The minimal Rickard complexes of braids on two strands

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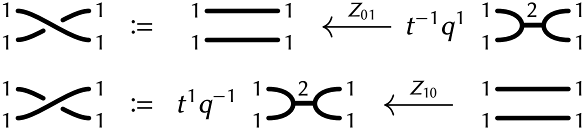
Outline

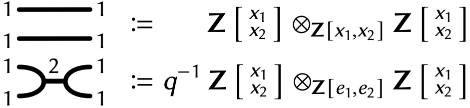
- Introduction
  - the minimal Rouquier complexes for braids on two strands
- Main result
  - the minimal Rickard complexes for braids on two strands
- An application related to the free loop space of the Grassmannian
  - also, how I found these minimal complexes

Introduction

Rouquier's categorification of the braid group assigns



The webs are shorthand for the Bott–Samelson bimodules

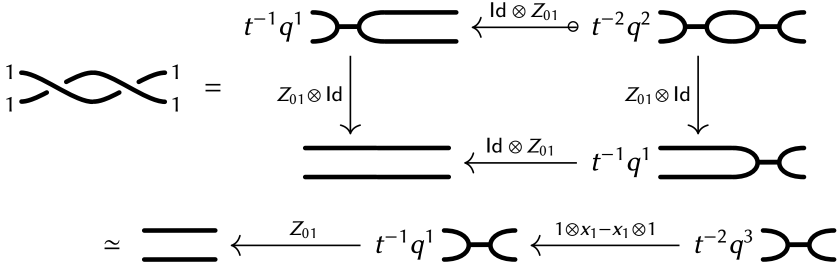


$Z_{01}\colon 1 \mapsto 1$ and $Z_{10}\colon 1 \mapsto 1 \otimes x_2 - x_1 \otimes 1$ are generators of their hom spaces. Tensor products satisfy the braid relations, so $\beta \in \mathrm{Br}_n$ determines a complex F_β. The triply-graded homology and the $\mathfrak{sl}_N$ homology of $\overline{\beta}$ are determined by F_β.

- $\mathrm{HHH}(\overline{\beta})$: take Hochschild homology of the bimodules, then homology.
- $\mathrm{KR}_N(\overline{\beta})$: pass to $\mathfrak{sl}_N$ foams category, form closure, then homology.

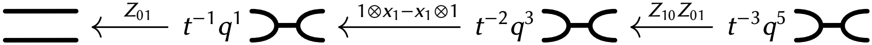
Introduction

Exercise 19.25 of Elias–Makisumi–Thiel–Williamson "Introduction to Soergel bimodules": There is a homotopy equivalence



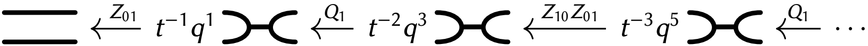
This three-term complex is *minimal*: it has no contractible direct summand. Any equivalent complex admits a deformation retract onto it.

The minimal complex of  is



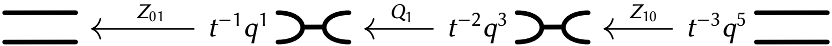
Introduction

The minimal complexes form an infinite complex $\mathscr{P}_1$



where $Q_1 = 1 \otimes x_1 - x_1 \otimes 1$. The leftmost $1 + k$ terms of $\mathscr{P}_1$ form a subcomplex that is the minimal complex of $\sigma^k \in \mathrm{Br}_2$. $\mathscr{P}_1$ is a categorified projector:

There is a related "compact" projector $\mathscr{K}_1$ (Hogancamp '18)



whose n-strand generalization plays an important role in the computation of HHH of torus links (Elias–Hogancamp '19, Hogancamp '17, Mellit '22, Hogancamp–Mellit '19)

