iQuantum Groups and Categorification, I

ICERM 10/21/25

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Quantum group (QG)

$$\mathcal{U} = \langle E_i, F_i, K_i^{\pm 1} \rangle_{i \in I} \geq \mathcal{U}^- = \langle F_i \rangle, \qquad \Delta: E_i \mapsto E_i \otimes 1 + K_i \otimes E_i$$

Modified quantum group

idempotents

e.g. $\dot{\mathcal{U}}_q(\mathfrak{sl}_2) = \dot{\mathcal{U}} = \langle E 1_\lambda, F 1_\lambda, 1_\lambda \rangle_{\lambda \in \mathbb{Z}}$

$\quad \vee\!/$

Integral form $\dot{\mathcal{U}}_{\mathbb{Z}} = \langle E^{(r)} 1_\lambda, F^{(r)} 1_\lambda, 1_\lambda \rangle_{r \geq 0, \lambda \in \mathbb{Z}}$

divided powers

$$E^{(r)} = \frac{E^r}{[r]!}$$

iQuantum Group (iQG)

- Quasi-split Satake diagram (I, τ) : $\tau \curvearrowright I$ automorphism τ : $\tau^2 = 1$
- $\boxed{\text{iQG } \mathcal{U}^\imath} := \langle B_i, K_i K_{\tau i}^{-1} \rangle_{i \in I} \leq \mathcal{U}$ $\qquad B_i := F_i + q^{\varsigma_i} E_{\tau i} K_i^{-1}$
- $(\mathcal{U}, \mathcal{U}^\imath)$: quantum symmetric pairs (QSP) [Letzter] [Kolb]

$\Delta : \mathcal{U}^\imath \to \mathcal{U}^\imath \otimes \mathcal{U}$ (coideal) $\qquad \xrightarrow[q \mapsto 1]{} (\mathfrak{g}, \mathfrak{g}^{\omega\tau})$

Example

(1) (QG as iQG of diagonal type)

$(I \sqcup I', \text{swap}) \rightsquigarrow$ QSP $(\mathcal{U} \otimes \mathcal{U}, \mathcal{U})$ $\qquad$ [Bao-W]

$$\mathcal{U} \xrightarrow{\Delta} \mathcal{U} \otimes \mathcal{U} \xrightarrow{\omega \otimes 1} \mathcal{U} \otimes \mathcal{U}$$

(2) (split iQG) $\tau = \mathrm{Id}$ $\quad \mathcal{U}^\imath = \langle B_i \mid i \in I \rangle$ $\qquad (\mathcal{U}(\mathfrak{sl}_n), \mathcal{U}^\imath) \xrightarrow{q \mapsto 1} (\mathfrak{sl}_n, \mathfrak{so}_n)$

split rank one $\quad \mathcal{U}^\imath = \mathbb{Q}(q)[B] \qquad B = F + q^{-1} E K^{-1}$

[~~Recall $F^{(n)} = \frac{F^n}{[n]!}$~~

iDivided Powers (iDP)

Example $\quad \lambda \in \mathbb{Z}_{\geq 0}$

$$B_{\bar 0}^{(2)} = \frac{B^2}{[2]} \curvearrowright L(2\lambda)_{\mathbb{Z}} \qquad B_{\bar 1}^{(2)} = \frac{B^2 - 1}{[2]} \curvearrowright L(2\lambda+1)_{\mathbb{Z}}$$

Define $\{B_{\bar t}^{(n)}\}_{n \geq 0}$, for $t \in \{0,1\}$ via recursion

$\quad \parallel$

$B^{(n)} 1_{\bar t}$

$$\boxed{B \cdot B_{\bar t}^{(n)} = [n+1]\, B_{\bar t}^{(n+1)} + \delta_{\bar n, \bar t}\, [n]\, B_{\bar t}^{(n-1)}} \qquad B_{\bar t}^{(1)} = B$$

