hogancamp–Rose–Wedrich '21 (Beliakova–Habiro '13).

$$\begin{array}{c} 2 \\ \diagdown \\ 2 \end{array} \begin{array}{c} 2 \\ \diagup \\ 2 \end{array} \simeq \begin{array}{c} t^{-4} q^8 \begin{array}{c} \diagup \\ \diagdown \end{array} \\ \downarrow \\ t^{-2} q^3 \begin{array}{c} \text{---} \\ \text{---} \end{array} \xleftarrow{\quad} t^{-3} q^4 \begin{array}{c} \diagup \\ \diagdown \end{array} \\ \downarrow \quad \oplus \quad \downarrow \\ \begin{array}{c} \text{---} \\ \text{---} \end{array} \xleftarrow{\quad} t^{-1} q^1 \begin{array}{c} \text{---} \\ \text{---} \end{array} \xleftarrow{\quad} t^{-2} q^2 \begin{array}{c} \diagup \\ \diagdown \end{array} \end{array}$$

# Main result

## Theorem (W '25)

*An explicit construction of the minimal complex homotopy equivalent to*



*for any number of crossings.*

- The challenge of finding the minimal complex was posed by Wedrich '14.
- For  $k \geq 0$  crossings, there are  $1 + k + k^2 + \dots + k^{\min(a,b)}$  indecomposable bimodules appearing in the minimal complex.
- They fit together into an infinite complex  $\mathcal{P}_b$  for which

$$\mathcal{P}_b \otimes \mathcal{P}_b \simeq \mathcal{P}_b \quad \mathcal{P}_1 \otimes \begin{array}{c} b-l \\ \hline l \quad l \\ \hline b+l \end{array} \begin{array}{c} b \\ l \\ b \end{array} \simeq 0 \text{ for } l \in \{1, \dots, b\}$$

generalizing  $\mathcal{P}_1$ .

For today, assume  $a = b$ .



# Main result

The minimal complex of  $\sigma^k$  with strands labeled by 2 consists of the leftmost  $1 + k$  columns of the bicomplex  $\mathcal{P}_2$

$$\begin{array}{ccccccc}
 & & & & & & t^{-8}q^{20} \text{---} \text{---} \text{---} \leftarrow \pi Z_{21}Z_{12}s_1 \dots \\
 & & & & & & \downarrow Q_2 t \\
 & & & & & & t^{-6}q^{14} \text{---} \text{---} \text{---} \leftarrow \pi Q_2 s_1 \quad t^{-7}q^{17} \text{---} \text{---} \text{---} \leftarrow s_1^* Z_{21}Z_{12} \partial_1 \dots \\
 & & & & & & \downarrow Z_{21}Z_{12}t \quad \downarrow Z_{21}Z_{12} \\
 & & & & & & t^{-4}q^8 \text{---} \text{---} \text{---} \leftarrow \pi Z_{21}Z_{12}s_1 \quad t^{-5}q^{11} \text{---} \text{---} \text{---} \leftarrow s_1^* Q_2 \partial_1 \quad t^{-6}q^{13} \text{---} \text{---} \text{---} \leftarrow \partial_1^* Z_{21}Z_{12}s_1 \dots \\
 & & & & & & \downarrow Q_2 t \quad \downarrow Q_2 \quad \downarrow Q_2 \\
 & & & & & & t^{-2}q^2 \text{---} \text{---} \text{---} \leftarrow \pi Q_2 s_1 \quad t^{-3}q^5 \text{---} \text{---} \text{---} \leftarrow s_1^* Z_{21}Z_{12} \partial_1 \quad t^{-4}q^7 \text{---} \text{---} \text{---} \leftarrow \partial_1^* Q_2 s_1 \quad t^{-5}q^9 \text{---} \text{---} \text{---} \leftarrow s_1^* Z_{21}Z_{12} \partial_1 \dots \\
 & & & & & & \downarrow Z_{12}t \quad \downarrow Z_{12} \quad \downarrow Z_{12} \quad \downarrow Z_{12} \\
 \text{---} \text{---} \text{---} \leftarrow \pi Z_{01} \quad t^{-1}q^1 \text{---} \text{---} \text{---} \leftarrow Q_1 \quad t^{-2}q^3 \text{---} \text{---} \text{---} \leftarrow Z_{10}s_1^* \partial_1 Z_{01} \quad t^{-3}q^5 \text{---} \text{---} \text{---} \leftarrow Q_1 \quad t^{-4}q^7 \text{---} \text{---} \text{---} \leftarrow Z_{10}s_1^* \partial_1 Z_{01} \dots
 \end{array}$$