Introduction

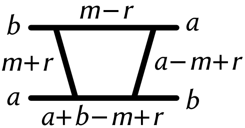
(Chuang–Rouquier)–Rickard complexes generalize Rouquier complexes:

$$:= \ \Gamma_0 \xleftarrow{\zeta_{01}} t^{-1}q^{1}\,\Gamma_1 \xleftarrow{\zeta_{12}} \cdots \xleftarrow{\zeta_{(m-1)m}} t^{-m}q^{m}\,\Gamma_m$$

$$:= \ t^{m}q^{-m}\,\Gamma_m \xleftarrow{\zeta_{m(m-1)}} \cdots \xleftarrow{\zeta_{21}} t^{1}q^{-1}\,\Gamma_1 \xleftarrow{\zeta_{10}} \Gamma_0$$

where $m := \min(a, b)$ and

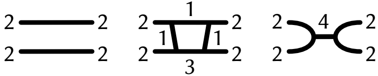
$\Gamma_r :=$

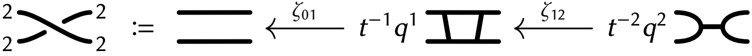


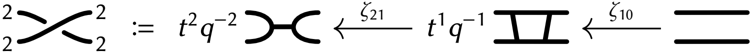
The webs $\Gamma_0, \ldots, \Gamma_m$ represent singular Bott–Samelson bimodules where

$$R_L := \begin{matrix}\mathrm{Sym}^b\\ \otimes \\ \mathrm{Sym}^a\end{matrix} \text{ and } R_R := \begin{matrix}\mathrm{Sym}^a\\ \otimes \\ \mathrm{Sym}^b\end{matrix} \qquad \mathrm{Sym}^c = \mathbf{Z}[x_1, \ldots, x_c]^{\mathfrak{S}_c}$$

i.e. $R_L = \mathbf{Z}[x_1, \ldots, x_{a+b}]^{\mathfrak{S}_b \times \mathfrak{S}_a}$ and $R_R = \mathbf{Z}[x_1, \ldots, x_{a+b}]^{\mathfrak{S}_a \times \mathfrak{S}_b}$. The labels represent 1-column partitions (exterior powers in the $\mathfrak{sl}_N$ category)

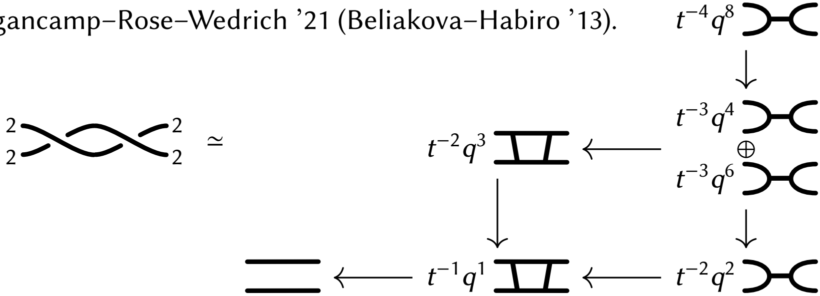
Example: $a = b = 2$. Webs 





The minimal complex for  was constructed by

Hogancamp–Rose–Wedrich ’21 (Beliakova–Habiro ’13).



Main result

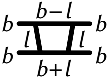
Theorem (W '25)

An explicit construction of the minimal complex homotopy equivalent to

for any number of crossings.

- The challenge of finding the minimal complex was posed by Wedrich '14.
- For $k \geq 0$ crossings, there are $1 + k + k^2 + \cdots + k^{\min(a,b)}$ indecomposable bimodules appearing in the minimal complex.
- They fit together into an infinite complex $\mathscr{P}_b$ for which

