Diagrammatic singular Soergel bimodules

or, the circles in ICERM corridor

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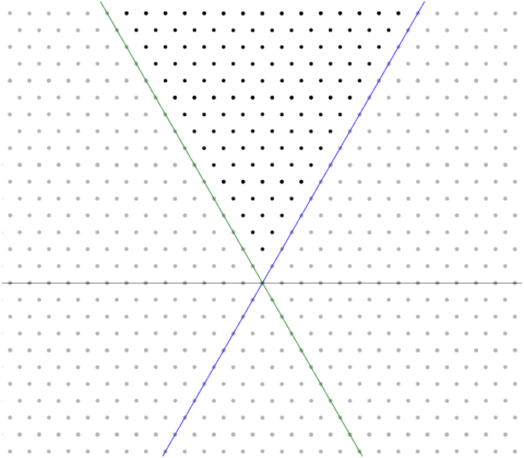
Some representation theory

Let G be a reductive group in characteristic $p > 0$.

Let $Rep = RepG$ be the category of rational representations of G.

The simples in $RepG$ are indexed by the dominant weights , for $G = SL_3$:



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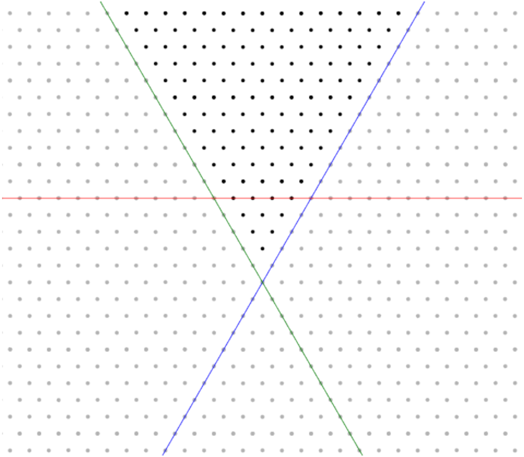
The affine Weyl group (W, S) acts on the weights.

For SL_3 ($p = 5$):

$S = \{r, g, b\}$



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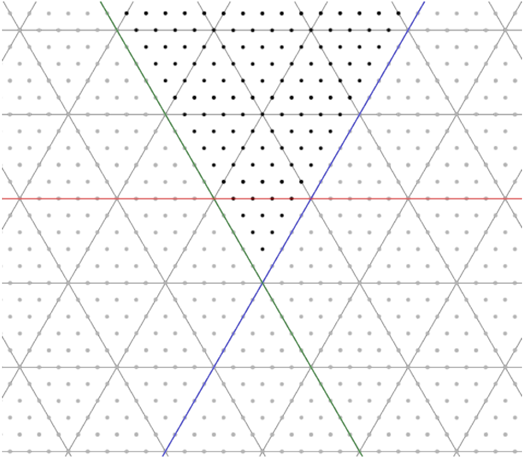
For SL_3 ($p = 5$):

$S = \{r, g, b\}$ generates
$W = \widetilde{S}_3$

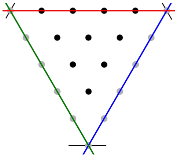


Coxeter presentation

$rgr = grg$
$rbr = brb$
$gbg = bgb$

$rr = e$
$gg = e$
$bb = e$

From weight lattice to (W, S)

For $\lambda \in \overline{C}$ (for SL_3 such λ is any weight/dot in ), let

Rep_λ = Serre subcategory generated by (dominant weights in)$W \cdot \lambda$.
Then

$$Rep = \oplus_{\lambda \in \overline{C}} Rep_\lambda$$

and

$$Rep_\lambda \cong Rep_\mu \qquad \text{when} \qquad I(\lambda) := \{s \in S \mid s \cdot \lambda = \lambda\} = I(\mu).$$

We thus get

$$Rep \cong \oplus_{I \subsetneq S} Rep_I^{\oplus m_I} \qquad \text{for} \qquad Rep_{I(\lambda)} :\cong Rep_\lambda.$$

For SL_3 with $(W, S) = (\widetilde{S}_3, \{r, g, b\})$ in characteristic 5 as above:

$$Rep \cong Rep_\varnothing^{\oplus 6} \oplus Rep_r^{\oplus 4} \oplus Rep_g^{\oplus 4} \oplus Rep_b^{\oplus 4} \oplus Rep_{r,g} \oplus Rep_{r,b} \oplus Rep_{g,b}$$

Between Rep_I, for various $I \subset S$, we have functors

$$U_s : Rep_I \to Rep_{I \sqcup s}, \quad D_s : Rep_{I \sqcup s} \to Rep_I$$

and their compositions.