# Main result

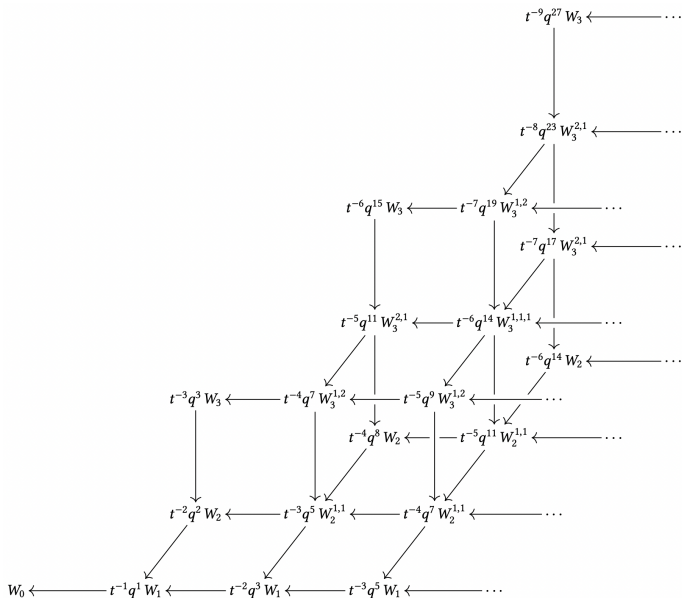
First three columns

$$\begin{array}{ccccc}
 & & & & t^{-4}q^8 \text{ (crossing)} \\
 & & & & \downarrow \\
 & & & & t^{-3}q^5 \text{ (crossing)} \\
 & & t^{-2}q^2 \text{ (crossing)} & \longleftarrow & \\
 & & \downarrow & & \downarrow \\
 \text{=} & \longleftarrow & t^{-1}q^1 \text{ (parallel)} & \longleftarrow & t^{-2}q^3 \text{ (parallel)}
 \end{array}$$

are a rearrangement of Hogancamp–Rose–Wedrich’s complex

$$\begin{array}{ccccc}
 & & & & t^{-4}q^8 \text{ (crossing)} \\
 & & & & \downarrow \\
 & & & & t^{-3}q^4 \text{ (crossing)} \\
 & & & & \oplus \\
 & & & & t^{-3}q^6 \text{ (crossing)} \\
 & & t^{-2}q^3 \text{ (parallel)} & \longleftarrow & \\
 & & \downarrow & & \downarrow \\
 \text{=} & \longleftarrow & t^{-1}q^1 \text{ (parallel)} & \longleftarrow & t^{-2}q^2 \text{ (crossing)}
 \end{array}$$

# Main result (Tricomplex $\mathcal{P}_3$ )



# Main result

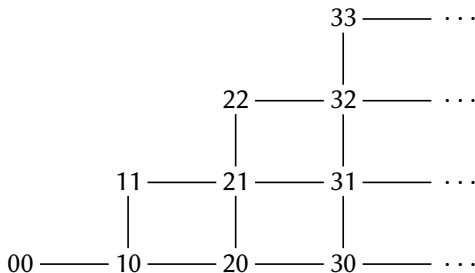
The infinite  $b$ -fold complex  $\mathcal{P}_b$  (for  $a = b$ ) has the following shape:

- There is a web  $W_\lambda$  for each partition  $\lambda$  with at most  $b$  parts.

