Categorification of iquantum groups II

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based on joint work with Jon Brundan and Weiqiang Wang

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PÉRIMÈTRE DE PHYSIQUE THÉORIQUE

Let's not beat around the bush. Consider a graph (I, E) with involution τ (potentially trivial). As discussed in Weiqiang's talk, we have an idempotented iquantum group $\dot{\mathbf{U}}^\imath$ associated to this data.

Theorem

There is a "2-iquantum group," that is, a 2-category $\mathfrak{U}^\imath$ whose Grothendieck group is $\dot{\mathbf{U}}^\imath$.

You'll note here that this is not the most general iquantum group I could have looked at (I've excluded all non-simply-laced cases). I'll get to that later.

To see the details:

I could just drop the definition here, but it's pretty complicated and wouldn't be that useful. Better to start by explaining logic of how it can be found:

- Important: quantum groups for I are a special case of iquantum groups, for $\mathbb{I} = I \cup -I$ with $\tau i = -i$. The "standard embedding" in this case is a twisted coproduct:

$$E_i \mapsto E_i \otimes 1 + 1 \otimes F_{-i} \qquad F_i \mapsto F_i \otimes 1 + 1 \otimes E_{-i}$$

  Our categorification will be a natural generalization of this case.
- In particular, the restriction of a highest weight module from $\mathbf{U} \otimes \mathbf{U}$ to $\mathbf{U}$ is a lowest $\otimes$ highest module $V_\lambda \otimes \Lambda_{-\mu}$.

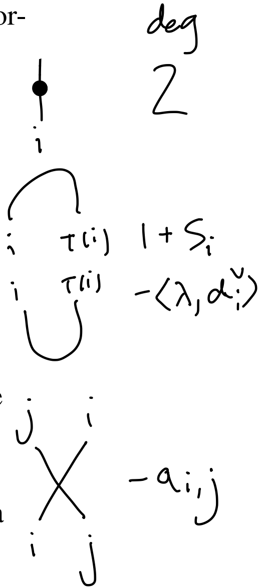
Just as in the quantum group case, we expect there to be objects λ for a weight lattice and 1-morphisms B_i for each of the generators b_i for $I \in I$. So we need to define morphisms between these and relations.

We have a bilinear form which tells us how many morphisms we should have $B_{\mathbf{i}} \to B_{\mathbf{j}}$.

1. We have $\langle b_i 1_\lambda, b_i 1_\lambda \rangle = 1/(1-q^{-2})$, so $\mathrm{Hom}(B_i, B_i) = \Bbbk[\bullet]$.
2. We have $\langle b_i b_j 1_\lambda, 1_\lambda \rangle = 0$ if $j \neq \tau i$, and there are $\varsigma_i \in \mathbb{Z}_{\geq 0}$ s. t.

$$\langle b_i b_{\tau(i)} 1_\lambda, 1_\lambda \rangle = q^{-1-\varsigma_i+\langle \lambda, \alpha_i^\vee \rangle}/(1-q^{-2})$$
$$\langle 1_\lambda, b_i b_{\tau(i)} 1_\lambda \rangle = q^{-1-\varsigma_i+\langle \lambda, \alpha_i^\vee \rangle}/(1-q^{-2})$$

   Thus, we just have cups and cups where opposite ends are labeled with i and τi. (Note, same as $\mathfrak{U}$: $E_i = B_i, F_i = B_{-i}$).
3. Examining $\langle b_i b_j 1_\lambda, b_j b_i 1_\lambda \rangle$ shows we also need a crossing.



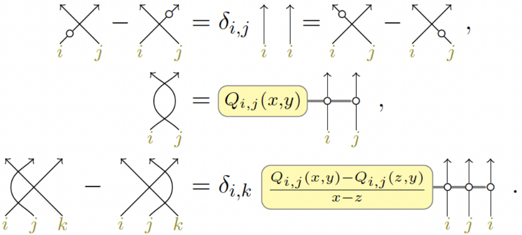
Relations?

So far, so good. Can guess the relations? A basis?