The **(singular) Hecke category** $\mathcal{SH}(W, S)$ is the category of these functors (more or less). It has

- objects $I \subsetneq S$
- 1-morphisms generated by '$+s$' (= '$I+s$') and '$-s$' (= '$I-s$') $s \in S$ (do $+s \mapsto U_s, -s \mapsto D_s$ when act on Rep)
- 2-morphisms (later).

Special examples:

$$\Theta_s : Rep_\varnothing \xrightarrow{U_s^\varnothing} Rep_s \xrightarrow{D_s^\varnothing} Rep_\varnothing$$

These Θ_s follow the combinatorics of the Kazhdan-Lusztig generators in the Hecke algebra.

The **regular Hecke category** $\mathcal{H}(W, S)$ encodes the natural transformations between and decompositions of the endofunctors $\Theta_s\Theta_t \cdots \Theta_u$ on $Rep_\varnothing$.

Parabolic cosets

Let (W, S) be a Coxeter group.
For $I \subset S$, denote by W_I the subgroup generated by $I \subset W$.
Then (W_I, I) is a Coxeter group.

Example. $(W_\varnothing = \{e\}, \varnothing)$ is the trivial Coxeter group.

We assume throughout: $I \subset S$ is *finitary* i.e., W_I is finite.

For $I, J \subset S$, consider the **finitary (parabolic) double cosets**:

$$\mathrm{p} = W_I w W_J = \{xwy \in W \mid x \in W_I, y \in W_J\}$$

- Each p contains a unique Bruhat maximal element $\overline{\mathrm{p}}$
- Each p contains a unique Bruhat minimal element $\underline{\mathrm{p}}$

Singular expressions in (W, S)

A singular *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \dots, I_r]$$

of finitary subsets of S such that, for each i, either

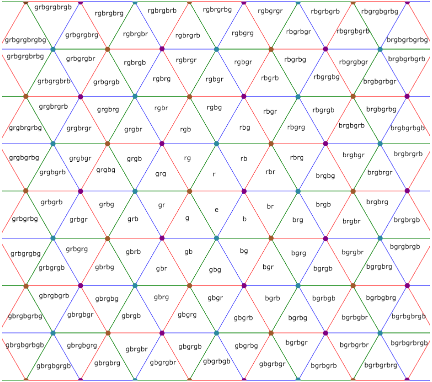
$$\text{either} \qquad I_i = I_{i-1} \setminus \{s\} \qquad \text{“}-s\text{“} \qquad \text{or} \qquad I_i = I_{i-1} \sqcup s \qquad \text{“}+s\text{”}$$

(That is, a singular expression is a 1-morphism in $\mathcal{SH}$.)

Equivalently, an expression is a forward path in the *Coxeter complex*

Max cells $\leftrightarrow W/W_\varnothing = W$;
codim. 1 cells $\leftrightarrow W/W_s$;
codim. 2 cells $\leftrightarrow W/W_{s,t}$;

⋮



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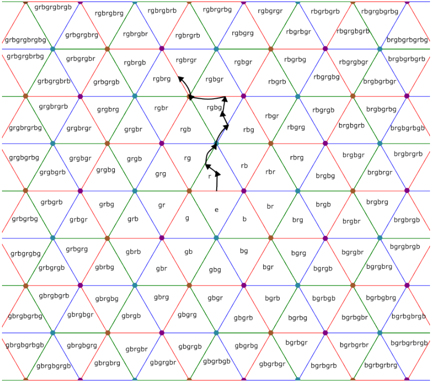


Example.

$[\{r\} - r + g + b - g - b + r + g - g]$
$= [r, \varnothing, g, \{g, b\}, b, \varnothing, r, \{r, g\}, r]$

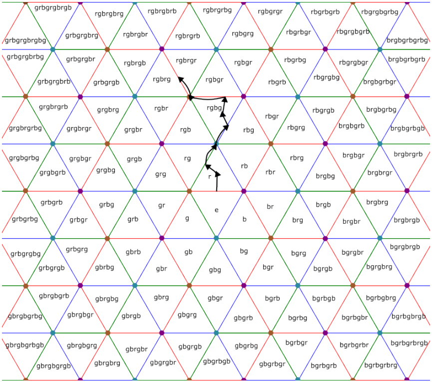
It expresses the coset $W_r rgbrg W_r$.