$$\mathscr{P}_b \otimes \mathscr{P}_b \simeq \mathscr{P}_b$$

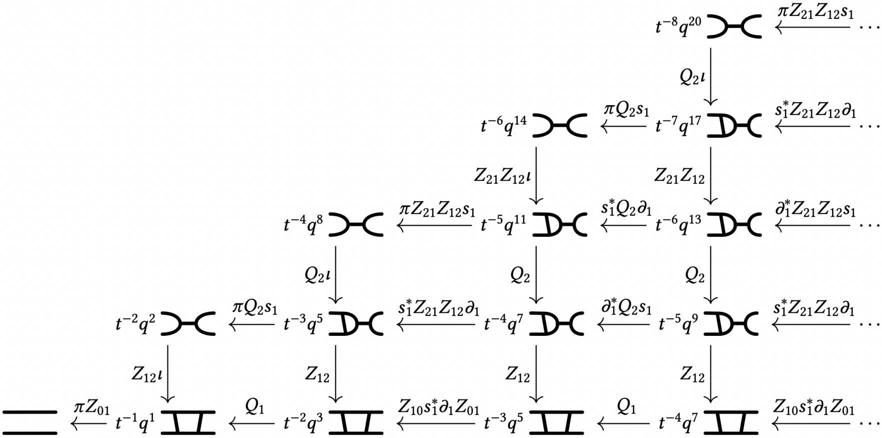
$\mathscr{P}_1 \otimes$  $\simeq 0$ for $l \in \{1, \ldots, b\}$

generalizing $\mathscr{P}_1$.

For today, assume $a = b$.

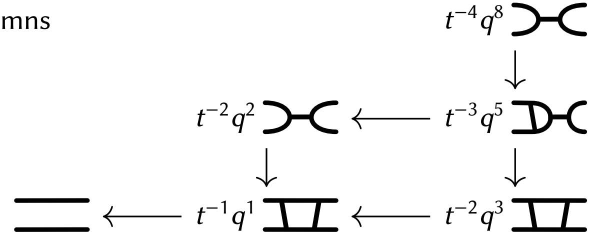
Main result

The minimal complex of σ^k with strands labeled by 2 consists of the leftmost $1 + k$ columns of the bicomplex $\mathscr{P}_2$

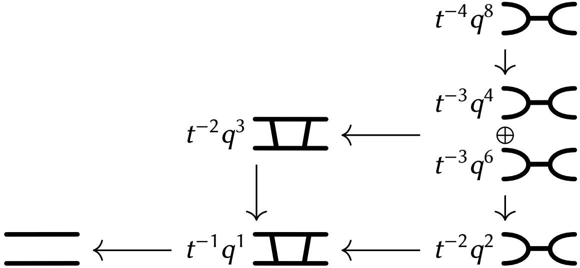


Main result

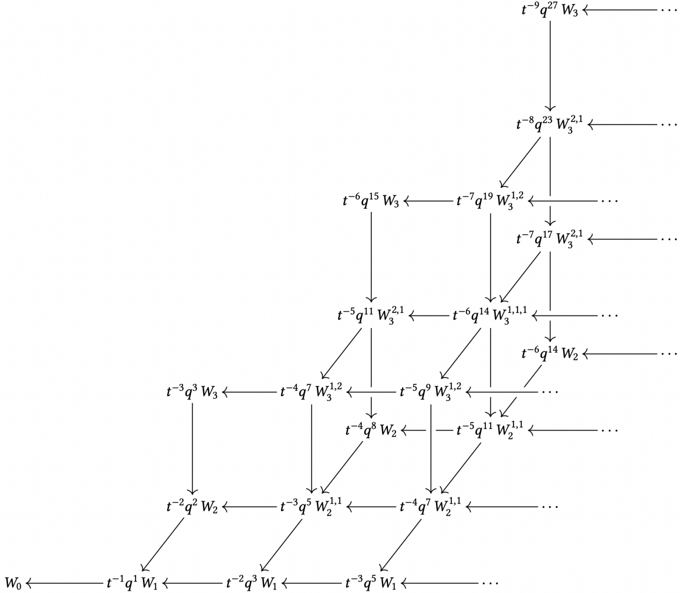
First three columns



are a rearrangement of Hogancamp–Rose–Wedrich's complex



Main result (Tricomplex $\mathscr{P}_3$)