Let's think about the original 2-quantum group **U**. This depends on a choice of parameters. For simplicity, geometric parameters:

$$Q_{i,j}(x,y) := (x-y)^{\#(i \to j)}(y-x)^{\#(j \to i)}.$$

Has generators like described above and enough to "tighten" a diagram to get rid of unnecessary crossings.



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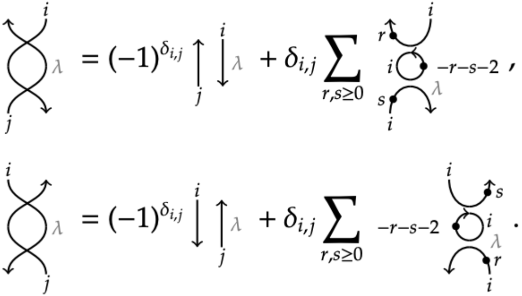

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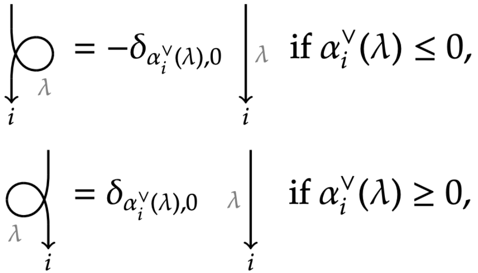
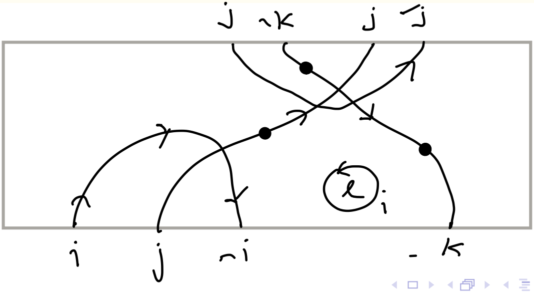


Theorem (Khovanov-Lauda)

Given two monomials $E_{\mathbf{i}}, E_{\mathbf{j}}$, the bilinear form $\langle E_{\mathbf{i}} 1_\lambda, E_{\mathbf{j}} 1_\lambda \rangle$ in $\mathbf{U}$ can be written as a sum over homotopy classes (rel boundary) of immersed 1-submanifolds D in $\mathbb{R} \times [0,1]$:

- *$\partial D \subset \mathbb{R} \times \{0,1\}$ with $|\mathbf{i}|$ points at $y = 0$ and $|\mathbf{j}|$ points at $y = 1$.*
- *We can consistently label bottom with $\mathbf{i}$ and top with $\mathbf{j}$.*

We weight each diagram with $q^{\deg D}(1-q^2)^{-(|\mathbf{i}|+|\mathbf{j}|)/2}$



Theorem (Brundan-Wang-W.)

Given two monomials $E_{\mathbf{i}}, E_{\mathbf{j}}$*, the bilinear form* $\langle E_{\mathbf{i}} 1_\lambda, E_{\mathbf{j}} 1_\lambda \rangle$ *in* $\mathbf{U}^\imath$ *can be written as a sum over homotopy classes (rel boundary) of immersed* 1*-submanifolds* D *in* $\mathbb{R} \times [0,1]$*:*

- $\partial D \subset \mathbb{R} \times \{0,1\}$ *with* $|\mathbf{i}|$ *points at* $y = 0$ *and* $|\mathbf{j}|$ *points at* $y = 1$*.*
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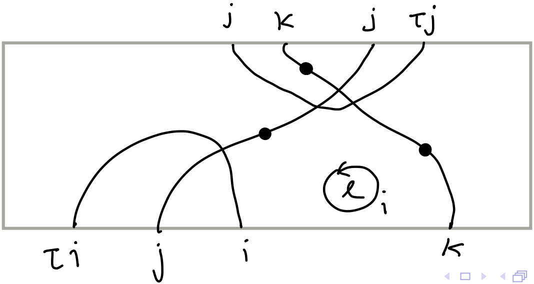

How do we guess the general relations?