Reduced expressions (rexs)

Definition. A singular expression is *reduced* if its forward path in the Coxeter complex goes away(bigger or equal in Bruhat order) from the origin at each step.

The expression
$[\{r\} - r + g + b - g - b + r + g - g]$
$= [r, \varnothing, g, \{g, b\}, b, \varnothing, r, \{r, g\}, r]$ is not reduced

at the penultimate step "$+g$".



Bott-Samelson bimodules

R is the polynomial ring $Sym(V)$ where W acts nicely on V (reflection faithful,...)

For $I \subset S$, let R^I be the invariants under $I \subset W$.

To a (singular) expression $I_\bullet = [I_0, I_1, I_2, I_3, \cdots, I_r]$

associate the *Bott-Samelson* bimodule

$$\mathrm{BS}(I_\bullet) = R^{I_0} \otimes R^{I_1} \otimes R^{I_2} \otimes \cdots \otimes R^{I_r}$$

where each $\otimes$ is over the smaller R^{I_i}, e.g.,

$$\mathrm{BS}([\varnothing, s, \varnothing]) = R \otimes_{R^s} R^s \otimes_{R^s} R \cong R \otimes_{R^s} R.$$

Note that $\mathrm{BS}(I_\bullet)$ is an (R^{I_0}, R^{I_r})-bimodule.

We will consider the 2-category $\mathcal{S}\mathbb{BS}\mathrm{Bim} = \mathcal{S}\mathbb{BS}\mathrm{Bim}(W, S)$ with

- objects $I \subset S$
- 1-morphisms (shifts of) $\mathrm{BS}(I_\bullet)$
- 2-morphisms (graded) bimodule maps.

Soergel bimodules

The 2-category of *singular Soergel bimodules* $\mathcal{S}\mathbb{S}\mathrm{Bim}$ is the additive idempotent completion of $\mathcal{S}\mathbb{B}\mathbb{S}\mathrm{Bim}$. It has

- objects $I \subset S$
- 1-morphisms direct sum of summands of (shifts of) $\mathrm{BS}(I_\bullet)$
- 2-morphisms (graded) bimodule maps.

Theorem (Williamson). The indecomposable 1-morphisms in $\mathcal{S}\mathbb{S}\mathrm{Bim}(I, J)$ are B_{p} indexed by the double cosets $\mathrm{p} \in W_I \backslash W / W_J$. If $I_\bullet$ is a reduced expression of p then

$$\mathrm{BS}(I_\bullet) \in B_{\mathrm{p}} \oplus \langle B_{\mathrm{q}} \rangle_{\overline{\mathrm{q}} < \overline{\mathrm{p}}}.$$

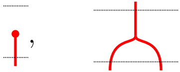
Regular Theorem (Soergel): The indecomposable Soergel bimodules in $\mathcal{S}\mathbb{S}\mathrm{Bim}(\varnothing, \varnothing)$ are B_w indexed by $w \in W$. If $w = st \cdots u$ is a reduced expression then $\mathrm{BS}(st \cdots u) \in B_w \oplus \langle B_x \rangle_{x<w}$.

These are not true for the same construction over a field of positive characteristic. We need another construction of the Hecke categories.

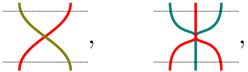
Regular answer

The **Elias-Williamson category** $\mathcal{H}(W,S)$ is a graded 2-category (with adjunctions, cyclic) category with one object and

- generating 1-morphisms B_s for $s \in S$ (other objects: $B_sB_tB_uB_uB_s, \cdots$);
- generating 2-morphisms



  for each $s \in S$ and



  $\cdots$

  for each braid relation $su = us$, $sts = tst$, $\cdots$, and polynomials in R;
- (an explicit list of) generating relations. "Soergel Calculus"

Then $\mathcal{H}(W,S) \cong \mathbb{BSB}\mathrm{im}$, but this definition works over $\mathbb{Z}$.

We have (over $\mathbb{Z}$ or any) that $[\mathcal{H}(W,S)]$ is the Hecke algebra, where $[B_s] = b_s$ is the Kazhdan-Lusztig generator.

Riche-Williamson, ... : decomposition numbers for $Rep_{\varnothing} G$ from this.