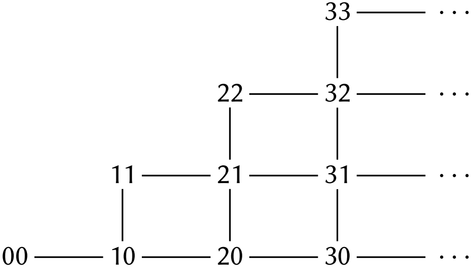
Main result

The infinite b-fold complex $\mathscr{P}_b$ (for $a = b$) has the following shape:

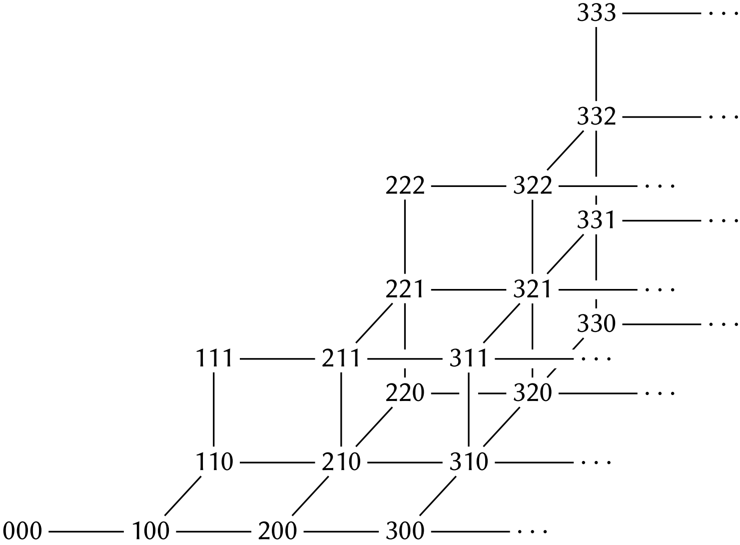
- There is a web W_λ for each partition λ with at most b parts.

  $$\lambda = (\lambda_1, \ldots, \lambda_b) \in \mathbf{Z}^b \qquad \lambda_1 \geq \cdots \geq \lambda_b \geq 0$$

  W_λ is in cohomological degree $-|\lambda| = -(\lambda_1 + \cdots + \lambda_b)$.
- There is a component of the differential connecting any two λ that differ in a single coordinate by 1.
- The minimal complex for σ^k with strands labeled by b is the subcomplex with objects indexed by λ with $k \geq \lambda_1 \geq \cdots \geq \lambda_b \geq 0$.



Main result


333 ⋯
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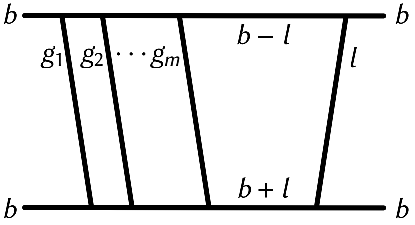
Main result

To define W_λ:

The *grouping* of $\lambda = (6, 6, 3, 2, 2, 2, 0, 0)$ is $(2, 1, 3)$.

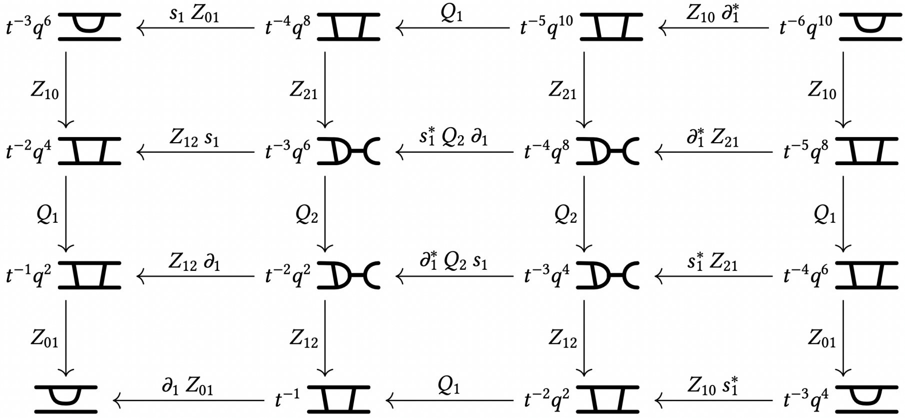
If λ has grouping $(g_1, g_2, \ldots, g_m)$ and length $l = g_1 + \cdots + g_m$, then W_λ is a q-shift of