That is, how do we find relations such that the Hom spaces have the dimension we expect?

There is basically one technique I know for checking this kind of linear independence for an algebra defined by generators and relations: mapping it into a space we already know a basis of.

Often we can think of this in terms of a representation, though that doesn't have to be the case.

Relations?

Meta-principle: Many categories that appear in this flavor of representation theory have deformations what make them simpler.

- Category $\mathcal{O}$ for $\mathfrak{g}$ deforms to category $\mathcal{O}$ for a Levi $\mathfrak{l}$.
- The Hecke algebra (Ariki-Koike algebra) at a special value deforms to be semisimple at a generic point.
- $\mathbb{F}_p G$ deforms to $\mathbb{Q}_p G$.

Prototypical lemma

finite rank

Assume A is a finite rank $\mathbb{k}[t,t^{-1}]$-algebra and B a $\mathbb{k}[t]$-algebra. If we have a surjective map $B[t^{-1}] \to A$, then $\dim_{\mathbb{k}} B \otimes_{\mathbb{k}[t]} \mathbb{k} \geq \dim_{\mathbb{k}(t)} A(t)$.

In particular, if a $\mathbb{k}[t]$-generating set of B maps to a basis of $A(t)$ over $\mathbb{k}(t)$, then it is a basis of ~~dim~~ $B \otimes_{\mathbb{k}[t]} \mathbb{k}$.

Fun with deformations

Let me explain how we can use this to prove non-degeneracy for the original quantum group.

- Consider the 2-quantum group $\mathfrak{U}$ for some Cartan datum on I.
- Given polynomials $m_i(u), n_i(u) \in \Bbbk[u]$, let λ be a weight such that $\langle \alpha_i^\vee, \lambda \rangle = \deg m_i - \deg n_i$. $i \in I$
- There is a representation $\mathcal{C}_{\mathbf{m},\mathbf{n}}$ of $\mathfrak{U}$ freely generated (up to Karoubi) by an object $\mathbb{V}$ killing the 2-morphisms

$$m_i(\bullet\!\downarrow) \colon F_i\mathbb{V} \to F_i\mathbb{V} \qquad n_i(\bullet\!\uparrow) \colon E_i\mathbb{V} \to E_i\mathbb{V}.$$

If $n_i = 1$ (or m_i) for all i, then the representations are the projectives over usual cyclotomic quotient. In particular, we know that (if $m_i(u) = u^{\langle \alpha_i^\vee, \lambda \rangle}$):

Theorem

The Grothendieck group of $C_{\mathbf{m},0}$ is V_λ, the simple of highest weight λ. The Grothendieck group of $C_{0,\mathbf{m}}$ is $\Lambda_{-\lambda}$, the simple of lowest weight $-\lambda$. In both categories, we know the dimension of Homs.

Natural guess is that $\mathcal{C}_{\mathbf{m},\mathbf{n}}$ categorifies $V_\mu \otimes \Lambda_{-\nu}$, making it close to the category $\mathcal{C}_{\mathbf{m},0} \otimes \mathcal{C}_{\mathbf{n},0}$.

That's not true in general. Certainly not if m_i and n_i are both monomials. On the other hand....

Theorem (W.)

If m_i and n_i are coprime, then we have an equivalence of categories

$$\mathcal{C}_{\mathbf{m},\mathbf{n}} \cong \mathcal{C}_{\mathbf{m},0} \otimes \mathcal{C}_{\mathbf{n},0}.$$

This is enough "big" representations of $\mathfrak{U}$ to prove that:

Theorem (W.)

The dimensions of Hom spaces in $\mathfrak{U}$ are given by the bilinear form.

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Theorem (W.)

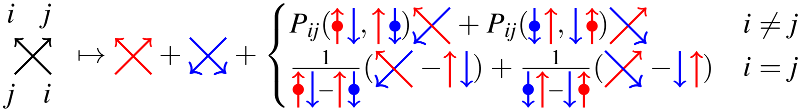
If $m_i(u) = u^{\mu_i}$ and $n_i(u) = (u-t)^{\nu_i}$, then we have an isomorphism $\mathcal{C}_{\mathbf{m},\mathbf{n}}[t^{-1}] \cong \mathcal{C}_{\mathbf{m},0} \otimes \mathcal{C}_{\mathbf{n},0}[t^{-1}]$.