Modified iQG $\quad \dot{\mathcal{U}}^\imath = \dot{\mathcal{U}}^\imath 1_{\bar 0} \oplus \dot{\mathcal{U}}^\imath 1_{\bar 1}$ admits integral form:

$${}_{\mathbb{Z}}\dot{\mathcal{U}}^\imath 1_{\bar t} = \mathbb{Z}[q^{\pm 1}]\text{-span of iDP}$$

ıweights $X^\imath := X/\langle \mu+\tau\mu \mid \mu\in X\rangle$, e.g. $\overset{\text{split rank 1}}{=} \mathbb{Z}/2\mathbb{Z} = \{\bar{0},\bar{1}\}$

Theorem [Bao-W] [Berman-W'18]

$\mathcal{U}^\imath \hookrightarrow \mathcal{U} = \mathcal{U}_q\mathfrak{sl}_2 \curvearrowright \overset{\bullet\,\eta}{\Lambda} = L(2\lambda)$, $\lambda\in\mathbb{Z}_{\geq 0}$ — QSP $(\mathcal{U}, \mathcal{U}^\imath)$

- $B_{\bar{0}}^{(n)}\eta = 0$, for $n > 2\lambda$.

$\approx$ CB formula [Lusztig] $(\mathcal{U}\otimes\mathcal{U}, \mathcal{U})$

- $B_{\bar{0}}^{(n)}\eta \in F^{(n)}\eta + \sum_{m<n} q^{-1}\mathbb{Z}_{\geq 0}[q^{-1}]\, F^{(m)}\eta$, $\quad 0\leq n\leq 2\lambda$.

$q^* \begin{bmatrix} ** \\ c \end{bmatrix}_{q^2}$

ıcanonical basis

$\{B_{\bar{0}}^{(n)} 1_{\bar{0}}\}_{n\geq 0}$ for $\dot{\mathcal{U}}^\imath 1_{\bar{0}}$

- App. to ıSerre

+ ıbraid group symmetries

An isometry

$\mathcal{U}^\imath \hookrightarrow \mathcal{U} \curvearrowright L(\lambda) =$ η_λ

Inner product $(\cdot,\cdot)$ $=$ $(\cdot,\cdot)^\imath$

anti-inv ρ on $\mathcal{U}$: $F_i \mapsto q_i^{-1} E_i K_i^{-1}$

$\varsigma \curvearrowright \mathcal{U}^\imath$ e.g. $B_i \mapsto B_i$ if $i = \tau i$

$(\cdot,\cdot)^-$ on $\mathcal{U}^- \overset{\lambda \to \infty}{\simeq}$

$\lambda \to \infty$, fixing $\bar{\lambda} \in X^\imath$

$\simeq \mathcal{U}^\imath 1_{\bar{\lambda}}$ w/ $(\cdot,\cdot)^\imath$

$\mathcal{U}^- \xrightarrow{j\ \simeq} \mathcal{U}^\imath 1_{\bar{\lambda}}$

[BWW25] isometry

$$\lim_{\lambda \to \infty} \left(j(x)\eta_\lambda - x\eta_\lambda \right) = 0$$

$$U^- \longrightarrow \dot{U}^{\imath}\mathbf{1}_{\bar\lambda}$$

PBW / Canonical basis $\overset{j}{\rightsquigarrow}$ PBW/standard basis

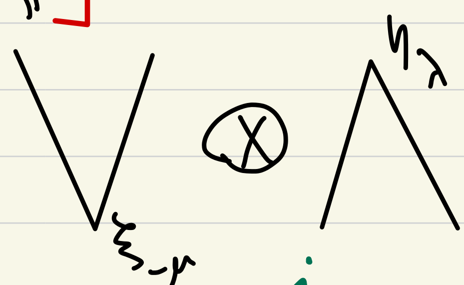
e.g. $\mathbb{Q}(q)[B]$ $\quad \Delta_n := j(F^{(n)})$ standard basis

$$\boxed{B^{(n)}_{\bar t} = \sum_{c=0}^{[\frac{n}{2}]} \frac{q^{-2(2c+1)}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4c})}\Delta_{n-2c},} \quad \text{if } n \not\equiv t\,(2)$$