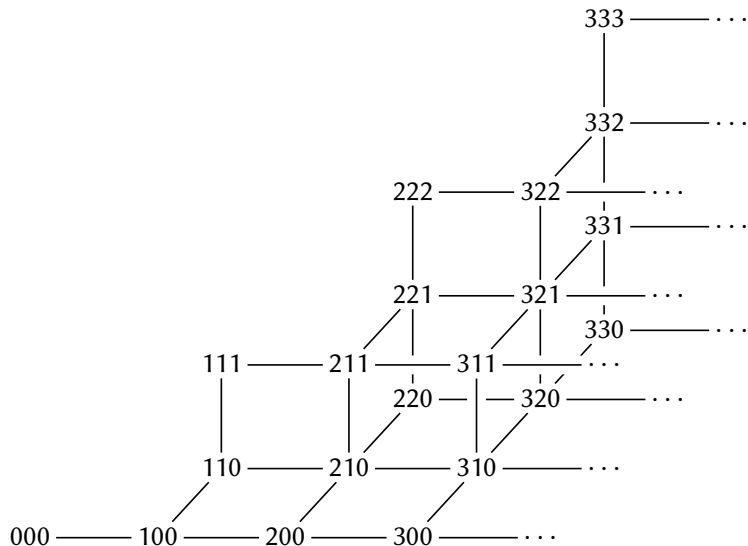
$$\lambda = (\lambda_1, \dots, \lambda_b) \in \mathbf{Z}^b \quad \lambda_1 \geq \dots \geq \lambda_b \geq 0$$

$W_\lambda$  is in cohomological degree  $-|\lambda| = -(\lambda_1 + \dots + \lambda_b)$ .

- There is a component of the differential connecting any two  $\lambda$  that differ in a single coordinate by 1.
- The minimal complex for  $\sigma^k$  with strands labeled by  $b$  is the subcomplex with objects indexed by  $\lambda$  with  $k \geq \lambda_1 \geq \dots \geq \lambda_b \geq 0$ .



# Main result

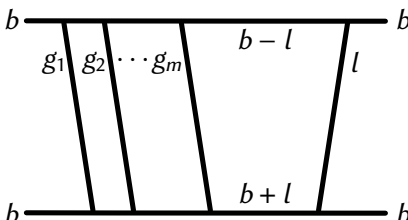


# Main result

To define  $W_\lambda$ :

The *grouping* of  $\lambda = (6, 6, 3, 2, 2, 2, 0, 0)$  is  $(2, 1, 3)$ .

If  $\lambda$  has grouping  $(g_1, g_2, \dots, g_m)$  and length  $l = g_1 + \dots + g_m$ , then  $W_\lambda$  is a  $q$ -shift of

$$W_l^{g_1, \dots, g_m} :=$$


# Main result

The differential is defined using another complex  $\mathcal{K}_b$  (new for  $b \geq 2$ ).  $\mathcal{K}_2$  is

$$\begin{array}{ccccccc}
 t^{-3}q^6 \text{ (cup) } & \xleftarrow{s_1 Z_{01}} & t^{-4}q^8 \text{ (II) } & \xleftarrow{Q_1} & t^{-5}q^{10} \text{ (II) } & \xleftarrow{Z_{10} \partial_1^*} & t^{-6}q^{10} \text{ (cup) } \\
 \downarrow Z_{10} & & \downarrow Z_{21} & & \downarrow Z_{21} & & \downarrow Z_{10} \\
 t^{-2}q^4 \text{ (II) } & \xleftarrow{Z_{12} s_1} & t^{-3}q^6 \text{ (DC) } & \xleftarrow{s_1^* Q_2 \partial_1} & t^{-4}q^8 \text{ (DC) } & \xleftarrow{\partial_1^* Z_{21}} & t^{-5}q^8 \text{ (II) } \\
 \downarrow Q_1 & & \downarrow Q_2 & & \downarrow Q_2 & & \downarrow Q_1 \\
 t^{-1}q^2 \text{ (II) } & \xleftarrow{Z_{12} \partial_1} & t^{-2}q^2 \text{ (DC) } & \xleftarrow{\partial_1^* Q_2 s_1} & t^{-3}q^4 \text{ (DC) } & \xleftarrow{s_1^* Z_{21}} & t^{-4}q^6 \text{ (II) } \\
 \downarrow Z_{01} & & \downarrow Z_{12} & & \downarrow Z_{12} & & \downarrow Z_{01} \\
 \text{ (cup) } & \xleftarrow{\partial_1 Z_{01}} & t^{-1} \text{ (II) } & \xleftarrow{Q_1} & t^{-2}q^2 \text{ (II) } & \xleftarrow{Z_{10} s_1^*} & t^{-3}q^4 \text{ (cup) }
 \end{array}$$