The dimensions of Hom spaces in $\mathcal{C}_{\mathbf{m},\mathbf{n}}$ are defined by the bilinear form on $V_\mu \otimes \Lambda_{-\nu}$.

Proof (BWW):

1. Consider the tensor product of two copies $\mathfrak{U} \otimes \mathfrak{U}$.
2. Localize by inverting $\uparrow\!\bullet \cdots \downarrow - \uparrow \cdots \downarrow\!\bullet$. If m_i and n_i are coprime, $\mathcal{C}_{\mathbf{m},0} \otimes \mathcal{C}_{\mathbf{n},0}$ is a module over this category.
3. Define a “twisted coproduct functor” $\Delta \colon \mathfrak{U} \to \mathfrak{U} \odot \mathfrak{U}$.
4. Show that the (non-obvious) $\mathfrak{U}$-action induced on $\mathcal{C}_{\mathbf{m},0} \otimes \mathcal{C}_{\mathbf{n},0}$ gives an isomorphism to $\mathcal{C}_{\mathbf{m},\mathbf{n}}$.

This functor sends $E_i \mapsto E_i \oplus F_i$. On generators, it can be chosen to send $\uparrow\!\bullet \mapsto \uparrow\!\bullet + \downarrow\!\bullet$ and



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Fun with deformations

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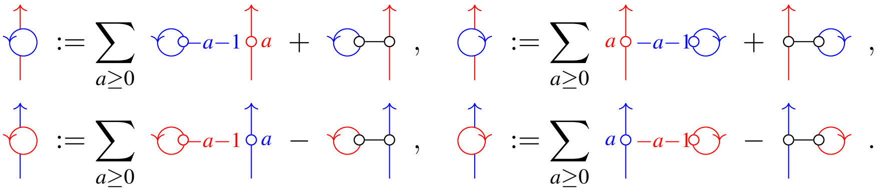
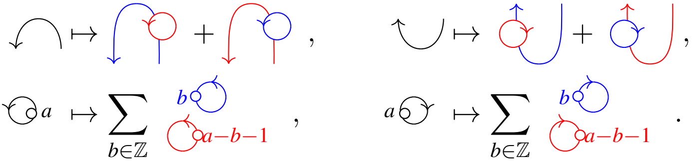


We can define the image of the leftward cup/cap under the functor Δ:



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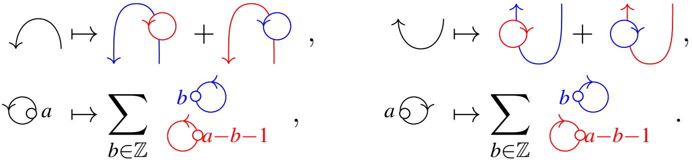

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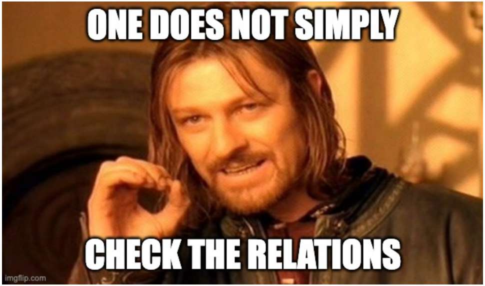

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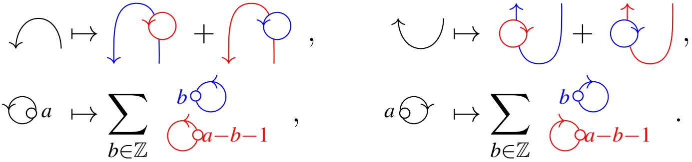

With this definition, the proof is "just" checking the relations.

Hopefully you guys thought that was interesting, but my intent was to give a template for how we prove this for a more general iquantum group.