- positivity, not integral
- q-Hermite (orth.) polynomials

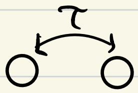
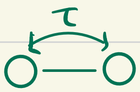
Special case $\quad$ [QSP of diagonal type]

[W'21] $\quad U \xrightarrow{\Delta} U \otimes U \curvearrowright$

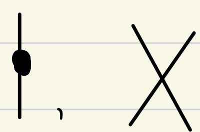


[L] $\quad \lambda, \mu \mapsto \infty \quad \lambda - \mu = \zeta$ fixed

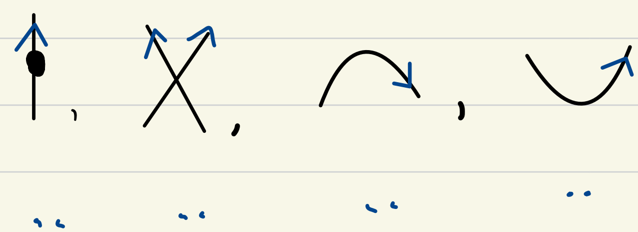
$$U^+ \otimes U^- \xrightarrow[\sim]{j} \dot{U}\mathbf{1}_\zeta$$

3 Rank one quasi-split iQGs	Geometric Type	ıCAT
 $\mathcal{U}^\imath \simeq U_q\mathfrak{sl}_2$	A	[L, R' 08]
	B/C	[BSWW 16]
SP_1  $\mathcal{U}^\imath \leq U_q\mathfrak{sl}_2$	D	[BWW'23] [BWW'25] : higher rank

Recall [Lauda] $\dot{\mathcal{U}}\mathfrak{sl}_2$ CAT'ed by NilHecke



$\dot{\mathcal{U}}^\imath(\mathfrak{sl}_2)$: add arrows



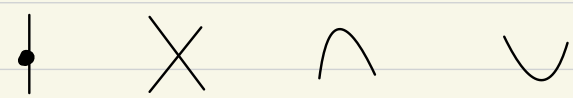
1 CATEGORIFICATION

Nil-Brauer NB_t is a ~~strict k-linear~~ monoidal category:

- generating object B

[objects $B^n \leftrightarrow n$, $1_{B^n} = \overbrace{| \, | \cdots |}^{n}$

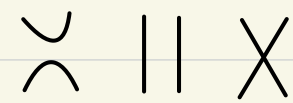
- generating morphisms



deg : 2 −2 0 0

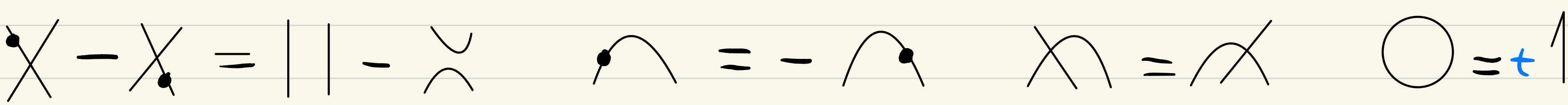
[like in $\mathbb{Z}/Sl_2$, but no orientation

[$\leftarrow (\cdot,\cdot)^2$ $\quad (B^2, B^2)^2 = \dfrac{2+q^2}{(1-q^{-2})^2}$



$q^0 \quad q^0 \quad q^{-2}$

- Relations [= nil version of affine Brauer category [R-Song]