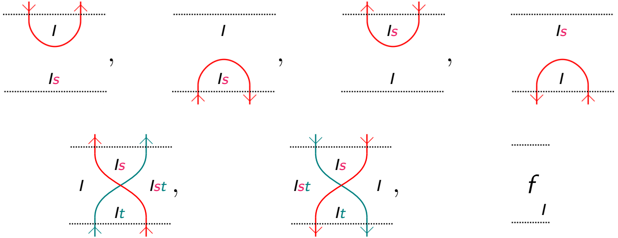
Singular Hecke category

The singular Hecke category $\mathcal{SH} = \mathcal{SH}(W, S)$ is a 2-category with

- objects $I \subset S$
- 1-generators '$\pm s$'(= '$I \pm s$') for ($I \subset S$ and $s \in S \setminus I$)

which, when R is appropriate over $\mathbb{R}$, is equivalent to $\mathcal{S}\mathbb{BS}\mathrm{Bim}$.

Some 2-morphisms in $\mathcal{S}\mathbb{BS}\mathrm{Bim}$:



for each $s, t \in S \setminus I$ and $f \in R^I$.

Theorem (Elias-K-Libedinsky-Patimo 2024) Certain diagrams made out of these (light leaves) form a basis of each graded hom space between $\mathrm{BS}(I_\bullet)$ and $\mathrm{BS}(J_\bullet)$. In particular, these form 2-generators of $\mathcal{SH}$.

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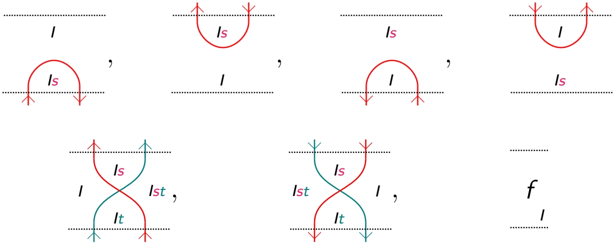
- objects $I \subset S$
- 1-generators '$\pm s$'(= '$I \pm s$') for ($I \subset S$ and) $s \in S \setminus I$
- 2-generators

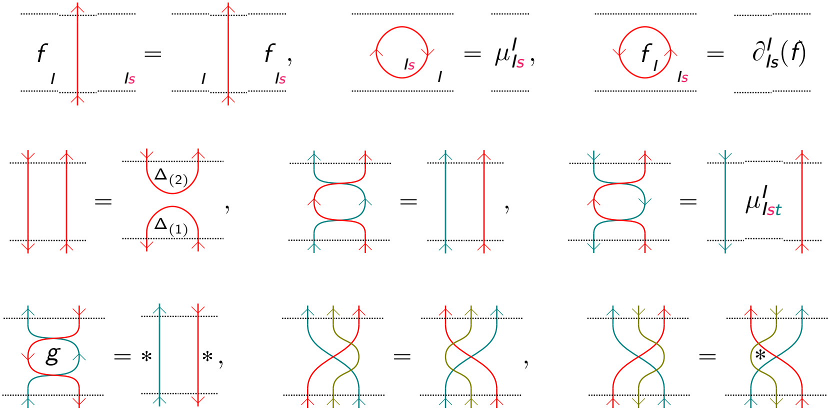


for each $s, t \in S \setminus I$ and $f \in R^I$

- relations (next slide)

Relations in $\mathcal{SH}$

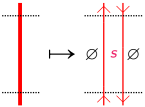
Some relations were given already by the 'Frobeinus diagrammatics' of Elias-Snyder-Williamson (2013):



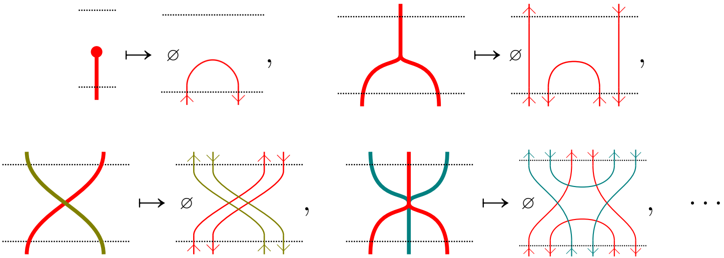
Some more relations known to Elias and Williamson around 2010
More work is needed for presentation (with ELP "Singular Soergel calculus" in progress at ICERM)

Comparison to regular diagrams

The natural embedding $\mathcal{H} \to \mathcal{SH}(\varnothing, \varnothing) \subset \mathcal{SH}$ is given by



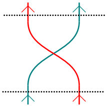
on 1-morphisms and



on 2-morphisms.