$$W_l^{g_1,\ldots,g_m} :=$$



Main result

The differential is defined using another complex $\mathscr{K}_b$ (new for $b \geq 2$). $\mathscr{K}_2$ is



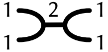
In general, $\mathscr{K}_b$ is modeled on the edges and vertices in $[0,3]^b \subset \mathbf{R}^b$.

Main result

I expect that $\mathscr{K}_b$ generalizes $\mathscr{K}_1$ in the following ways.

Known:

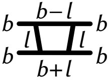
- $\mathscr{K}_1 \otimes$ 

$\simeq \mathscr{K}_1$
- $\mathscr{K}_1 \otimes$ 

$\simeq 0$

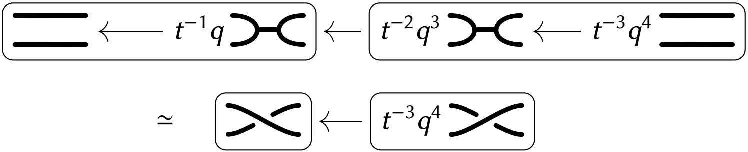
Conjecture:

- $\mathscr{K}_b \otimes$ 

$\simeq \mathscr{K}_b$
- $\mathscr{K}_b \otimes$ 

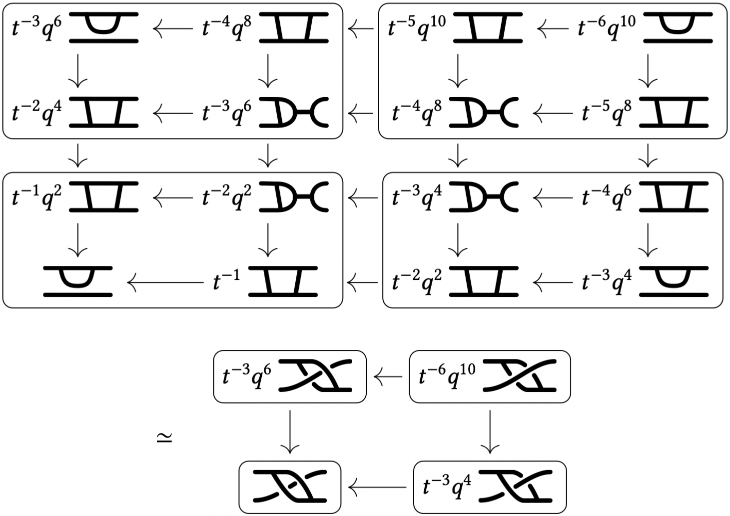
$\simeq 0$ for $l \in \{1, \ldots, b\}$

Known:



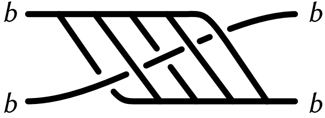
Main result

Conjecture:



Main result

$\mathscr{K}_b$ is shaped like the cube $[0, 3]^b$. The cube has 2^b disjoint subcubes of size $[0, 1]^b$. The 2^b corresponding subquotient complexes of $\mathscr{K}_b$ are conjecturally