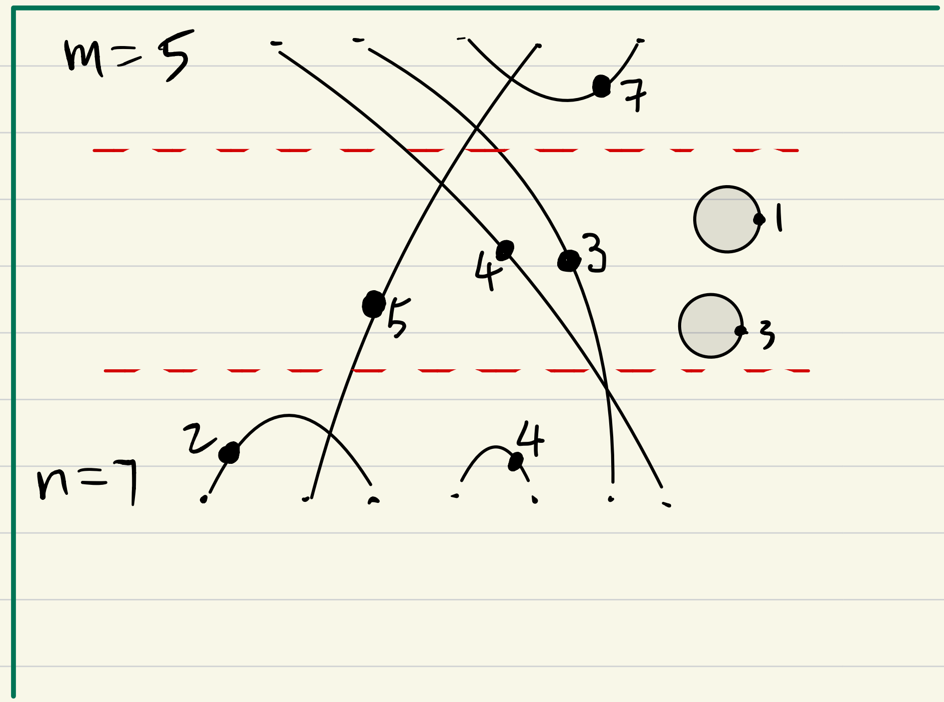
Remark Nil-Brauer $\simeq$ 2-category w/ objects $t \in \{0,1\} = X^2$ $\quad \boxed{t = 0,1}$

Theorem [BWW'23]

(1) $\mathrm{End}_{\mathcal{NB}_t}(\mathbb{1}) \xrightarrow{\simeq} \Gamma$ (Small) Sym Fun

[bubbles $2(-1)^{r+1} \bigcirc\!\!\bullet_r \longleftrightarrow g_r$

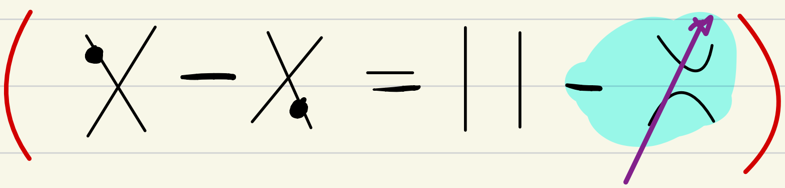
(2) $\mathrm{Hom}_{\mathcal{NB}_t}(B^n, B^m)$ is free Γ-module
w/ basis $\boxed{\text{dotted reduced diagrams}}$



& w/ $\mathrm{rank}_q = (B^n, B^m)^\imath$ in ${}_{\mathbb{Z}}\dot{\mathrm{U}}^\imath$

not subalg

The middle part $\neq$ nilHecke over Γ



(3) $K_0(NB_t) \xrightarrow{\sim} {}_{\mathbb{Z}}\mathcal{U}^{\iota}_t$

[B] Graded triangular basis $\approx$ [GRS] weakly triangular category

Sub/quotient $\Rightarrow$ NB admits "Cartan" $\simeq$ Nil-Hecke

$NB \twoheadrightarrow NB_{\geq n} := NB/\langle 1_m \mid m<n \rangle$

"Cartan" $\bar{1}_n NB_{\geq n} \bar{1}_n \simeq NH_n \otimes \Gamma$: projective module $P^{NH}(n)$

std $\Delta(n) \twoheadrightarrow L(n)$

$P(n) \twoheadrightarrow \Delta(n)$

L $j_!^n = NB_{\geq n}\bar{1}_n \otimes_{NB_n} (-)$ standardization functor

$\Delta(n) := j_!^n(P^{NH}(n))$

(4) $[P(n)] \longmapsto B_t^{(n)}$, $[\Delta(n)] \longmapsto \Delta_n$ standard basis

$P(n)$ admits a Δ-filtration

Categorification in higher rank II

Ben !