In general,  $\mathcal{K}_b$  is modeled on the edges and vertices in  $[0, 3]^b \subset \mathbf{R}^b$ .

# Main result

I expect that  $\mathcal{K}_b$  generalizes  $\mathcal{K}_1$  in the following ways.

Known:

$$\bullet \mathcal{K}_1 \otimes \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 1 \end{array} \simeq \mathcal{K}_1 \qquad \bullet \mathcal{K}_1 \otimes \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 1 \end{array} \simeq 0$$

Conjecture:

$$\bullet \mathcal{K}_b \otimes \begin{array}{c} b \\ \diagup \quad \diagdown \\ b \end{array} \simeq \mathcal{K}_b \qquad \bullet \mathcal{K}_b \otimes \begin{array}{c} b-l \\ \diagup \quad \diagdown \\ b+l \end{array} \simeq 0 \text{ for } l \in \{1, \dots, b\}$$

Known:

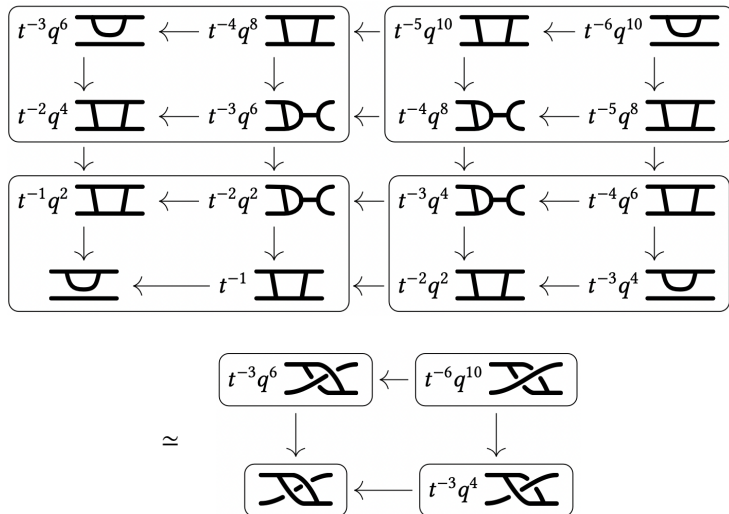
$$\boxed{\begin{array}{c} \text{---} \\ \text{---} \end{array} \leftarrow t^{-1}q \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}} \leftarrow \boxed{t^{-2}q^3 \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \leftarrow t^{-3}q^4 \begin{array}{c} \text{---} \\ \text{---} \end{array}}$$

$$\simeq \boxed{\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}} \leftarrow \boxed{t^{-3}q^4 \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}}$$



# Main result

Conjecture:



# Main result

$\mathcal{K}_b$  is shaped like the cube  $[0, 3]^b$ . The cube has  $2^b$  disjoint subcubes of size  $[0, 1]^b$ . The  $2^b$  corresponding subquotient complexes of  $\mathcal{K}_b$  are conjecturally