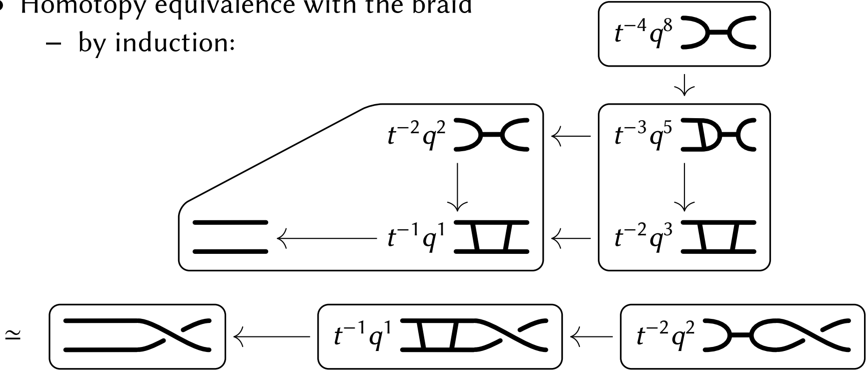


There are 2^b ways for the weft string to weave through the b warp strings.

Main result

Key steps in proving the main result:

- Construction of the complex
  - hardest part: educated guesswork, reverse engineering, luck
- Homotopy equivalence with the braid
  - by induction:



- Minimality
  - from the construction + the perverse filtration

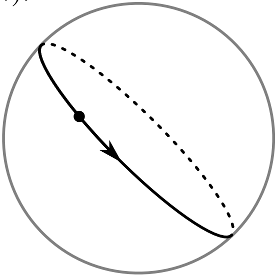
An application related to the free loop space of $\mathrm{Gr}(b, N)$

Consider the complex Grassmannian $\mathrm{Gr}(b, N)$ where $N \geq 2b$ for simplicity. The infinite simplex

$$\{(\lambda_1, \ldots, \lambda_b) \in \mathbf{Z}^b \mid \lambda_1 \geq \cdots \geq \lambda_b \geq 0\}$$

also shows up in the Riemannian geometry of $\mathrm{Gr}(b, N)$.

A *closed geodesic* $\gamma \colon S^1 \to \mathrm{Gr}(b, N)$ locally minimizes length. The unitary action $U \colon \mathrm{Gr}(b, N) \to \mathrm{Gr}(b, N)$ for $U \in \mathrm{U}(N)$ sends a closed geodesic γ to another $U \circ \gamma$.



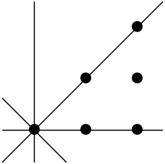
Let $\mathcal{G}(\mathrm{Gr}(b, N))$ be the space of closed geodesics. It has many connected components, and $\mathrm{U}(N)$ turns out to be transitive on each component.

$$\mathcal{G}(\mathbf{CP}^1) = \mathbf{CP}^1 \sqcup \frac{\mathrm{U}(2)}{\Delta \mathrm{U}(1)} \sqcup \frac{\mathrm{U}(2)}{\Delta \mathrm{U}(1)} \sqcup \frac{\mathrm{U}(2)}{\Delta \mathrm{U}(1)} \sqcup \cdots$$
$$= S^2 \sqcup UTS^2 \sqcup UTS^2 \sqcup UTS^2 \sqcup \cdots$$

An application related to the free loop space of $\mathrm{Gr}(b, N)$

$\mathscr{G}(\mathrm{Gr}(b, N))$ = space of closed geodesics. There is a bijection

$$\{(\lambda_1, \ldots, \lambda_b) \in \mathbf{Z}^b \mid \lambda_1 \geq \cdots \geq \lambda_b \geq 0\} \longleftrightarrow \begin{array}{c}\text{connected components}\\ \text{of } \mathscr{G}(\mathrm{Gr}(k, N))\end{array}$$