Singular braid morphisms

We have *upup* morphisms and we have *downdown* morphisms



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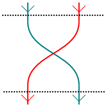
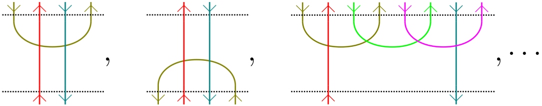
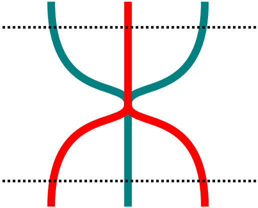


The *switchback* morphisms are of the form



The singular braid morphisms generates the regular braid morphisms:



Singular braid morphisms

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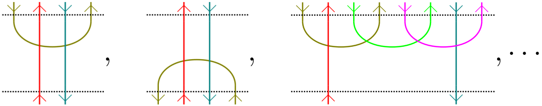
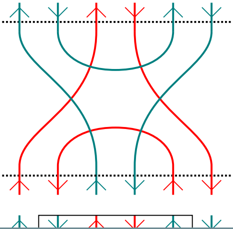


The singular braid relations generates the regular braid relations:

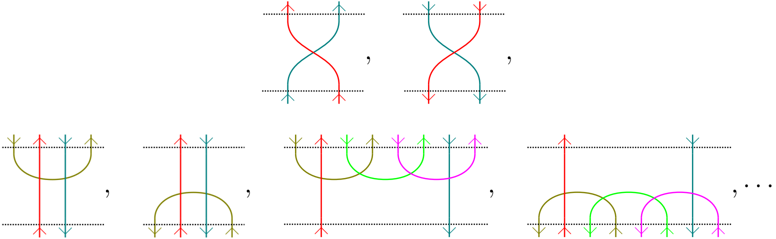


R(educed) ex(pression) moves

Singular Matsumoto Theorem [Elias-K.]

For $I, J \subset S$ and a double coset $\mathrm{p} \in W_I \backslash W / W_J$, any two reduced expressions of p are related by braid relations (upup, downdown, and switchback).

That is, the *rex moves*, generated by



give morphisms between any two $\mathrm{BS}(I_\bullet)$ for rexs $I_\bullet$ of p.

Recall: For any rex $I_\bullet$ of p, we have $\mathrm{BS}(I_\bullet) = B_\mathrm{p} \oplus$ some lower terms.

Proposition (EKLP). A rex move restricts to isomorphism on B_p.

A light leaf

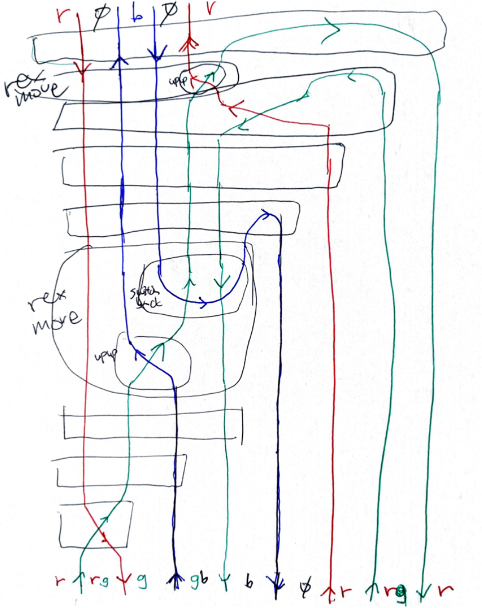
$[\{r\} - r + b - b + r]$ a rex for q

B_{q}

$\uparrow$

$\mathrm{BS}(\{r\} + g - r + b - g - b + r + g - g)$

(modulo lower terms)



$[\{r\} + g - r + b - g - b + r + g - g]$

The series (with Elias, Libedinsky, Patimo)

Elias-K., **A Singular Coxeter presentation**, 2105.08563

Circleland, Season 1 (2023-2024)

EKLP, Demazure operators for double cosets, 2307.15021

EKLP, Subexpressions and the Bruhat order for double cosets, 2307.15726

EKLP, On reduced expressions for core double cosets, 2402.08673

EKLP, **Singular Light Leaves**, 2401.03053

K, An Atomic Coxeter presentation, 2312.16666

Circleland, Season 2 (2024-)

EKLP, The Atomic Leibniz rule, 2407.13128

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EKLP, **Singular Soergel calculus (in progress)**

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