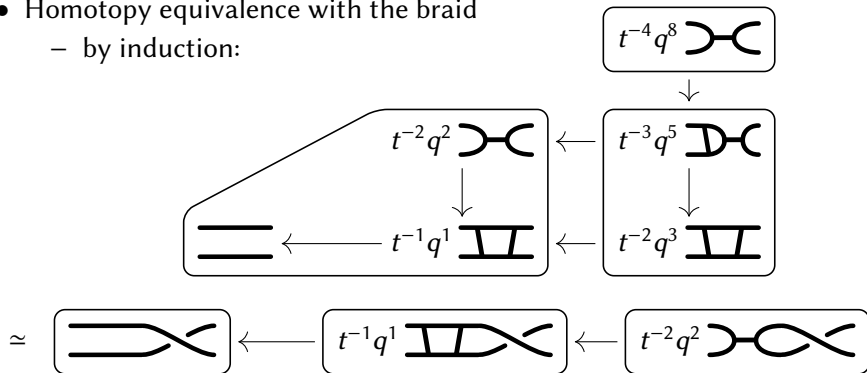


There are  $2^b$  ways for the weft string to weave through the  $b$  warp strings.

# Main result

Key steps in proving the main result:

- Construction of the complex
  - hardest part: educated guesswork, reverse engineering, luck
- Homotopy equivalence with the braid
  - by induction:



- Minimality
  - from the construction + the perverse filtration

# An application related to the free loop space of $\mathrm{Gr}(b, N)$

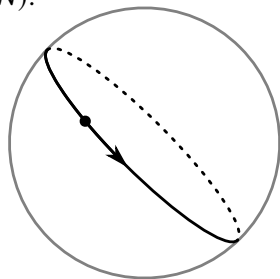
Consider the complex Grassmannian  $\mathrm{Gr}(b, N)$  where  $N \geq 2b$  for simplicity. The infinite simplex

$$\{(\lambda_1, \dots, \lambda_b) \in \mathbf{Z}^b \mid \lambda_1 \geq \dots \geq \lambda_b \geq 0\}$$

also shows up in the Riemannian geometry of  $\mathrm{Gr}(b, N)$ .

A closed geodesic  $\gamma: S^1 \rightarrow \mathrm{Gr}(b, N)$  locally minimizes length. The unitary action  $U: \mathrm{Gr}(b, N) \rightarrow \mathrm{Gr}(b, N)$  for  $U \in \mathrm{U}(N)$  sends a closed geodesic  $\gamma$  to another  $U \circ \gamma$ .

Let  $\mathcal{G}(\mathrm{Gr}(b, N))$  be the space of closed geodesics. It has many connected components, and  $\mathrm{U}(N)$  turns out to be transitive on each component.

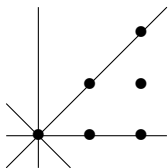


$$\begin{aligned}\mathcal{G}(\mathbf{CP}^1) &= \mathbf{CP}^1 \sqcup \frac{\mathrm{U}(2)}{\Delta \mathrm{U}(1)} \sqcup \frac{\mathrm{U}(2)}{\Delta \mathrm{U}(1)} \sqcup \frac{\mathrm{U}(2)}{\Delta \mathrm{U}(1)} \sqcup \dots \\ &= S^2 \sqcup UTS^2 \sqcup UTS^2 \sqcup UTS^2 \sqcup \dots\end{aligned}$$

# An application related to the free loop space of $\text{Gr}(b, N)$

$\mathcal{G}(\text{Gr}(b, N))$  = space of closed geodesics. There is a bijection

$$\{(\lambda_1, \dots, \lambda_b) \in \mathbf{Z}^b \mid \lambda_1 \geq \dots \geq \lambda_b \geq 0\} \longleftrightarrow \begin{array}{c} \text{connected components} \\ \text{of } \mathcal{G}(\text{Gr}(b, N)) \end{array}$$