For a general iquantum group, the inclusion $\mathbf{U}^\imath \hookrightarrow \mathbf{U}$ generalizes the coproduct $\mathbf{U} \hookrightarrow \mathbf{U} \otimes \mathbf{U}$, thought of as the iquantum group for $\mathbb{I} = I \cup -I$.

Thus, we should expect that $\mathfrak{U}^\imath$ has a "standard embedding" into a localization $\underline{\mathfrak{U}}$ of the 2-quantum group. In fact, we should be able to *find* the relations of $\mathfrak{U}^\imath$ by guessing the image of the generators under this map, and then seeing what 2-morphisms are sent to 0.

So, in theory, we “just” write down the images of our generators.

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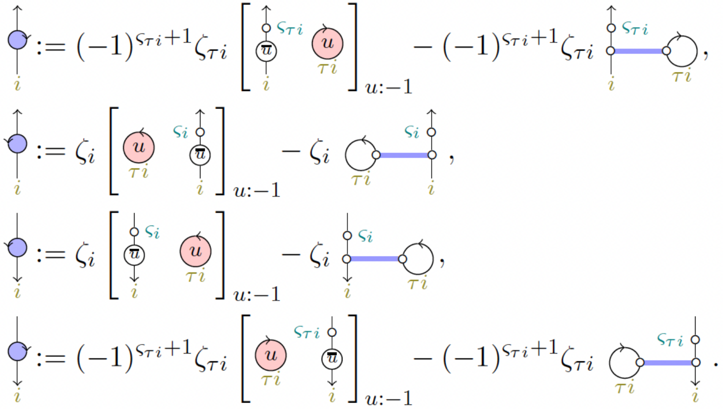
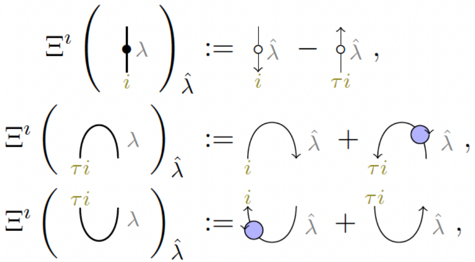
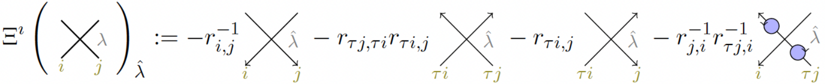
The tough part is the definition of the internal bubble—we can't get around writing it out, since can't use a partition into two components.



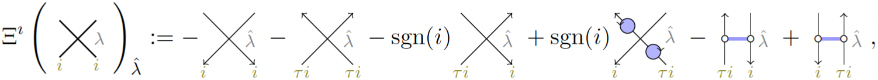
Once we have that, the rest of the images are relatively to guess based interactions with adjunction.

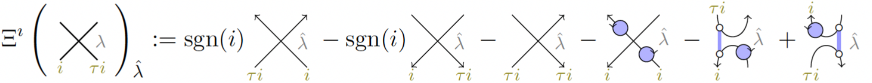


The standard embedding

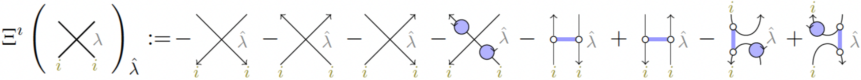


if $i \neq j$ *and* $i \neq \tau j$,

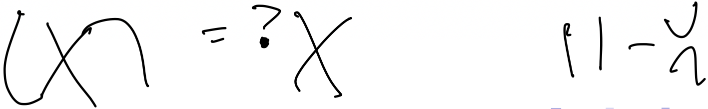




if $i \neq \tau i$, *and*

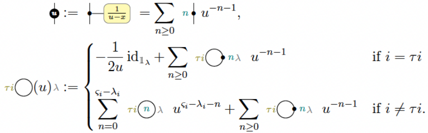


if $i = \tau i$.