The connected component corresponding to λ is a homogeneous space $\mathrm{U}(N)/K_\lambda$ for some subgroup $K_\lambda \subset \mathrm{U}(N)$. So

$$\mathscr{G}(\mathrm{Gr}(b, N)) = \bigsqcup_\lambda \mathrm{U}(N)/K_\lambda.$$

A closed geodesic $\gamma_\lambda \colon S^1 \to \mathrm{Gr}(b, N) = \mathrm{U}(N)/(\mathrm{U}(b) \times \mathrm{U}(N-b))$ is given by $\gamma_\lambda(t) = \exp(tX_{B(\lambda)})$ for $t \in [0, 1]$ where

$$X_{B(\lambda)} := \begin{pmatrix} 0 & B(\lambda) \\ -B(\lambda)^* & 0 \end{pmatrix} \in \mathfrak{u}(N) \qquad B(\lambda) := \mathrm{diag}(\lambda_1\pi, \ldots, \lambda_b\pi) \in \mathbf{C}^{b\times(N-b)}$$

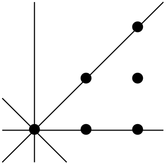
An application related to the free loop space of $\mathrm{Gr}(b, N)$

Given the minimal complex for σ^{2k+1} with strands labeled b, we may interpret it in the $\mathfrak{sl}_N$ webs and foams category. Form the braid closure to obtain the Λ^b-colored $\mathfrak{sl}_N$ complex of $T(2, 2k+1)$.

$$
\begin{array}{ccccccccccc}
&&&&&&&&&& \overline{W_{55}} \\
&&&&&&&&&& \downarrow \\
&&&&&&&& \overline{W_{44}} & \longleftarrow & \overline{W_{54}} \\
\\
&&&&&& \overline{W_{33}} && \overline{W_{43}} & \longleftarrow & \overline{W_{53}} \\
&&&&&& \downarrow && \downarrow && \downarrow \\
&&&& \overline{W_{22}} & \longleftarrow & \overline{W_{32}} && \overline{W_{42}} & \longleftarrow & \overline{W_{52}} \\
\\
&& \overline{W_{11}} && \overline{W_{21}} & \longleftarrow & \overline{W_{31}} && \overline{W_{41}} & \longleftarrow & \overline{W_{51}} \\
&& \downarrow && \downarrow && \downarrow && \downarrow && \downarrow \\
\overline{W_{00}} & \longleftarrow & \overline{W_{10}} && \overline{W_{20}} & \longleftarrow & \overline{W_{30}} && \overline{W_{40}} & \longleftarrow & \overline{W_{50}}
\end{array}
$$

The components of the differential decrementing a part of λ of even size vanish. The complex splits into a direct sum of complexes, indexed by partitions with even sized parts, all at most $2k + 1$.

An application related to the free loop space of $\mathrm{Gr}(b, N)$



$$
\begin{array}{ccccccccc}
 & & & & & & & & \overline{W_{55}} \\
 & & & & & & & & \downarrow \\
 & & & & & & \overline{W_{44}} & \longleftarrow & \overline{W_{54}} \\
 & & & & & & & & \\
 & & & & & \overline{W_{33}} & \overline{W_{43}} & \longleftarrow & \overline{W_{53}} \\
 & & & & & \downarrow & \downarrow & & \downarrow \\
 & & & \overline{W_{22}} & \longleftarrow & \overline{W_{32}} & \overline{W_{42}} & \longleftarrow & \overline{W_{52}} \\
 & & & & & & & & \\
 & & \overline{W_{11}} & \overline{W_{21}} & \longleftarrow & \overline{W_{31}} & \overline{W_{41}} & \longleftarrow & \overline{W_{51}} \\
 & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\
\overline{W_{00}} & \longleftarrow & \overline{W_{10}} & \overline{W_{20}} & \longleftarrow & \overline{W_{30}} & \overline{W_{40}} & \longleftarrow & \overline{W_{50}}
\end{array}
$$

The homology of the summand complex corresponding to 2λ is isomorphic to $H^*(\mathrm{U}(N)/K_\lambda)$! By Gugenheim–May '74, the cohomology of $U(N)/K_\lambda$ is

$$H^*(\mathrm{U}(N)/K_\lambda) \cong \mathrm{Tor}_{H^*(\mathrm{BU}(N))}(H^*(\mathrm{B}K_\lambda), \mathbf{Z})$$

The summand complex is $\mathbf{Z} \otimes$ a $H^*(\mathrm{BU}(N))$-free resolution of $H^*(\mathrm{B}K_\lambda)$!