The connected component corresponding to  $\lambda$  is a homogeneous space  $U(N)/K_\lambda$  for some subgroup  $K_\lambda \subset U(N)$ . So

$$\mathcal{G}(\text{Gr}(b, N)) = \bigsqcup_{\lambda} U(N)/K_\lambda.$$

A closed geodesic  $\gamma_\lambda: S^1 \rightarrow \text{Gr}(b, N) = U(N)/(U(b) \times U(N-b))$  is given by  $\gamma_\lambda(t) = \exp(tX_{B(\lambda)})$  for  $t \in [0, 1]$  where

$$X_{B(\lambda)} := \begin{pmatrix} 0 & B(\lambda) \\ -B(\lambda)^* & 0 \end{pmatrix} \in \mathfrak{u}(N) \quad B(\lambda) := \text{diag}(\lambda_1\pi, \dots, \lambda_b\pi) \in \mathbf{C}^{b \times (N-b)}$$

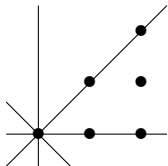
# An application related to the free loop space of $\mathrm{Gr}(b, N)$

Given the minimal complex for  $\sigma^{2k+1}$  with strands labeled  $b$ , we may interpret it in the  $\mathfrak{sl}_N$  webs and foams category. Form the braid closure to obtain the  $\Lambda^b$ -colored  $\mathfrak{sl}_N$  complex of  $T(2, 2k+1)$ .

$$\begin{array}{ccccc}
 & & & & \overline{W_{55}} \\
 & & & & \downarrow \\
 & & & \overline{W_{44}} \longleftarrow & \overline{W_{54}} \\
 & & & & \\
 & & \overline{W_{33}} & & \overline{W_{43}} \longleftarrow \overline{W_{53}} \\
 & & \downarrow & & \downarrow \\
 \overline{W_{22}} \longleftarrow & \overline{W_{32}} & & \overline{W_{42}} \longleftarrow & \overline{W_{52}} \\
 & & & & \\
 & \overline{W_{11}} & \overline{W_{21}} \longleftarrow \overline{W_{31}} & \overline{W_{41}} \longleftarrow \overline{W_{51}} \\
 & \downarrow & \downarrow & \downarrow & \downarrow \\
 \overline{W_{00}} \longleftarrow & \overline{W_{10}} & \overline{W_{20}} \longleftarrow \overline{W_{30}} & \overline{W_{40}} \longleftarrow & \overline{W_{50}}
 \end{array}$$

The components of the differential decrementing a part of  $\lambda$  of even size vanish. The complex splits into a direct sum of complexes, indexed by partitions with even sized parts, all at most  $2k+1$ .

# An application related to the free loop space of $\mathrm{Gr}(b, N)$



$$\begin{array}{ccccc}
 & & & & \overline{W_{55}} \\
 & & & & \downarrow \\
 & & & & \overline{W_{44}} \longleftarrow \overline{W_{54}} \\
 & & & \overline{W_{33}} & \overline{W_{43}} \longleftarrow \overline{W_{53}} \\
 & & & \downarrow & \downarrow \\
 & & & \overline{W_{22}} \longleftarrow \overline{W_{32}} & \overline{W_{42}} \longleftarrow \overline{W_{52}} \\
 & & & & \downarrow \\
 & & & & \overline{W_{41}} \longleftarrow \overline{W_{51}} \\
 & & & & \downarrow \\
 & & & & \overline{W_{40}} \longleftarrow \overline{W_{50}} \\
 & & \overline{W_{11}} & \overline{W_{21}} \longleftarrow \overline{W_{31}} & \\
 & & \downarrow & \downarrow & \\
 & & \overline{W_{00}} \longleftarrow \overline{W_{10}} & \overline{W_{20}} \longleftarrow \overline{W_{30}} & 
 \end{array}$$

The homology of the summand complex corresponding to  $2\lambda$  is isomorphic to  $H^*(U(N)/K_\lambda)!$  By Gugenheim–May '74, the cohomology of  $U(N)/K_\lambda$  is

$$H^*(U(N)/K_\lambda) \cong \mathrm{Tor}_{H^*(\mathrm{BU}(N))}(H^*(\mathrm{BK}_\lambda), \mathbf{Z})$$