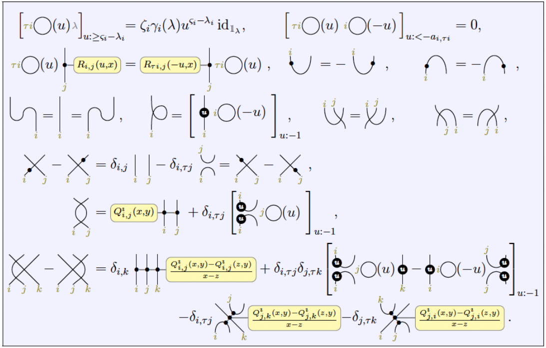


The final relations are pretty complicated.

Relations are expressed using dot and bubble generating functions (formal series in u^{-1}):

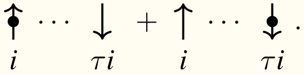


Also x, y, z denote dots on strings in order from left to right. Defining relations:



Proof (BWW):

1. Localize $\underline{\mathfrak{U}}$ by inverting

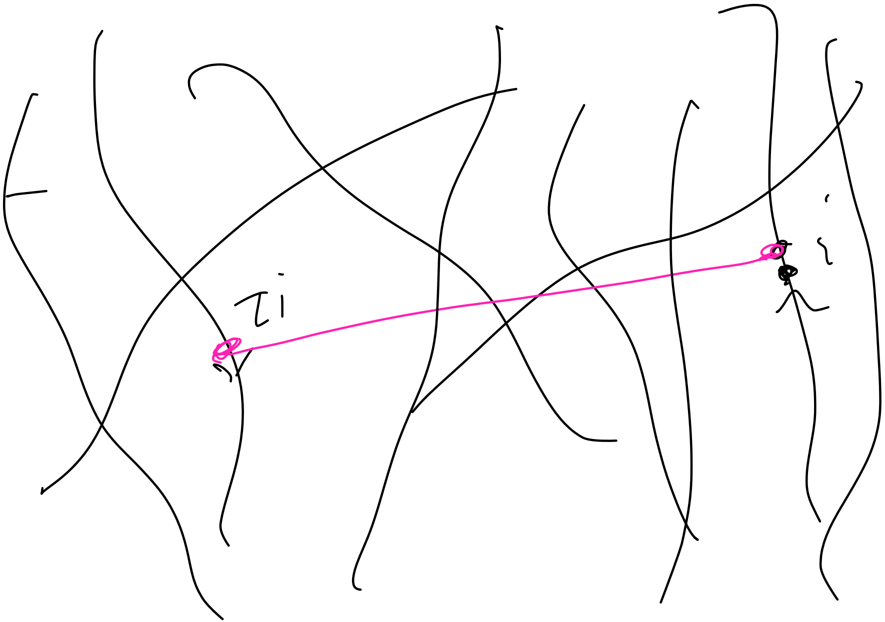


   For $\mathbf{m}, \mathbf{n}$ generic, $\mathcal{C}_{\mathbf{m},\mathbf{n}}$ is a module over this localization (here's where we need geometric parameters!)
2. Define a "twisted coproduct functor" $\Delta : \mathfrak{U}^\imath \to \underline{\mathfrak{U}}$.

With this categorified standard embedding we can prove:

Theorem

The 2-category $\mathfrak{U}^\imath$ is non-degenerate—its Hom-spaces match the bilinear form, and its Grothendieck group is $\mathbf{U}^\imath$.


τi

Nondegeneracy

We have only proven this thus far for geometric parameters in the symmetric Cartan case. The proof that $\mathcal{C}_{\mathbf{m},\mathbf{n}}$ is a module over the localization $\underline{\mathfrak{U}}$ has eluded us thus far.

The definition above makes sense much more generally, though there is a slightly odd condition that comes up: if i and j both τ-fixed then $a_{ij} \equiv a_{ji} \pmod 2$.

It doesn't seem like this should be necessary, but we need to make some kind of adjustment in our framework to avoid it.

Q/ What basis of G_0 comes from indecomposables?

For U, ADE, Lusztig's canonical basis ✓

?

Q/ In what generality do we get canonical basis?