An application related to the free loop space of $\mathsf{Gr}(b, N)$

For the connection to the free loop space $L\,\mathsf{Gr}(b, N) := \mathsf{Maps}(S^1, \mathsf{Gr}(b, N))$:

$$\mathsf{KR}_N(T(2, \infty), \Lambda^b) = \mathsf{KR}_N(\overline{\mathscr{P}_b}) \cong \bigoplus_{\lambda} H^*(\mathsf{U}(N)/K_\lambda) \cong H^*(L\,\mathsf{Gr}(b, N))$$

The space of closed geodesics $\mathscr{G}(\mathsf{Gr}(b, N))$ is a subspace of $L\,\mathsf{Gr}(b, N)$. It is the critical set of the energy functional $E\colon L\,\mathsf{Gr}(b, N) \to \mathbf{R}$, which is a perfect Morse–Bott function by Ziller '77.

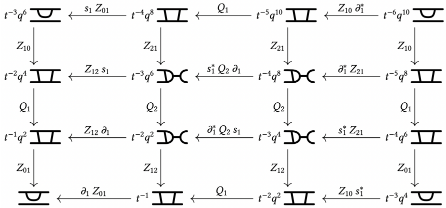
An application related to the free loop space of $\mathrm{Gr}(b, N)$

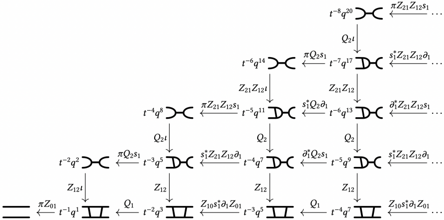
$\mathscr{K}_b$ modeled on $[0, 3]^b$.

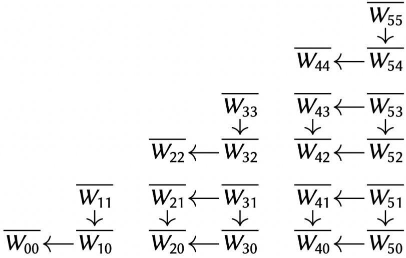
$\mathscr{P}_b$ modeled on partitions with at most b parts.

Braid closure in $\mathfrak{sl}_N$ category computes H^* of connected components of $\mathscr{G}(\mathrm{Gr}(b, N))$.

Discovered in reverse.







An application related to the free loop space of $\mathrm{Gr}(b, N)$

Why did I believe that the Λ^b-colored $\mathfrak{sl}_N$ homology of $T(2, 2k+1)$ should be related to connected components of $\mathscr{G}(\mathrm{Gr}(b, N))$?

For $b = 1$ and $N = 2$

$$\mathrm{Kh}(L) \Longrightarrow \mathrm{I}^\sharp(L) \dashleftarrow H^*(\mathscr{R}_2(L))$$

$\mathrm{I}^\sharp(L)$ is Kronheimer–Mrowka's $\mathrm{SU}(2)$ instanton Floer homology: an "instanton deformation" of the cohomology of the meridian-traceless $\mathrm{SU}(2)$ representation space $\mathscr{R}_2(L)$ of Lin '92. For $L = T(2, 2k+1)$, all three are isomorphic.

$$\mathrm{KR}_N(L, \Lambda^b) \Longrightarrow ??? \dashleftarrow H^*(\mathscr{R}_N(L, \Lambda^b))$$

I computed $\mathscr{R}_N(T(2, 2k+1), \Lambda^b)$ and noticed that it matches the connected components of $\mathscr{G}(\mathrm{Gr}(k, N))$.

Thanks!

Thanks for listening!