The summand complex is  $\mathbf{Z} \otimes$  a  $H^*(\mathrm{BU}(N))$ -free resolution of  $H^*(\mathrm{BK}_\lambda)!$

# An application related to the free loop space of $\mathrm{Gr}(b, N)$

For the connection to the free loop space  $L\mathrm{Gr}(b, N) := \mathrm{Maps}(S^1, \mathrm{Gr}(b, N))$ :

$$\mathrm{KR}_N(T(2, \infty), \Lambda^b) = \mathrm{KR}_N(\overline{\mathcal{P}_b}) \cong \bigoplus_{\lambda} H^*(\mathrm{U}(N)/K_{\lambda}) \cong H^*(L\mathrm{Gr}(b, N))$$

The space of closed geodesics  $\mathcal{G}(\mathrm{Gr}(b, N))$  is a subspace of  $L\mathrm{Gr}(b, N)$ . It is the critical set of the energy functional  $E: L\mathrm{Gr}(b, N) \rightarrow \mathbf{R}$ , which is a perfect Morse–Bott function by Ziller '77.



# An application related to the free loop space of $\text{Gr}(b, N)$

$\mathcal{K}_b$  modeled on  $[0, 3]^b$ .

$\mathcal{P}_b$  modeled on partitions with at most  $b$  parts.

Braid closure in  $\mathfrak{sl}_N$  category computes  $H^*$  of connected components of  $\mathcal{G}(\text{Gr}(b, N))$ .

Discovered in reverse.

$$\begin{array}{ccc}
 & & \overline{W_{55}} \\
 & & \downarrow \\
 \overline{W_{44}} & \leftarrow & \overline{W_{54}} \\
 & & \downarrow \\
 & & \overline{W_{33}} \\
 & & \downarrow \\
 \overline{W_{22}} & \leftarrow & \overline{W_{32}} \\
 & & \downarrow \\
 & & \overline{W_{11}} \\
 & & \downarrow \\
 \overline{W_{00}} & \leftarrow & \overline{W_{10}}
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \overline{W_{33}} \\
 & & \downarrow \\
 \overline{W_{21}} & \leftarrow & \overline{W_{31}} \\
 & & \downarrow \\
 \overline{W_{20}} & \leftarrow & \overline{W_{30}}
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \overline{W_{53}} \\
 & & \downarrow \\
 \overline{W_{43}} & \leftarrow & \overline{W_{53}} \\
 & & \downarrow \\
 \overline{W_{42}} & \leftarrow & \overline{W_{52}} \\
 & & \downarrow \\
 \overline{W_{41}} & \leftarrow & \overline{W_{51}} \\
 & & \downarrow \\
 \overline{W_{40}} & \leftarrow & \overline{W_{50}}
 \end{array}$$

# An application related to the free loop space of $\mathrm{Gr}(b, N)$

Why did I believe that the  $\Lambda^b$ -colored  $\mathfrak{sl}_N$  homology of  $T(2, 2k+1)$  should be related to connected components of  $\mathcal{G}(\mathrm{Gr}(b, N))$ ?

For  $b = 1$  and  $N = 2$

$$\mathrm{Kh}(L) \rightrightarrows I^\#(L) \xleftarrow{\cdots} H^*(\mathcal{R}_2(L))$$

$I^\#(L)$  is Kronheimer–Mrowka’s  $\mathrm{SU}(2)$  instanton Floer homology: an “instanton deformation” of the cohomology of the meridian-traceless  $\mathrm{SU}(2)$  representation space  $\mathcal{R}_2(L)$  of Lin ’92. For  $L = T(2, 2k+1)$ , all three are isomorphic.

$$\mathrm{KR}_N(L, \Lambda^b) \rightrightarrows ??? \xleftarrow{\cdots} H^*(\mathcal{R}_N(L, \Lambda^b))$$

I computed  $\mathcal{R}_N(T(2, 2k+1), \Lambda^b)$  and noticed that it matches the connected components of  $\mathcal{G}(\mathrm{Gr}(k, N))$ .

# Thanks!

Thanks for listening!

